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Unify Steiner Wiener Distance for Some Class of m -Polar Fuzzy Graphs

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ABSTRACT

Sometimes information in a network model is based on multi-agent, multi-attribute, multi-object, multi-polar information or uncertainty rather than a single bit. An m -polar fuzzy model is useful for such network models which gives more and more precision, flexibility, and comparability to the system as compared to the classical, fuzzy models. On the other, The Steiner tree problem in networks, and particularly in graphs, was formulated by Hakimi [25] and Levi [21] by definition minimal size connected tree sub graph that contains the vertices in S . Steiner trees have applications to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a sub network that uses the fewest communication links. In this paper, we extend Steiner distance $SW_k(G)$ for m -polar fuzzy graphs and give this parameter for Join, composition and Cartesian product of two m -polar fuzzy graphs.

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1. Introduction

A fuzzy set is an important mathematical structure to represent a collection of objects whose boundary is vague [1]. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional models used in engineering and science. Rosenfeld discussed the idea of fuzzy graph in 1975 [2]. Several concepts on fuzzy graphs were introduced by Mordeson [3]. Lately Ameri and his co-authors explained a new concepts of product interval-valued fuzzy graph [4]. Rashmanlou, Borzooei and their co-authors studied on the bipolar fuzzy graphs with categorical properties [5, 6] and Talebi and Rashmanlou [7] studied the complement and isomorphism of bipolar fuzzy graphs. Juanjuan Chen [8] introduced the notion of the m -polar fuzzy set as a generalisation of bipolar fuzzy sets. Hayat and co-authors studied on bipolar Anti Fuzzy h -ideals in Hemi-rings [9]. Ghorai and Pal studied on some properties of m -polar fuzzy graphs as a bipolar fuzzy graphs [10,11]. Akram and Younas studied certain types of irregular m -polar fuzzy graphs in [12]. In 2016 Akram and Adeel studied on m -polar fuzzy labelling graphs [13] and Akram and Waseem introduced certain metrics in m -polar fuzzy graphs in [14]. Lately Prem Kumar studied on m -polar fuzzy graph representation of concept lattice [15–18]. On the other the

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Steiner distance of a set S of vertices in a connected graph G , is the number of edges in a smallest connected subgraph of G containing S , called a Steiner tree for S . If $|S| = 2$, then the Steiner distance of S is the distance between the two vertices of S . Steiner trees have applications to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the fewest communication links. A Steiner tree for the vertices that need to be connected corresponds to such a subnetwork. Lately, Gutman and co-authors introduced the concept of the Steiner Wiener index of a graph [19,20]. The Steiner k -Wiener index $SW_k(G)$ of G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d(S)$$

where the steiner distance $d(S)$ of the vertices of S is the minimum size of a connected subgraph of G whose vertex set is S . In this paper, we expand this concept to m -polar fuzzy graphs and compute Steiner Wiener index for some class of m -polar fuzzy graphs.

Definition 1.1: [1] A fuzzy set μ in a universe X is a mapping $\sigma : X \rightarrow [0, 1]$. A fuzzy relation on X is a fuzzy set μ in $X \times X$. Let σ be a fuzzy set in X and μ fuzzy relation on X . We call μ is a fuzzy relation on σ if $\mu(x, y) \leq \min\{\sigma(x), \sigma(y)\} \forall x, y \in X$.

Definition 1.2: [2] A fuzzy graph is a pair $G : (\sigma, \mu)$ where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ . We assume that S is finite and nonempty, μ is reflexive and symmetric.

Definition 1.3: [8] An m -polar fuzzy set (or a $[0, 1]^m$ -set) on X is exactly a mapping $A : X \rightarrow [0, 1]^m$. Note that $[0, 1]^m$ (m th-power of $[0, 1]$) is considered as a poset with the point-wise order \leq , where m is an arbitrary ordinal number (we make an appointment that $m = \{n | n < m\}$, when $m > 0$), \leq is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ and $(x, y \in [0, 1]^m)$ and $p_i : [0, 1]^m \rightarrow [0, 1]$ is the i th projection mapping ($i \in m$). $0 = (0, 0, \dots, 0)$ is the smallest element in $[0, 1]^m$ and $1 = (1, 1, \dots, 1)$ is the largest element in $[0, 1]^m$.

