International Journal of Analysis and Applications
Volume 16, Number 6 (2018), 793-808
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-16-2018-793

VARIOUS KINDS OF FREENESS IN THE CATEGORIES OF KRASNER HYPERMODULES

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Abstract. The purpose of this paper is to study the concept of freeness in the categories of Krasner hypermodules over a Krasner hyperring. In this regards first we construct various kinds of categories of hypermodules based on various kinds of homomorphisms of hypermodules, such as homomorphisms, good homomorphisms, multivalued homomorphisms and etc. Then we investigate the notion of free hypermodule in these categories. This leads us to introduce different types of free, week free, ϵ^*-free and fundamental free hypermodules and obtain the relationship among them.

1. Introduction

The concept of hyperstructure is the generalization of the concept of algebraic structure. As a matter of fact, the hyperstructures are more natural and general than the algebraic structures. For the first time, hypergroups, as a suitable generalization of groups, were defined by Marty in 1934 [11]. Recently, many hyperstructures, for example, hypergroups, hyperrings, hyperfield, hypermodules and hypervector spaces, have been introduced and studied by many authors, e.g., [1], [3], [4], [5], [6], [7], [9], and [13]. For the first time, the concepts of hyperring and hyperfield were introduced by Krasner in connection with his work on valued fields. One of the most important hyperstructures satisfying the module-like axioms as a generalization of module is a type of hypermodule over a Krasner hyperring that we call it *Krasner hypermodule* (see [14]).

Received 2017-11-07; accepted 2018-01-17; published 2018-11-02.

2010 *Mathematics Subject Classification.* 18D35.

Key words and phrases. Krasner hyperring; Krasner hypermodule; free hypermodule.

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Interested readers can find a wide generalization of Krasner hypermodules in [15]. In this paper, we study these hyperstructures and focus on various freeness for a Krasner hypermodule. Next Section is a summary and reminder of [14].

2. Preliminaries

We start this section with some basic and fundamental concepts of category theory, and then we proceed to recall some requirements from hyperstructures theory.

Definition 2.1. *A category denoted by $\mathcal{C}$ consists of*

(1) *A class of objects: $A, B, C, \dots$*

(2) *A class of morphisms or arrows: $f, g, h, \dots$*

with the following data:

- *Given morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, that is, with: $cod(f) = dom(g)$ there is given a morphism: $g \circ f : A \longrightarrow C$ called the composition of morphisms f and g.*
- *Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : A \longrightarrow C$.*
- *Identity: $f \circ id_A = f = id_B \circ f$ for all $f : A \longrightarrow B$.*

Notation 2.1. *The classes of objects and morphisms of a category are denoted by $Ob(\mathcal{C})$ and $Mor(\mathcal{C})$, respectively. The class of all morphisms from A to B of category $\mathcal{C}$ is denoted by $Mor_{\mathcal{C}}(A, B)$. Note that a morphism $f : A \longrightarrow B$ in 2.1 is not necessarily a function from A to B.*

Recall that a zero object in an arbitrary category $\mathcal{C}$ is an object denoted by 0 such that $|Mor_{\mathcal{C}}(M, 0)| = |Mor_{\mathcal{C}}(0, M)| = 1$ for every $M \in Ob(\mathcal{C})$. The morphism $A \longrightarrow 0 \longrightarrow B$ of $Mor_{\mathcal{C}}(A, B)$ in a category $\mathcal{C}$ is called the *zero morphism* (see [2] or [8]).

Throughout this paper, $P(X)$ denotes the set of all subsets of X and $P^*(X) = P(X) \setminus \{\emptyset\}$. Here, $\mathcal{S}ets$ denotes the category of sets as objects with functions between sets as morphisms.

Now we state some basic definitions related to hyperstructures theory. Let H be a non-empty set. Then H together with the map

$$\cdot : H \times H \longrightarrow P^*(H)$$

$$(a, b) \mapsto a \cdot b$$

denoted by $(H, \cdot)$ is called a *hypergroupoid* and $\cdot$ is called a *hyperproduct* or *hyperoperation* on H. Let $A, B \subseteq H$. The hyperproduct $A \cdot B$ is defined as

$$A \cdot B = \bigcup_{(a,b) \in A \times B} a \cdot b.$$

If there is no confusion, for simplicity $\{a\}$, $A \cdot \{b\}$ and $\{a\} \cdot B$ are denoted by a, $A \cdot b$ and $a \cdot B$, respectively. Also we use ab instead of $a \cdot b$ for $a, b \in H$.

A non-empty set S together with the hyperoperation $\cdot$, denoted by $(S, \cdot)$ is called a *semihypergroup* if for all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. A semihypergroup $(H, \cdot)$ satisfying $x \cdot H = H \cdot x = H$ for every $x \in H$, is called a *hypergroup*.

Let x be an element of semihypergroup $(H, +)$ (resp., $(H, \cdot)$) such that $e+y = y+e = y$ (resp., $e \cdot y = y \cdot e = y$). Then x is called a scalar identity (resp., unit).

Every scalar identity or scalar unit in a semihypergroup H is unique. We denote the scalar identity (resp., unit) of H by 0_H (resp., 1_H).

Let 0_H (resp., 1_H) be the scalar identity (resp., unit) of hypergroup $(H, +)$ (resp., $(H, \cdot)$) and $x \in H$. An element $x' \in H$ is called an *inverse* of x in $(H, +)$ (resp., $(H, \cdot)$) if $0_H \in x+x' \cap x'+x$ (resp., $1_H \in x \cdot x' \cap x' \cdot x$).

A semihypergroup with a scalar identity is called a *hypermonoid*.

A non-empty set H together with the hyperoperation $+$ is called a *canonical hypergroup* if the following axioms hold:

(1) $(H, +)$ is a semihypergroup (associativity);
(2) $(H, +)$ is commutative (commutativity);
(3) there is a scalar identity 0_H (existence of scalar identity);
(4) for every $x \in H$, there is a unique inverse denoted by $-x$ such that $0_H \in x + (-x)$, which for simplicity we write $0_H \in x - x$ (existence of inverse);
(5) $\forall x, y, z \in H : \quad x \in y + z \Longrightarrow y \in x - z$ (reversibility).

Definition 2.2. *A non-empty set R together with the hyperoperation $+$ and the operation $\cdot$ is called a Krasner hyperring if the following axioms hold:*

(1) *$(R, +)$ is a canonical hypergroup;*
(2) *$(R, \cdot)$ is a semigroup including 0_R as a bilaterally absorbing element, that is $0_R \cdot x = x \cdot 0_R = 0_R$ for all $x \in R$;*
(3) *$(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.*

We say R has a unit (element) 1_R when $1_R \cdot r = r \cdot 1_R = r$ for all $r \in R$.

Example 2.1. *Let $(R, +, \cdot)$ be a ring and N a normal subgroup of semigroup $(R \setminus \{0\}, \cdot)$. Let $R' = \frac{R}{N}$ be the set of classes of the form $\bar{x} = x \cdot N$. If for all $\bar{x}, \bar{y} \in \bar{R}$, we define $\bar{x} +' \bar{y} = \{\bar{z} | \quad z \in \bar{x} + \bar{y}\}$, and $\bar{x} \cdot' \bar{y} = \overline{x \cdot y}$, then $(R, +', \cdot')$ is a Krasner hyperring.*

Now we state the concept of hypermodule over a hyperring. One of the most important and well-behaved classes of hypermodules is a class induced by the structure of a Krasner hyperring. To study such hyperstructures, we start with the following concept.

Definition 2.3. *Let X and Y be two non-empty sets. A map $*: X \times Y \longrightarrow Y$ sending (x, y) to $x * y \in Y$ is called a left external multiplication on Y. If $U \subseteq X$ and $V \subseteq Y$, then we define $u * V := \cup_{v \in V} u * v$ and $U * v := \cup_{u \in U} u * v$.*

Analogously, a right external multiplication on Y is defined by $*: Y \times X \longrightarrow Y$ sending (y, x) to $y * x \in Y$.

Definition 2.4. *Let $(R, +, \cdot)$ be a Krasner hyperring. A canonical hypergroup $(A, +)$ together with the left external multiplication $*: R \times A \longrightarrow A$ on A is called a left Krasner hypermodule over R if for all $r_1, r_2 \in R$ and for all $a_1, a_2 \in A$ the following axioms hold:*

(1) $r_1 * (a_1 + a_2) = r_1 * a_1 + r_1 * a_2$;

(2) $(r_1 + r_2) * a_1 = r_1 * a_1 + r_2 * a_1$;

(3) $(r_1 \cdot r_2) * a_1 = r_1 * (r_2 * a_1)$;

(4) $0_R * a_1 = 0_A$.

Remark 2.1. (i) *If A is a left Krasner hypermodule over a Krasner hyperring R, then we say that A is a left Krasner R-hypermodule. Clearly, a right Krasner R-hypermodule is defined with the map $*: A \times R \longrightarrow A$ possessing the smilar properties.*

(ii) *If R is a Krasner hyperring with 1_R and A is a Krasner R-hypermodule satisfying $1_R * a = a$ (resp. $a * 1_R = a$) for all $a \in A$, then A is said a unitary left (resp. right) Krasner R-hypermodule.*

(iii) *Throughout the paper, for convenience, by hyperring R we mean a Krasner hyperring with 1_R and by R-hypermodule A we mean a unitary left Krasner R-hypermodule unless otherwise stated.*

Definition 2.5. *A non-empty subset B of an R-hypermodule A is said to be an R-subhypermodule of A if B is an R-hypermodule itself, that is for all $x, y \in B$ and all $r \in R$, $x - y \subseteq B$ and $r * x \in B$.*

Proposition 2.1. *[14, Remark 3.2]*

(i) *In every (unitary) R-hypermodule A, $(-1_R) * a = -a$ for every $a \in A$.*

(ii) *In every hyperring R, $(-1_R) \cdot r = -r$ for every $r \in R$.*

Remark 2.2. *Let A be an R-hypermodule, $r, s \in R$ and $a \in A$. In the sequel, when there is no confusion, we use rs and ra instead of $r \cdot s$ and $r * a$, respectively.*

Unlike the category of modules, there are various types of homomorphisms in the categories of hypermodules.

Definition 2.6. *Let A and B be two R-hypermodules. A function $f : A \longrightarrow B$ that satisfies the conditions:*

(i) $f(x + y) \subseteq f(x) + f(y)$;

(ii) $f(rx) = rf(x)$;

for all $r \in R$ and all $x, y \in A$, is said to be an (inclusion) R-homomorphism from A into B.

Remark 2.3. *If in (i) of Definition 2.6 the equality holds, then f is called a strong (or good) R-homomorphism.*

The category whose objects are all R-hypermodules and whose morphisms are all R-homomorphisms is denoted by ${}_R h\mathbf{mod}$. The class of all R-homomorphisms from A into B is denoted by $hom_R(A, B)$.
Also, ${}_{R_s} h\mathbf{mod}$ is the category of all R-hypermodules whose morphisms are strong R-homomorphisms. The class of all strong R-homomorphisms from A into B is denoted by $hom^s_R(A, B)$. It is easy to see that ${}_{R_s} h\mathbf{mod}$ is a subcategory of ${}_R h\mathbf{mod}$, and we write ${}_{R_s} h\mathbf{mod} \preceq {}_R h\mathbf{mod}$ and read ${}_{R_s} h\mathbf{mod}$ is a subcategory of ${}_R h\mathbf{mod}$.