Definition 1.4: [14] Let σ be an m -polar fuzzy subset of a non-empty set V . An m -polar fuzzy relation on σ is an m -polar fuzzy subset μ of $V \times V$ defined by the mapping $\mu : V \times V \rightarrow [0, 1]^m$ such that for all $x, y \in V$, $p_i \circ \mu(xy) \leq \inf\{p_i \circ \sigma(x), p_i \circ \sigma(y)\}$, $i = 1, 2, \dots, m$ where $p_i \circ \sigma(x)$ denotes the i th degree of membership of the vertex x and $p_i \circ \mu(xy)$ denotes the i th degree of membership of the edge xy .

An m -polar fuzzy graph was introduced by Chen et al. [8] and modified by Akram and Waseem [14].

Definition 1.5: [14] An m -polar fuzzy graph is a pair $G = (\sigma, \mu)$, where $\sigma : V \rightarrow [0, 1]^m$ is an m -polar fuzzy set in V and $\mu : V \times V \rightarrow [0, 1]^m$ is an m -polar fuzzy relation on V such that $p_i \circ \mu(xy) \leq \inf\{p_i \circ \sigma(x), p_i \circ \sigma(y)\}$ for all $x, y \in V \times V$ note that $p_i \circ \mu(xy) = 0$ for all $xy \in V \times V - E$ for all $i = 1, 2, 3, \dots, m$. σ is called the m -polar fuzzy vertex set of G and μ is called the m -polar fuzzy edge set of G , respectively. An m -polar fuzzy relation μ on V is called symmetric if $p_i \circ \mu(xy) = p_i \circ \mu(yx)$ for all $x, y \in V$.

Definition 1.6: [16] Let G be a connected m -polar fuzzy graph. The length of an m -polar fuzzy path $P: v_1, v_2, \dots, v_n$ in G , is denoted by $l(P)$ and defined as $l(P) = ((l_1(P), l_2(P), \dots, l_m(P)))$ where $l_i(P) = \sum_{j=1}^{n-1} \frac{1}{p_i \circ \mu(v_j v_{j+1})}$, $i = 1, 2, 3, \dots, m$. If $n = 0$, define $l_i(P) = 0$ and for $n \geq 1$, $l_i(P) > 0$.

Also if G is disconnected then $l_i(P)$ may be zero. Let G be a connected m -polar fuzzy graph. The distance, $d(u, v)$ is the smallest length of any $u - v$ m -polar fuzzy path P in G , where $u, v \in V$.

That is, $d(u, v) = (d_1(u, v), d_2(u, v), \dots, d_m(u, v))$ where $d_i(u, v) = \inf\{l_i(P) : P \text{ is an } u - v \text{ path}\}$.

For m -polar fuzzy graph $G = (\sigma, \mu)$, we defined μ -Unify as follows:

Definition 1.7: If G is a m -polar fuzzy graph with $|e(G)| = m$, then μ -Unify is as follow

$$\mu_U(e) = (\min(p_1 \circ \mu(e_i)), \dots, \min(p_m \circ \mu(e_i))), i = 1, 2, \dots, m$$

Definition 1.8: [21] Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs.

The Cartesian product $G^* = G_1^* \times G_2^* = (V, E)$ of graphs G_1^* and G_2^* , Where $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) : x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) : z \in V_2, x_1 y_1 \in E_1\}$.

Then, the composition of the graph G_1^* with G_2^* is denoted by $G_1^*[G_2^*] = (V_1 \times V_2, E^0)$, where

$E^0 = E \cup \{(x_1, x_2)(y_1, y_2) : x_1 y_1 \in E_1, x_2 \neq y_2\}$ and E is defined in $G_1^* \times G_2^*$. Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

The join of two simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is the simple graph with the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E'$, where E' is the set of all edges joining the nodes of V_1 and V_2 and assume that $V_1 \cap V_2 = \emptyset$. The join G_1^* and G_2^* is denoted by $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$.

Ghorai and Pal extension above definition for m -polar fuzzy graph.