So far we have considered the morphisms or arrows, as usual, the functions between objects. But one can consider a morphism from A to B as a function from A into $P^*(B)$ called a multivalued function from A to B. Considering multivalued functions between sets, we have the following definition:

Definition 2.7. *Category of hypersets denoted by $\mathcal{HS}ets$ is a category with the following data:*

(1) $Ob(\mathcal{HS}ets) = Ob(\mathcal{S}ets)$,

(2) $Mor(\mathcal{HS}ets) =$ *the class of all multivalued functions between objects,*

that the composition $g \circ f$ is defined as the following:

$$(g \circ f)(a) = \bigcup_{b \in f(a)} g(b), \quad \forall a \in A, \tag{2.1}$$

and an identity morphism for an object A is $id_A(x) = \{x\}$ for all $x \in A$.

Now we are ready to define a generalization of usual morphisms of ${}_R h\mathbf{mod}$.

Definition 2.8. *If A and B are two R-hypermodules, then multivalued function f from A into B is a mapping $f : A \longrightarrow P^*(B)$ satisfying the following conditions:*

(i) $f(x + y) \subseteq f(x) + f(y)$;

(ii) $f(rx) = rf(x)$;

for all $r \in R$ and all $x, y \in A$, is said to be a multivalued R-homomorphism, for short R_{mv}-homomorphism.

Remark 2.4. *In (i) of Definition 2.8, if the equality holds, then f is called a strong (or good) multivalued R-homomorphism, for short an R_{smv}-homomorphism.*

Notation 2.2. *The class of all R_{mv}-homomorphisms (resp., R_{smv}-homomorphisms) from A into B is denoted by $Hom_R(A, B)$ (resp., $Hom^s_R(A, B)$).*

Proposition 2.2. *[14, Remark 3.9]*

(i) *For every $f \in hom_R(A,B)$, $f(0_A) = 0_B$.*

(ii) *For every $f \in hom_R(A,B)$, $f(-x) = -f(x)$ and $f(x-y) = f(x) - f(y)$.*

Let $f \in Hom_R(A,B)$ and $h \in Hom_R(B,C)$. The composition $h \circ f$ is defined as Equation 2.1. Also, for every R-hypermodule A, the R-homomorphism id_A with definition $id_A(x) = \{x\}$ for all $x \in A$ is the identity morphism as before. Hereafter, ${}_R\mathcal{H}\mathbf{mod}$ (resp., ${}_{R_s}\mathcal{H}\mathbf{mod}$) denotes the category whose objects are all R-hypermodules and whose morphisms from A to B are all R_{mv}-homomorphisms (resp., R_{smv}-homomorphisms) from A into B. Clearly, ${}_{R_s}\mathcal{H}\mathbf{mod}$ is a subcategory of ${}_R\mathcal{H}\mathbf{mod}$, i.e., ${}_{R_s}\mathcal{H}\mathbf{mod} \preceq {}_R\mathcal{H}\mathbf{mod}$.

Remark 2.5. (i) *Hereafter, we identify a singleton $X = \{a\}$ by its element a. Also, we sometimes write $f(a) = b$ instead of $f(a) = \{b\}$.*
So every single-valued morphism $f \in Hom_R(A,B)$ (resp., $f \in Hom^s_R(A,B)$) is an element of $hom_R(A,B)$ (resp., $hom^s_R(A,B)$), and conversely, every element of $hom_R(A,B)$ (resp., $hom^s_R(A,B)$) can be considered as an element of $Hom_R(A,B)$ (resp., $Hom^s_R(A,B)$), So ${}_Rh\mathbf{mod} \preceq {}_R\mathcal{H}\mathbf{mod}$ (resp., ${}_{R_s}h\mathbf{mod} \preceq {}_{R_s}\mathcal{H}\mathbf{mod}$).

(ii) *Let $f,g \in Hom_R(A,B)$. Define the relation $\leq$ on $Hom_R(A,B)$ in which $f \leq g$ means $f(x) \subseteq g(x)$ for all $x \in A$. Clearly $(Hom_R(A,B), \leq)$ is a poset.*

For convnience and distinguishing, we call ${}_Rh\mathbf{mod}$ and ${}_{R_s}h\mathbf{mod}$ *primary categories of Krasner R-hypermodules*. Also, ${}_R\mathcal{H}\mathbf{mod}$ and ${}_{R_s}\mathcal{H}\mathbf{mod}$ are called *secondary categories of Krasner R-hypermodules*.

3. Freeness of hypermodules

As it is well-known free objects play an important role in the study of modules theory. In [12] it was shown that free object does not exist in the category of hypergroups. Also, in [10] the notion of free hypermodules in the category of Krasner hypermodules was introduced. However, it is not clear that whether this definition is suitable in view point of category theory. Here we give various types of freeness in the categories of R-hypermodules and investigate the relationship between them.
Fix a hyperring $(R,+,\cdot)$. Let $\mathcal{U}(R)$ denote the set of all expressions of the form $\sum\limits_{i\in I}(\prod\limits_{j\in J_i} r_j)$ in which $r_j \in R$ where I and all J_i's are finite. The relation γ is defined on R is defined as follows:
for all $x,y \in R$,

$$x\gamma y \Longleftrightarrow \exists u \in \mathcal{U}(R): \quad x,y \in u.$$

The transitive closure of the relation γ is called the *fundamental relation* of R denoted by γ^*. Let $\gamma^*(r)$ denote the equivalence class containing $r \in R$. Then it is shown that $\frac{R}{\gamma^*}$ with the sum $\oplus$ and product $\otimes$ is a ring as follows:

for all $x, y \in R$,

$$\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(z) \quad \forall z \in \gamma^*(x) + \gamma^*(y);$$

$$\gamma^*(x) \otimes \gamma^*(y) = \gamma^*(x \cdot y).$$

The fundamental relation γ^* is the smallest equivalence relation such that $\frac{R}{\gamma^*}$ is a ring. The ring $\frac{R}{\gamma^*}$ is called the *fundamental ring* of R.

Also, the fundamental relation of an R-hypermodule A can be defined similar to above denoted by ϵ_A^* that $\frac{A}{\epsilon_A^*}$ is a *fundamental module* over the ring $\frac{R}{\gamma^*}$ with operations:

$$\epsilon_A^*(x) \oplus \epsilon_A^*(y) = \epsilon_A^*(z) \quad \forall z \in \epsilon_A^*(x) + \epsilon_A^*(y);$$

$$\gamma^*(r) \odot \epsilon_A^*(x) = \epsilon_A^*(r * x),$$

for all $x, y \in A$ and $r \in R$. The fundamental relation ϵ_A^* is the smallest equivalence relation such that $\frac{A}{\epsilon_A^*}$ is a module over the ring $\frac{R}{\gamma^*}$. (For more details, see [16] and [17]).

Now we introduce the following concept:

Definition 3.1. *An R-hypermodule F is said to be free on $X \subseteq F$ if for every R-hypermodule A and for any morphism $f : X \longrightarrow A$ in $\mathcal{H}Sets$, there exists a unique $\bar{f} \in Hom_R(F, A)$ such that $\bar{f} \circ i = f$ in which $i = id_F|_X$, i.e., the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & F \arrow[dl, "\exists! \bar{f}"] \\ A & \end{tikzcd}$$

Remark 3.1. *In [10] Ch. G. Massouros defined a free R-hypermodule differently. In the sense of Massouros, an R-hypermodule F is said to be free on $X \subseteq F$ if X generates F (see Definition 3.5) and for every R-hypermodule A and for an arbitrary morphism $f : X \longrightarrow A$ in $\mathcal{S}ets$, there exists $\bar{f} \in Hom_R(F, A)$ such that $\bar{f}(x) = \{f(x)\}$ for every $x \in X$, i.e., the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & F \arrow[dl, "\exists \bar{f}"] \\ A & \end{tikzcd}$$

*By this definition, the morphism $\bar{f}$ is not necessarily unique R_{mv}-homomorphism, but in [10], it was shown that $\bar{f}$ is a maximum R_{mv}-homomorphism, that is $\bar{f}$ is a maximum element in the poset $(Hom_R(F, A), \leq)$ such that $\bar{f} \circ i = f$. Clearly, a free R-hypermodule on $X \subseteq F$ based on this definition, is not really free on X in ${}_R\mathcal{H}$**mod** or ${}_Rh$**mod**.*

In the following, we introduce the concept of *weak freeness over $\mathcal{H}Sets$ or $\mathcal{S}ets$* motivated by Massouros definition.

Definition 3.2. *An R-hypermodule F is said to be weak free on $X \subseteq F$ over $\mathcal{H}Sets$ (resp., $\mathcal{S}ets$) if for every R-hypermodule A and for every morphism $f : X \longrightarrow A$ in $\mathcal{H}Sets$ (resp., $\mathcal{S}ets$), there exists a maximum $\bar{f} \in Hom_R(F, A)$ such that $\bar{f} \circ i = f$ in which $i = id_F|_X$, i.e., the diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & F \arrow[dl, "\exists \bar{f}"] \\ A & \end{tikzcd}$$

commutes.

Remark 3.2. (i) *Every free R-hypermodule on X is weak free on X over $\mathcal{H}Sets$.*

(ii) *Every free R-hypermodule on X in the sense of Massouros is really weak free on X over $\mathcal{S}ets$.*

Notation 3.1. *Let A be an R-hypermodule and $X \subseteq A$ and set*

$$\epsilon^*_A(X) := \cup_{x \in X} \epsilon^*_A(x).$$

For $X = \{x\}$, we write $\epsilon^*_A(x)$ instead of $\epsilon^*_A(\{x\})$.

Definition 3.3. *Two morphisms f and g in $Hom_R(A, B)$ is said to be ϵ^*-equivalent, and write $f \sim_{\epsilon^*} g$ if and only if $\epsilon^*_B(f(x)) = \epsilon^*_B(g(x))$ for every $x \in A$.*

Clearly $\sim_{\epsilon^*}$ is an equivalence relation on $Hom_R(A, B)$. We denote the equivalence class of f with respect to $\sim_{\epsilon^*}$ by $[f]$.

Definition 3.4. *An R-hypermodule F is said to be ϵ^*-free on $X \subseteq F$ over $\mathcal{H}Sets$ (resp., $\mathcal{S}ets$) if for every R-hypermodule B and for every morphism $f : X \longrightarrow B$ in $\mathcal{H}Sets$ (resp., $\mathcal{S}ets$), there exists an ϵ^*-unique $\bar{f} \in Hom_R(F, B)$ such that $\bar{f} \circ i = f$ in which $i = id_F|_X$, i.e., if there exists $\bar{f}' \in Hom_R(F, B)$ such that $\bar{f}' \circ i = f$, then $[f] = [g]$.*

Now we recall the notion of a *generating set*.

Definition 3.5. *Let R be a hyperring not necessarily with 1_R and A be an R-hypermodule and $X \subseteq A$. $\langle X \rangle$ denotes the smallest R-subhypermodule of A containing X or the intersection of all R-subhypermodules of A containing X.*

Notation 3.2. *Let A be an R-hypermodule. Then for $x \in A$ and $m \in \mathbb{Z}$,*

$$mx = \begin{cases} \underbrace{x + x + \cdots + x}_{m \text{ times } x} & \text{if } n > 0 \\ 0_A & \text{if } n = 0 \\ \underbrace{-x - x - \cdots - x}_{-m \text{ times } x} & \text{if } n < 0. \end{cases}$$