Definition 1.9: [19] The Cartesian product $G_1 \times G_2$ of two m -polar fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, is defined as a pair $(V_1 \times V_2, \sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$, such that for $i = 1, 2, \dots, m$

- (i) $p_i \circ (\sigma_1 \times \sigma_2)(x_1, x_2) = \min\{p_i \circ \sigma_1(x_1), p_i \circ \sigma_2(x_2)\}$ for all $(x_1, x_2) \in V_1 \times V_2$.
- (ii) $p_i \circ (\mu_1 \times \mu_2)((x, x_2)(x, y_2)) = \min\{p_i \circ \sigma_1(x), p_i \circ \mu_2(x_2, y_2)\}$ for all $x \in V_1$, for all $x_2 y_2 \in E_2$.
- (iii) $p_i \circ (\mu_1 \times \mu_2)((x_1, z)(y_1, z)) = \min\{p_i \circ \mu_1(x_1, y_1), p_i \circ \sigma_2(z)\}$ for all $z \in V_2$, for all $x_1 y_1 \in E_1$.

Definition 1.10: The composition $G_1[G_2] = (V_1 \times V_2, \sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ of two m -polar fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, is defined as follows: for $i = 1, 2, \dots, m$

- (i) $p_i \circ (\sigma_1 \circ \sigma_2)(x_1, x_2) = \min\{p_i \circ \sigma_1(x_1), p_i \circ \sigma_2(x_2)\}$ for all $(x_1, x_2) \in V_1 \times V_2$

- (ii) $pi \circ (\mu \circ \mu_2) = ((x, x_2)(x, y_2)) = \min\{pi \circ \sigma_1(x), pi \circ \mu_2(x_2y_2)\}$ for all $x \in V_1$, for all $x_2y_2 \in E_2$.
- (iii) $pi \circ (\mu_1 \circ \mu_2) = ((x_1, z)(y_1, z)) = \min\{pi \circ \mu_1(x_1, y_1), pi \circ \sigma_2(z)\}$ for all $z \in V_2$, for all $x_1y_1 \in E_1$
- (iv) $pi \circ (\mu_1 \circ \mu_2)((x_1, x_2)(y_1, y_2)) = \min\{pi \circ \sigma_2(x_2), pi \circ \sigma_2(y_2), pi \circ \mu_1(x_1y_1)\}$, for all $(x_1, x_2)(y_1, y_2) \in E^0 - E$.

Definition 1.11: The join $G_1 + G_2 = (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ of two m -polar fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, is defined as follows:

- (i) $pi \circ (\sigma_1 + \sigma_2)(x) = pi \circ (\sigma_1 \cup \sigma_2)(x)$ if $x \in V_1 \cup V_2$
- (ii) $pi \circ (\mu_1 + \mu_2)(xy) = pi \circ (\mu_1 \cup \mu_2)(xy)$ if $xy \in E_1 \cup E_2$
- (iii) $pi \circ (\mu_1 + \mu_2)(xy) = \min\{pi \circ \sigma_1(x), pi \circ \sigma_2(y)\}$ if $xy \in E'$

where E' is the set of all of the edges joining the nodes of V_1 and V_2 assuming that $V_1 \cap V_2 = \emptyset$.

Definition 1.12: [19] The Steiner k -Wiener index $SW_k(G)$ of G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d(S)$$

S -Steiner fuzzy tree is a subgraph $T(V', E')$ of G with $S \subseteq V'$. Such that the Steiner distance $d(S)$ is the minimum size of edges of tree $T(V', E')$ among all connected subgraphs whose vertex sets contain S ($S \subseteq V'$). Observe that $d(S) = \min\{|e(T)| \mid S \subseteq V(T)\}$ for fuzzy graph $G = (\sigma, \mu)$, we defined k -Unify Steiner Wiener distance as follows

$$USW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d_w(S)$$

such that $d_w(S) = d(S) \cdot \mu_U(e)$.

Example 1.1: An example the graph G in Figure 1, which is the molecular graph of 1, 1, 3-trimethyl-cyclobutane. Its vertices are labelled by $u_1, u_2, u_3, u_4, u_5, u_6, u_7$. The sequence 4, 3, 5, 6, 7 is a path of G . In this graph the Steiner Wiener for $k = 2$ are computed:

$$\begin{aligned} d(1, 2) &= 2, d(1, 3) = 1, d(1, 4) = 2, d(1, 5) = 2, d(1, 6) = 3, d(1, 7) = 4 \\ d(2, 3) &= 1, d(2, 4) = 2, d(2, 5) = 2, d(2, 6) = 3, d(2, 7) = 4 \\ d(3, 4) &= 1, d(3, 6) = 2, d(3, 5) = 1, d(3, 7) = 3 \\ d(4, 6) &= 1, d(4, 5) = 2, d(4, 7) = 2 \\ d(5, 6) &= 1, d(6, 7) = 1 \\ d(5, 7) &= 2 \\ \text{so: } USW_2(G) &= 42 \end{aligned}$$

Example 1.2: The molecular graph of 2, 3, 4 trimethyl -3-ethyl-hexane Figure 2; its Steiner Wiener index for $k = 2$ is 148.

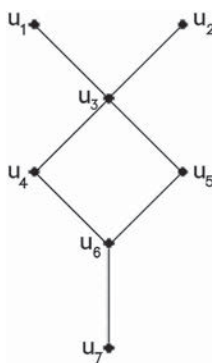


Figure 1. The molecular graph of 1; 1; 3-trimethyl-cyclobutane.

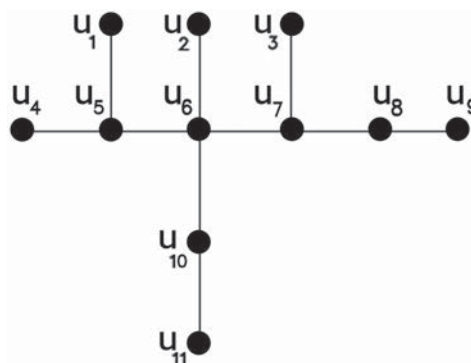


Figure 2. The molecular graph of 2,3,4-trimethyl-3-ethyl-hexane.

2. Main Result

Mao, Wang and Gutman [18] introduced Steiner Wiener index for join, composition and product of two graphs. Now we expand this concept to m -polar fuzzy graphs.

Theorem 2.1: Let G be a connected m -polar fuzzy graph with n vertices and let H be a connected m -polar fuzzy graph with n' vertices ($n \geq n'$) and k be an integer with $3 \leq k \leq n + n'$. Then the k -Unify Steiner Wiener distance for join of two m -polar fuzzy graphs is as follow

(1) If $k \geq n$ then

$$USW_k(G + H) = \binom{n + n'}{k} (k - 1) \mu_{U(G+H)} \quad (1)$$

(2) If $k \leq n'$ then

$$\begin{aligned} USW_k(G + H) &= x_{kG}(k - 1) \mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] (k - 1) \mu_{UG} \\ &\quad + x_{kH}(k - 1) \mu_{UH} + \left[\binom{n'}{k} - x_{kH} \right] (k - 1) \mu_{UH} \end{aligned}$$

$$+ (k-1)\mu_{U(G+H)} \left[\binom{n+n'}{k} - \binom{n}{k} - \binom{n'}{k} \right] \quad (2)$$

where x_{kG} is the number of the k -subsets of $V(G)$, such that the fuzzy subgraph induced by each k -subset is connected, and x_{kH} is the number of the k -subsets of $V(H)$ such that the subgraph induced by each k -subset is connected.

(3) If $n' \leq k \leq n$ Then

$$\begin{aligned} USW_k(G+H) &= x_{kG}(k-1)\mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] k\mu_{UG} \\ &\quad + x_{kG}(k-1)\mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] k\mu_{UG} \end{aligned} \quad (3)$$

Proof: (1) Since $k > n$, it Follows That $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$ for any $S \subseteq V(G+H)$, Suppose That $S \cap V(G) = \{g_1, g_2, \dots, g_l\}$ and $S \cap V(H) = \{h_1, h_2, \dots, h_{k-l}\}$, Then one of the Fuzzy Tree Induced by the Edges is $\{g_1, h_1\} \cup \{g_1 h_j | 2 \leq j \leq k-l\} \cup \{g_i h_1 | 2 \leq i \leq l\}$ is a Spanning Fuzzy Tree with $|e(T)| = k-1$, Then This Contribution is