Proposition 3.1. *For a subset X of a not necessarily unitary R-hypermodule A, $\langle X\rangle$ is the set*

$$\{a \in \sum_{i=1}^{m} r_i x_i + \sum_{i=1}^{n} n_i x_i + \sum_{i=1}^{k} k_i(x_i - x_i) \mid r_i \in R,\ x_i \in X,\ m,n,k,k_i \in \mathbb{N},\ n_i \in \mathbb{Z}\}.$$

Proof. It is clear to straightforward. □

Definition 3.6. *The set X is said to be a generating set for an R-hypermodule A, or X generates A, if $A = \langle X\rangle$. Here, A is called finitely generated if it has a finite generating set.*

Let $X = \{x\}$. For simplicity, we use $\langle x\rangle$ instead of $\langle X\rangle$. It is easy to see that

$$\langle x\rangle = \{a \in rx + mx + \sum_{i=1}^{n} n_i(x - x) \mid r \in R,\ m \in \mathbb{Z},\ n, n_i \in \mathbb{N}\}.$$

Let $Rx = \{rx \mid r \in R,\ x \in A\}$.

Remark 3.3. *Let R be with identity 1_R and A be a unitary R-hypermodule. Then*

(i) $\langle x\rangle = Rx$. *Indeed, since*

$$\begin{aligned} mx &= \underbrace{x + x + \cdots + x}_{m \text{ times } x} = \underbrace{1_R * x + 1_R * x + \cdots + 1_R * x}_{m \text{ times } x} \\ &= (\underbrace{1_R + 1_R + \cdots + 1_R}_{m \text{ times } 1_R}) * x \subseteq Rx \end{aligned}$$

and (by Proposition 2.1 (i))

$$x - x = x + (-x) = 1_R * x + ((-1_R) * x) = (1_R + (-1_R)) * x = (1_R - 1_R) * x \subseteq Rx,$$

we have $\langle x\rangle = Rx$.

(ii) *Letting $X = \{x_i\}_{i\in I} \subseteq A$, $A = \langle X\rangle$ if and only if for every $a \in A$, there exists a finite $J \subseteq I$ such that $a \in \sum_{j\in J} r_j x_j$ which $r_j \in R$ and $x_j \in X$.*

Definition 3.7. *Let A be an R-hypermodule and $X \subseteq A$. X is said linearly independent if for all $n \in \mathbb{N}$ and all $x_1, x_2, \ldots, x_n \in X$, $0_A \in \sum_{i=1}^{n} r_i x_i$ implies $r_1 = r_2 = \cdots = r_n = 0_R$.*

Definition 3.8. *Let F be an R-hypermodule and X be a generating set which is linearly independent. Then X is called a basis for F.*

Remark 3.4. *The empty set is linearly independent and is a basis for the trivial R-hypermodule $\mathbf{0} = \{0\}$ (based on Definition 3.5).*

If X is a basis for F, then for every $a \in F$ there are $x_1, x_2, \ldots, x_n \in X$ and unique $r_1, r_2, \ldots, r_n \in R$ such that $a \in \sum_{i=1}^{n} r_i x_i$. In order to show the uniqueness of r_i's, let $a \in \sum_{i=1}^{m} r_i' x_i$ for some $m \in \mathbb{N}$. Without loss of generality, assume $m = n$. Then

$$0_F \in a - a \in \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} r_i' x_i$$

$$\Longrightarrow 0_F \in \sum_{i=1}^{n}(r_i - r')x_i$$

$$\Longrightarrow \exists c_i \in r_i - r'_i : \quad 0_F \in \sum_{i=1}^{n} c_i x_i$$

Since X is a basis, $c_i = 0_R$. So $0_R \in r_i - r'_i$ implies $r_i = r'_i$.

Every r_i $(i = 1, 2, \ldots, n)$ is called the *ith coordinate* of a in R. In fact, every coordinate of a can be considered as a function from F into R mapping a to an appropriate r_i denoted by f_i. Indeed, $f_i(x_j) = 1_R$ if $i = j$, $f_i(x_j) = 0_R$ if $i \neq j$ and $f_i(a) = r_i$ for $a \in \sum\limits_{i=1}^{n} r_i x_i$. Every f_i is called *ith coordinating function* of a. Clearly for all $a, b \in F$ and all $r \in R$, we have $f_i(a + b) \subseteq f_i(a) + f_i(b)$ and $f_i(ra) = rf_i(a)$.

For an arbitrary morphism $f : X \longrightarrow B$ in $\mathcal{HS}ets$, define the morphism

$$\bar{f} \colon F \longrightarrow B$$

$$\bar{f}(a) = \sum_i f_i(a) * f(x_i) \tag{3.1}$$

that the summation is indeed taken finite for an appropriate n, and $f_i(a) = r_i$ is the ith coordinate of a. Since X is a basis and the ith coordinate of a is uniquely determined, so $\bar{f}$ is well-defined. It is easy to see that $\bar{f}(x_i) = f(x_i)$ for all $x_i \in X$.

$$\begin{aligned}
\bar{f}(a+b) = \bigcup_{c \in a+b} \bar{f}(c) &= \bigcup_{c \in a+b} \left(\sum_i (f_i(c) * f(x_i)) \right) \quad \text{(by Equation 3.1)} \\
&\subseteq \sum_i \left(\bigcup_{c \in a+b} \{f_i(c)\} \right) * f(x_i) \\
&\subseteq \sum_i f_i(a+b) * f(x_i) \\
&\subseteq \sum_i f_i(a) * f(x_i) + \sum_i f_i(b) * f(x_i) \\
&= \bar{f}(a) + \bar{f}(b) \qquad \text{(by Equation 3.1)}.
\end{aligned}$$