$$USW_k(G+H) = \binom{n+n'}{k} (k-1)\mu_{U(G+H)} \quad (4)$$

(2) For any $S \subseteq V(G+H)$, we have $S \subseteq V(G)$ or $S \subseteq V(H)$ or $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Suppose that $S \subseteq V(G)$, if $G[S]$ is connected then $G[S]$ contains $nSP(G[S])$ spanning fuzzy trees. If x_{kG} is the number of k -subset of $V(G)$ such that the fuzzy subgraph induced by each k -subset is connected, then this contribution is $(k-1)x_{kG}\mu_{UG}$, and if $G[S]$ is not connected then spanning fuzzy tree has more than k edges. Assume that set $= \{g_1, g_2, \dots, g_k\}$, clearly the fuzzy tree induced by the edges in $\{g_i, h_1 | 1 \leq i \leq k\}$ is one of the spanning fuzzy tree and has less than k edges, So $d_{G+H}(S) = k$. Since x_{kG} is the number of the k -subsets of $V(G)$ such that the fuzzy subgraph induced by each k -subset is connected, it follows that this contribution is $\left[\binom{n}{k} - x_{kG} \right] k\mu_{UG}$, Similarly if $S \subseteq V(H)$, the contribution for two case connected and disconnected for $H[S]$ is

$$(k-1)x_{kH}\mu_{UH} + \left[\binom{n'}{k} - x_{kH} \right] k\mu_{UH}$$

For other case suppose that $S \cap V(G) \neq \emptyset$ and $\cap V(H) \neq \emptyset$, clearly spanning fuzzy tree has $k-1$ edges, then this contribution is

$$\begin{aligned} &(k-1)\mu_{U(G+H)} \left[\sum_{i=1}^{k-1} \binom{n}{i} \binom{n'}{k-i} \right] \\ &= (k-1)\mu_{U(G+H)} \left[\binom{n+n'}{k} - \binom{n}{k} - \binom{n'}{k} \right] \end{aligned}$$

From the above argument, we get

$$\begin{aligned}
 USW_k(G+H) &= x_{kG}(k-1)\mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] k\mu_{UG} \\
 &\quad + x_{kH}(k-1)\mu_{UH} + \left[\binom{n'}{k} - x_{kH} \right] k\mu_{UH} \\
 &\quad + (k-1)\mu_{U(G+H)} \left[\binom{n+n'}{k} - \binom{n}{k} - \binom{n'}{k} \right] \quad (5)
 \end{aligned}$$

(3) Since $n' \leq k \leq n$, it follows that for any $S \subseteq V(G+H)$, either $S \subseteq V(G)$ or $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, suppose that $S \subseteq V(G)$, similarly to proof of 2, in this case the contribution is

$$x_{kG}(k-1)\mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] k\mu_{UG}$$

To consider $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, the spanning fuzzy tree has $k-1$ edge, then the portion is

$$\begin{aligned}
 &(k-1)\mu_{U(G+H)} \sum_{i=1}^{k-1} \binom{n}{i} \binom{n'}{k-i} \\
 &= (k-1)\mu_{U(G+H)} \left[\binom{n+n'}{k} - \binom{n}{k} - \binom{n'}{k} \right]
 \end{aligned}$$

From the above argument it follows

$$\begin{aligned}
 USW_k(G+H) &= x_{kG}(k-1)\mu_{UG} + \left[\binom{n}{k} - x_{kG} \right] k\mu_{UG} \\
 &\quad + (k-1)\mu_{U(G+H)} \left[\binom{n+n'}{k} - \binom{n}{k} - \binom{n'}{k} \right] \quad (6)
 \end{aligned}$$

Lemma 2.1: [20] Let $S = \{(g_1, h_1), (g_2, h_2), \dots, (g_k, h_k)\}$ be a set of distinct vertices of $G \times H$. Let $S_G = \{g_1, g_2, \dots, g_k\}$ and $S_H = \{h_1, h_2, \dots, h_k\}$. Then $d_{G \times H}(S) \geq d_G(S_G) + d_H(S_H)$.

Theorem 2.2: Let G be a connected m -polar fuzzy graph with n vertices, and let H be a connected m -polar fuzzy graph with n' vertices. Let k be an integer with $2 \leq k \leq nn'$. Then the bounds of k -Unify Steiner Wiener distance for Cartesian product of two m -polar fuzzy graphs is

$$\begin{aligned}
 &\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} [(l-1)x_{lG}\mu_{UG} \\
 &\quad + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG}] \\
 &\quad + \sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}'} [(l'-1)x_{l'H}\mu_{UH} \\
 &\quad + l' \left[\binom{n}{l'} - x_{l'H} \right] \mu_{UH}]
 \end{aligned}$$

$$\begin{aligned}
&\leq USW_k(G \times H) \\
&\leq \frac{k}{2} \left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \cdots \binom{n'}{k_l} [(l-1)x_{lG}\mu_{UG} \right. \\
&\quad \left. + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG} \right] \\
&\quad \sum_{l'=2}^k \binom{n}{k'_1} \binom{n}{k'_2} \cdots \binom{n}{k'_{l'}} [(l'-1)x_{l'H}\mu_{UH} \\
&\quad + l' \left[\binom{n}{l'} - x_{l'H} \right] \mu_{UH}] \tag{7}
\end{aligned}$$