Note that the first inclusion is obtained from $f_i(a + b) \subseteq f_i(a) + f_i(b)$. Clearly $\bar{f}(ra) = r\bar{f}(a)$ for all $r \in R$. So f is an R_{mv}-homomorphism and thus $\bar{f} \in Hom_R(F, B)$. Now let $\bar{f}' \in Hom_R(F, B)$ be another R_{mv}-homomorphism with $\bar{f}'(x_i) = \bar{f}(x_i)$ for every $x_i \in X$. Then for every $a \in F$, we have $a \in \sum_i f_i(a) * x_i$ and thus

$$\begin{aligned}
\bar{f}'(a) &\subseteq \bar{f}' \left(\sum_i (f_i(a) * x_i) \right) \\
&= \sum_i f_i(a) * \bar{f}'(x_i) \\
&= \sum_i f_i(a) * f(x_i) = \bar{f}(a).
\end{aligned} \tag{3.2}$$

Hence $\bar{f}' \leq \bar{f}$. Thus we have the following statement:

Theorem 3.1. *Let F be an R-hypermodule with basis X. Then F is weak free on X over $\mathcal{H}\mathcal{S}ets$.*

Theorem 3.2. *Let F be an R-hypermodule with basis X. If F is weak free on X over $\mathcal{S}ets$, then it is ϵ^*-free on X over $\mathcal{S}ets$.*

Proof. Following the proof of Theorem 3.1, consider Equation 3.2. If f is a morphism of $\mathcal{S}et$, then $\bar{f}'(a)$ and $\bar{f}(a)$ are contained in the finite linear combination $\sum\limits_{i} f_i(a) * f(x_i)$ in which $f_i(a) \in R$ and $f(x_i) \in B$. Thus $\epsilon^*_B(\bar{f}'(a)) = \epsilon^*_B(\bar{f}(a))$. Hence F is ϵ^*-free on X. □

Definition 3.9. *If $A = \sum\limits_{i\in I} A_i$ in which every A_i is an R-subhypermodule of A and $A_j \cap \sum\limits_{i\neq j} A_i = \{0_A\}$ for every j $(j \in J)$, then we write $A = \oplus_{i\in I} A_i$, and A is said to be the direct sum of $\{A_i\}_{i\in I}$.*

Proposition 3.2. *Let $X = \{x_i\}_{i\in I}$ be a subset of an R-hypermodule F which I is an index set. If for every $a \in F$, there exist a finite set I_0, unique $r_j \in R$ and $x_j \in X$ such that $a \in \sum\limits_{j\in I_0} r_j x_j$, then $F = \oplus_{i\in I} Rx_i$.*

Proof. Suppose every element of F is contained in a uniquely expressed linear combination of the form $\sum\limits_{i=1}^{n} r_i x_i$ in which $r_i \in R$ and $x_i \in X$ for an appropriate $n \in \mathbb{N}$. Consequently, $rx_i = 0_F$ for $r \in R$ implies $r = 0_R$. Also, for every element $a \in F$, $a \in \sum\limits_{i=1}^{n} Rx_i$ for some $n \in \mathbb{N}$. So $F = \sum\limits_{i\in I} Rx_i$. Suppose $a \in Rx_j$ and $a \in \sum\limits_{i\neq j} Rx_i$ that the summation is taken finite. So assume $a = r_j x_j$ and $a \in \sum\limits_{i=1, i\neq j}^{n} r_i x_i$ in which $r_1, r_2, \ldots, r_n \in R$ for an appropriate $n \in \mathbb{N}$ by a new indexing. Thus

$$0_F \in a - a \in \left(\sum_{i=1, i\neq j}^{n} r_i x_i\right) - (r_j x_j).$$

But

$$-(r_j * x_j) = (-1_R) * (r_j * x_j) = (-1_R \cdot r_j) * x_j = (-r_j) * x_j,$$

from Proposition 2.1. So $0_F \in \left(\sum\limits_{i=1, i\neq j}^{n} r_i x_i\right) + (-r_j)x_j$. Since X is a basis for F, we obtain $-r_j = 0_R = r_i$ for all $1 \leq i \leq n$ and $i \neq j$. Consequently $a = 0_F$. □

Corollary 3.1. *Let F be an R-hypermodule with basis X. Then $F = \oplus_{i\in I} Rx_i$.*

Proof. It is clear from Proposition 3.2. □

The following result shows that the converse of Proposition 3.2 holds: Indeed,

Theorem 3.3. *Let $X = \{x_i\}_{i\in I}$ be a subset of an R-hypermodule F which I is an index set. For every $a \in F$, there exist a finite set I_0, unique $r_j \in R$ and $x_j \in X$ such that $a \in \sum\limits_{j\in I_0} r_j x_j$ if and only if $F = \oplus_{i\in I} Rx_i$.*

Proof. According to Proposition 3.2, we must suppose $F = \oplus_{i\in I} Rx_i$ and prove for every $a \in F$, there exist a finite set I_0, unique $r_j \in R$ and $x_j \in X$ such that $a \in \sum\limits_{j\in I_0} r_j x_j$.

Clearly $a \in F = \oplus_{i\in I} Rx_i$ implies that there exist $n \in \mathbb{N}$, $r_j \in R$ and $x_j \in X$ such that $a \in \sum\limits_{j=1}^{n} r_j x_j$. To show the uniqueness of $r_j \in R$, let $a \in \sum\limits_{j=1}^{m} r'_j x_j$ for some $m \in \mathbb{N}$. Without loss of generality, $n = m$. Then $0_F \in a - a \in \sum\limits_{j=1}^{n} r_j x_j - \sum\limits_{j=1}^{n} r'_j x_j$. Thus $0_F \in \sum\limits_{j=1}^{n} d_j x_j$ where $d_j \in r_j - r'_j$. If $d_j = 0_R$ for every $1 \le j \le n$, then by the reversibility of R, $r_j = r'_j$ for every $1 \le j \le n$. Without loss of generality, suppose $d_1 \neq 0_R$ and $d_1 x_1 \neq 0_F$. Then $0_F \in \sum\limits_{j=1}^{n} d_j x_j$, and thus $d_1 x_1 \in \sum\limits_{j=2}^{n} d_j x_j$ by the reversibility of F. So $d_1 x_1 \in Rx_i \cap \sum\limits_{j=2}^{n} Rx_j$ that is a contradiction. Hence the proof is complete. □

Theorem 3.4. *Given a set X, there exists an R-hypermodule F and some $Y \subseteq F$ with $|X| = |Y|$ such that F is weak free on Y over $\mathcal{HS}ets$.*

Proof. Clearly, R can be regarded as an R-hypermodule, and so, one can form the direct sum $F = \oplus_{i\in X} R_i$, where for all $i \in X$, $R_i = R$. Define $l\colon X \longrightarrow F$ as follows: $l(x) = (r_{i,x})_{i\in X}$ where $r_{i,x} = \delta_{i,x}$. It can be easily shown that $\{(r_{i,x})_{i\in X} \mid x \in X\}$ is a basis for F. Denote $(r_{i,x})_{i\in X}$ as e_x. Then we can write $F = \oplus_{x\in X} Re_x$ and every element of $F$is contained in a unique finite linear combination $\sum\limits_{x\in X} r_x e_x$ where $r_x \in R$. Indeed, every element of F has the form $(r_x)_{x\in X}$ in which all but only a finitely many r_x's are zero. So the subset $\{e_x\}_{x\in X}$ is a basis for F. Thus by Theorem 3.1, F is weak free on $\{e_x\}_{x\in X}$. Consequently, considering the injective map $l\colon X \longrightarrow F$ with $x \mapsto e_x$ and letting $Y = l(X)$, F is weak free on Y. □

Theorem 3.5. *For every R-hypermodule A, there is some surjective $\bar{f} \in Hom_R(F, A)$ in which F is weak free R-hypermodule on some $Y \subseteq F$ over $\mathcal{HS}ets$.*

Proof. Let $A = \langle X\rangle$. Also, let F, $Y \subseteq F$ and $l\colon X \longrightarrow Y$ as in the proof of Theorem 3.4. So F is a weak free R-hypermodule on $Y = l(X)$ over $\mathcal{HS}ets$. Let $a \in A = \langle X\rangle$. Acording to Proposition 3.1, suppose

$$a \in \sum_{i=1}^{m} r_i x_i + \sum_{i=1}^{n} n_i x_i + \sum_{i=1}^{k} k_i(x_i - x_i) \;:\; r_i \in R,\; x_i \in X,\; m, n, k, k_i \in \mathbb{N},\; n_i \in \mathbb{Z}\}.$$

Define $\bar{f}(z) = \sum_{i=1}^{m} r_i l^{-1}(e_{x_i})$ or $\bar{f}(z) = \sum_{i=1}^{m} r_i x_i$. The surjectivity of $\bar{f}$ is clear.

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "l^{-1}\cong"'] & F \arrow[ddl, "\exists \bar{f}"] \\
X \arrow[d, "i"'] & \\
A &
\end{tikzcd}$$

(Note that since Y is a basis for F, we have $z \in \sum\limits_{x\in X} r_x e_x$ for every $z \in F$ where $r_x \in R$ as in the proof of Theorem 3.4.) □

Now we state a new notion by using the fundamental module of an R-hypermodule.

Definition 3.10. *An R-hypermodule F is called fundamental free if its fundamental module, $\frac{F}{\epsilon_F^*}$, is free $\frac{R}{\gamma^*}$-module.*

Example 3.1. *[13] Let $(G, \cdot)$ be a group with $|G| \geq 4$, and define a hyperaddition and a multiplication on $R = G \cup \{0\}$, by:*

$$a + 0 = 0 + a = a \quad \text{for all} \quad a \in R;$$

$$a + a = \{a, 0\} \quad \text{for all} \quad a \in G;$$

$$a + b = b + a = G \setminus \{a, b\} \quad \text{for all} \quad a, b \in G : a \neq b;$$

$$a \odot 0 = 0 \odot a = 0 \quad \text{for all} \quad a \in R;$$

$$a \odot b = a \cdot b \quad \text{for all} \quad a, b \in G.$$

Then $(R, +, \odot)$ is a hyperring. Clearly every hyperring R is an R-hypermodule and $\epsilon_R^* = \gamma^*$. On the other hand, for every $x, y \in R$ we have $x\gamma 0\gamma y$, since $x + x = \{x, 0\}$ and $y + y = \{y, 0\}$. So $x\gamma^* y$, and indeed we have only one equivalence class. Hence $R' = \frac{R}{\gamma^*}$ is the trivial ring $\mathbf{0} = \{0_{R'}\}$. Clearly $R' = \frac{R}{\gamma^*}$ is a free R'-module. Thus R is fundamental free as an R-hypermodule.

Remark 3.5. *Note that in 3.1, every R-hypermodule is fundamental free, since every R'-module is free. Indeed, every module over the trivial ring $R' = \mathbf{0}$ is free.*

In general, we state the following proposition:

Proposition 3.3. *For every hyperring R with trivial fundamental ring, all objects of ${}_R\mathcal{H}\mathbf{mod}$ (or ${}_Rh\mathbf{mod}$) are fundamental free.*

Proof. The proof is clear, since every $\frac{R}{\gamma^*}$-module is free. □

Definition 3.11. *An R-homomorphism $f \in Hom_R(A, B)$ is ϵ^*-inverse of $g \in Hom_R(B, A)$ if $\epsilon_B^*((f \circ g)(b)) = \epsilon_B^*(b)$ and $\epsilon_A^*((g \circ f)(a)) = \epsilon_A^*(a)$ for all $(a, b) \in A \times B$, or equivalently $[f \circ g] = [id_B]$ and $[g \circ f] = [id_A]$. In this case, we say f is an ϵ^*-isomorphism and A is ϵ^*-isomorphic to B denoted by $A \overset{\epsilon^*}{\cong} B$.*