Proof: For any $S \subseteq V(G \times H)$ and $|S| = k$, all the vertices in S belong to some copies of G and some copies of H . Without loss of generality, let $H(g_1), H(g_2), \dots, H(g_l)$ be all the copies of H such that $S \cap V(H(g_i)) \neq \emptyset$ ($1 \leq i \leq l$) and $S \cap V(H(g_i)) = \emptyset$, $l+1 \leq i \leq n$ and let $G(h_1), G(h_2), \dots, G(h_{l'})$ be all the copies of G such that $S \cap V(G(h_j)) \neq \emptyset$ ($1 \leq j \leq l'$) and $S \cap V(G(h_j)) = \emptyset$ ($l'+1 \leq j \leq n'$).

Observe that $1 \leq l \leq k$ and $1 \leq l' \leq k$. Let $S = \{(g_1, h_1), (g_2, h_2), \dots, (g_k, h_k)\}$ be a set of distinct vertices of $G \times H$. Then $S_G = \{g_1, g_2, \dots, g_k\} \subseteq \bigcup_{i=1}^l V(H(g_i))$ and $S_H = \{h_1, h_2, \dots, h_k\} \subseteq \bigcup_{i=1}^{l'} V(G(h_i))$. Note that g_i and g_j are not necessarily different for $1 \leq i \neq j \leq k$, and h_i and h_j are not necessarily different for $1 \leq i \neq j \leq k$. Without loss of generality, let $|S_G \cap V(H(g_i))| = k_i$ and $|S_H \cap V(G(h_j))| = k'_j$, where $1 \leq i \leq l$ and $1 \leq j \leq l'$. It is clear that $\sum_{i=1}^l k_i = k$ and $\sum_{j=1}^{l'} k'_j = k$. From Lemma 2.1 $d_{G \times H}(S) \geq d_G(S_G) + d_H(S_H)$ for any $S \subseteq V(G \times H)$. Note that we have $\sum_{l=1}^k \binom{n'}{k_1} \binom{n'}{k_2} \cdots \binom{n'}{k_l}$ ways to determine $S_G = \{g_1, g_2, \dots, g_l\}$. So in this case the contribution to $USW_k(G \times H)$ is at least

$$\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \cdots \binom{n'}{k_l} USW_l(G)$$

So by compute $USW_l(G)$ for two case connect and disconnect sub m -polar fuzzy graph induced by k -subset, then

$$USW_l(G) = (l-1)x_{lG}\mu_{UG} + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG}$$

such that x_{lG} is the number of the l - subsets of $V(G)$ such that the subgraph induced by each l - subsets is connected. Similarly, since we have $\sum_{l'=2}^k \binom{n}{k'_1} \binom{n}{k'_2} \cdots \binom{n}{k'_{l'}}$ ways to determine. $S_H = \{h_1, h_2, \dots, h_k\}$, from above reason the contribution to

$USW_k(G \times H)$ is at least

$$\sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \cdots \binom{n}{k_{l'}} USW_{l'}(H)$$

so similarly

$$USW_{l'}(H) = (l' - 1)x_{l'H}\mu_{UH} + l' \left[\binom{n}{l'} - x_{l'H} \right] \mu_{UH}$$

by Lemma [2.1] $d_{G \times H}(S) \geq d_G(S_G) + d_H(S_H)$, then

$$\begin{aligned} USW_k(G \times H) &\geq \sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \cdots \binom{n'}{k_l} USW_l(G) \\ &\quad + \sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \cdots \binom{n}{k_{l'}} USW_{l'}(H) \end{aligned}$$

Therefore

$$\begin{aligned} USW_k(G \times H) &\geq \sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \cdots \binom{n'}{k_l} [(l - 1)x_{lG}\mu_{UG} \\ &\quad + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG}] \\ &\quad + \sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \cdots \binom{n}{k_{l'}} [(l' - 1)x_{l'H}\mu_{UH} \\ &\quad + l' \left[\binom{n}{l'} - x_{l'H} \right] \mu_{UH}] \end{aligned} \quad (8)$$