Theorem 3.6. *If F and F' are two ϵ^*-free R-hypermodules on the sets X and X' over $\mathcal{H}\mathcal{S}ets$, respectively, and $|X| = |X'|$, then $F \overset{\epsilon^*}{\cong} F'$.*

Proof. Since $|X| = |X'|$, we have a bijection $h : X \longrightarrow X'$ in $\mathcal{S}ets$. Now consider the inclusion $i' : X' \longrightarrow F'$ and set $f = i' \circ h$ as a morphism in $\mathcal{HS}ets$. Since F is ϵ^*-free on X over $\mathcal{HS}ets$, we have an R_{mv}-homomorphism $\bar{f}$ such that $\bar{f} \circ i = f$ as the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "h"] & F \arrow[d, "\bar{f}"] \\ X' \arrow[r, "i'"'] & F' \end{tikzcd}$$

commutes. Also, consider the inclusion $i : X \longrightarrow F$ and set $g = i \circ h^{-1}$ as a morphism in $\mathcal{HS}ets$. Since F' is ϵ^*-free on X' over $\mathcal{HS}ets$, we have an R_{mv}-homomorphism $\bar{g}$ such that $\bar{g} \circ i' = g$ as the diagram

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "h^{-1}"] & F' \arrow[d, "\bar{g}"] \\ X \arrow[r, "i"'] & F \end{tikzcd}$$

commutes. Thus we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "h"] & F \arrow[d, "\bar{f}"] \\ X' \arrow[r, "i'"] \arrow[d, "h^{-1}"] & F' \arrow[d, "\bar{g}"] \\ X \arrow[r, "i"'] & F \end{tikzcd}$$

So $\bar{g} \circ \bar{f} \circ i = i \circ h^{-1} \circ h = i \circ id_X = i$. On the other hand, $id_F \circ i = i$. Since F is ϵ^*-free on X over $\mathcal{HS}ets$, we have $[\bar{g} \circ \bar{f}] = [id_F]$. Similarly, one can check that $[\bar{f} \circ \bar{g}] = [id_{F'}]$. Hence $F \overset{\epsilon^*}{\cong} F'$. □

According to Theorem 3.2, we have the following result:

Corollary 3.2. *Let $X \subseteq F$ and $X' \subseteq F'$ be two bases for R-hypermodules F and F', respectively. If F and F' are weak free on X and X' over $\mathcal{S}ets$, respectively, and $|X| = |X'|$, then $F \overset{\epsilon^*}{\cong} F'$.*

Lemma 3.1. *Let X be a basis for an R-hypermodule. If every summation $\sum_{i=1}^{n} r_i x_i$ for $r_i \in R$ and $x_i \in X$, which x_i's are not necessarily distinct, is a singleton, then R is a ring and F is an R-module .*

Proof. Clearly, for all $r_i, r_j \in R$ and $x_i, x_j \in X$, $r_i x_i + r_j x_j$ is a singleton. Also, $(r_i + r_j)x_i = r_i x_i + r_j x_i$ is a singleton. Without loss of generality, assume $x \in \sum_{i=1}^{n} r_i x_i$ and $y \in \sum_{i=1}^{n} r'_i x_i$ for $r_i, r'_i \in R$ and $x_i \in X$. Then we can write $x + y \subseteq \sum_{i=1}^{n} t_i x_i$ in which $t_i = r_i + r'_i$. According to the assumption, if we prove t_i is a singleton, then $x+y$ is a singleton and F is an R-module. Let $r, s \in t_i$. Since $t_i x_i = r_i x_i + r'_i x_i$ is a singleton and $r x_i, s x_i \in t_i x_i$, we have $r x_i = s x_i = t_i x_i$. Clearly, $0_F \in r x_i - s x_i = (r - s)x_i$. On the other hand,

$0_F \in 0_R * x_i$. So $0_R \in r - s$. Hence $r = s$. Consequently, t_i is a singleton.

Now we prove R is ring. Let $r, s \in R$ and $t, t' \in r + s$. Clearly, $tx_i, t'x_i \in (r+s)x_i = rx_i + sx_i$ every $x_i \in X$. Then $tx_i = t'x_i$. Then $0_F \in tx_i - t'x_i = (t - t')x_i$. On the other hand, $0_F \in 0_R * x_i$. So $0_R \in t - t'$. Hence $t = t'$. Consequently, $r + s$ is a singleton. □

Proposition 3.4. *Let F be a free R-hypermodule on basis $X \subseteq F$ such that for all $r, s \in R$ and all $x \in X$, we have $|rx + sx| = 1$. Then R is a ring and F is a free R-module on X in the category ${}_R\mathcal{H}\mathbf{mod}$.*

Proof. Note that every free R-hypermodule on $X \subseteq F$ is a weak free R-hypermodule on X over $\mathcal{S}ets$. Thus if $i = id_F|_X$, then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & F \arrow[dl, "\exists! f"] \\ F & \end{tikzcd}$$

i.e., $f \circ i = i$. Since X is a basis, for an arbitrary $x \in F$, we have $f(x) = \sum\limits_{x_i \in X} r_i^x x_i$ in which every $r_i^x \in R$ depends on x (by Equation 3.1).

Note that $\sum\limits_{x_i \in X} r_i^x x_i$ is the unique presentation of x by the elements of the basis X, i.e., $x \in \sum\limits_{x_i \in X} r_i^x x_i$ and r_i^x's are unique.

On the other hand,

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & F \arrow[dl, "id_F"] \\ F & \end{tikzcd}$$

and indeed, $id_F \circ i = i$. So $f = id_F$ and thus $f(x) = id_F(x)$. This implies $\sum\limits_{x_i \in X} r_i^x x_i = x$. Consequently, every unique presentation of every element of F is a singleton. According to the assumption, every summation $\sum\limits_{i=1}^{n} r_i x_i$, which x_i's are not necessarily distinct, is a singleton. Thus the result is followed by Lemma 3.1. □

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