On the other suppose that $l \leq k - 1$. For each i ($1 \leq i \leq l$), there is an Steiner tree in $H(g_i)$, say T_i' . Similarly, there is an Steiner tree in $G(h_1)$, say T . Then the tree induced by the edges in $E\left(\bigcup_{i=1}^l T_i'\right) \cup E(T)$ is an Steiner tree in $G \times H$, and thus

$$d_{G \times H}(S) \leq d_G(S_G) + ld_H(S_H) \leq d_G(S_G) + (k - 1)d_H(S_H)$$

Suppose that $l = k$ and $l' = k$, all the vertices in $S = \{(g_1, h_1), (g_2, h_2), \dots, (g_k, h_k)\}$ belong to different copies of G and different copies of H . For each i ($1 \leq i \leq l - 1$), there is an Steiner tree in $H(g_i)$, say T_i' . Similarly, there is an Steiner tree in $G(h_1)$, say T . Then the tree induced by the edges in $E\left(\bigcup_{i=1}^{l-1} T_i'\right) \cup E(T)$ is an Steiner tree in $G \times H$, and therefore

$$\begin{aligned} d_{G \times H}(S) &\leq d_G(S_G) + (l - 1)d_H(S_H) \\ &\leq d_G(S_G) + (k - 1)d_H(S_H) \end{aligned}$$

In the other case suppose that $l = k$ and $l' \leq k - 1$. Since $l = k$ it follows that each $H(g_i)$ contains exactly one vertex of S . Since $l' \leq k - 1$. It follows that there exists some $G(h_i)$ such

that $G(h_i)$ contains at least two vertices of S . For each i ($1 \leq i \leq l-1$), there is an Steiner tree in $H(g_i)$, say T_i' .

Similarly there is an Steiner tree in $G(h_1)$, say T . Then the tree induced by the edges in $E\left(\bigcup_{i=1}^{l-1} T_i'\right) \cup E(T)$ is an Steiner tree in $G \times H$, and thus

$$\begin{aligned} d_{G \times H}(S) &\leq d_G(S_G) + (l-1)d_H(S_H) \\ &\leq d_G(S_G) + (k-1)d_H(S_H) \end{aligned}$$

Since there are $\sum_{l=1}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l}$ ways to determine $S_G = \{g_1, g_2, \dots, g_k\}$ it follows that in this case, the contribution to $USW_k(G \times H)$ is at least

$$\left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} USW_l(G) \right]$$

On the other there are $\sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}'}$ ways to determine $S_H = \{h_1, h_2, \dots, h_k\}$, the respective contribution to $USW_k(G \times H)$ is at least

$$\left[\sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}'} \right] USW_{l'}(H)$$

Combining these results, we have

$$\begin{aligned} USW_k(G \times H) &\leq \left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} \right] USW_l(G) + (k-1) \cdot \\ &\quad + \left[\sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}'} \right] USW_{l'}(H) \end{aligned}$$

and similarly,

$$\begin{aligned} USW_k(G \times H) &\leq (k-1) \left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} \right] USW_l(G) \\ &\quad + \left[\sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}'} \right] USW_{l'}(H) \end{aligned}$$

This yields

$$USW_k(G \times H) \leq \frac{k}{2} \left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} USW_l(G) \right. \\ \left. + \sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}} USW_{l'}(H) \right]$$

According to the above reason we have

$$USW_k(G \times H) \leq \frac{k}{2} \left[\sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} (l-1)x_{lG}\mu_{UG} \right. \\ \left. + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG} \right. \\ \left. + \sum_{l'=2}^k \binom{n}{k_1'} \binom{n}{k_2'} \dots \binom{n}{k_{l'}} (l'-1)x_{l'H}\mu_{UH} \right. \\ \left. + l' \left[\binom{n}{l'} - x_{l'H} \right] \mu_{UH} \right] \quad (9)$$

Combining (8) and (9), we arrive at Inequality(7).

Theorem 2.3: Let G be a connected m -polar fuzzy graph with n vertices, and let H be a connected m -polar fuzzy graph with n' vertices. Let k be an integer, $2 \leq k \leq nn'$. Then the Unify Steiner Wiener distance for composition of two m -polar fuzzy graph is

$$USW_k(G_1[G_2]) = nk \binom{n'}{k} \mu_{UH} - nx_{kH}\mu_{UH} \\ + \sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} [(l-1)x_{lG}\mu_{UG} + \left[\binom{n}{l} - x_{lG} \right] \mu_{UG}] \\ \sum_{l=2}^k (k-l)\mu_{UG[H]} + \binom{n}{l} (l-1)x_{lG}\mu_{UG} + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG} \quad (10)$$

where $\sum_{i=1}^l k_i = k$ and x_{kH} (x_{kG}) is the number the k -subsets of $V(H)$ ($V(G)$) such that the subgraph induced by each k -subset is connected in $H(G)$.

Proof: For any S and $|S| = k$, if there exists some $H(g_i)$ such that $S \subseteq V(H(g_i))$, then the contribution to $USW_k(G[H])$ is

$$(k-1)x_{kH}\mu_{UH} + k \left[\binom{n'}{k} - x_{kH} \right] \mu_{UH} = k \binom{n'}{k} \mu_{UH} - x_{kH}\mu_{UH}$$

Such that x_{kH} is the number of k -subsets of $V(H)$ that the subgraph induced by each k -subset is connected. Since in $G[H]$ there are n copies of H , $H(g_i)$ ($1 \leq i \leq n$), it follows that

the contribution to $USW_k(G[H])$ is

$$nk \binom{n'}{k} \mu_{UH} - nx_{kH} \mu_{UH} \quad (11)$$

From now on, we assume that the vertices in S belong to at least two copies of H in $G[H]$. Without loss of generality, we assume that $H(g_1), H(g_2), \dots, H(g_l)$ satisfy $S \cap V(H(g_i)) \neq \emptyset$ for each g_i , $1 \leq i \leq l$, and $S \cap V(H(g_i)) = \emptyset$ for each g_i , $(l+1 \leq i \leq n)$. Set $|S \cap V(H(g_i))| = k_i$. Then $\sum_{i=1}^l k_i = k$.

Let $S_i = S \cap V(H(g_i)) = \{(g_i, h_j) | 1 \leq j \leq k_i\}$ for each g_i , $(1 \leq i \leq l)$. For one desired vertex, say (g_i, h_{ji}) , form $H(g_i)$, where $1 \leq i \leq l$ and $h_{ji} \in \{h_1, h_2, \dots, h_{k_i}\}$. Set $S' = \{(g_i, h_{ij}) | 1 \leq i \leq l\}$. Then $|S'| = l$, $d_{G[H]}(S') = d_G(\{h_1, h_2, \dots, h_l\})$. Since in G there exists a Steiner tree connecting $\{g_1, g_2, \dots, g_l\}$ of size $d_G(\{g_1, g_2, \dots, g_l\})$, it follows that in $G[H]$ there is a Steiner tree T' connecting S' of size $d_G(g_1, g_2, \dots, g_l)$. The S' -Steiner tree T' is a subtree of the S -Steiner tree T , where $S' \subseteq S$.

We first consider the contribution of these subtrees. Note there are $\binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l}$ ways to choose S_1, S_2, \dots, S_l . For every S_i , there is a Steiner tree T' connecting S' of size $d_G(\{g_1, g_2, \dots, g_l\})$. When $\{g_1, g_2, \dots, g_l\}$ takes over all l —

Subset of $V(G)$, the contribution of these subtrees to $USW_k(G[H])$ is

$$\begin{aligned} & \sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} USW_l(G) \\ &= \sum_{l=2}^k \binom{n'}{k_1} \binom{n'}{k_2} \dots \binom{n'}{k_l} [(l-1)x_{lG} \mu_{UG} + \left[\binom{n}{l} - x_{lG} \right] \mu_{UG}] \end{aligned} \quad (13)$$

For other contribution to $USW_k G[H]$, for given $S_1, S_2 \dots S_l$.

There is a Steiner tree T' connecting S' of size $d_G(\{g_1, g_2, \dots, g_l\})$. We now extend this subtree T' to a Steiner tree connecting S in $G[H]$. for each vertex (g, h) in $\left(\bigcup_{i=1}^l S_i \right) \setminus S'$, there exists a vertex in $V(T')$, say (g', h') , such that $(g, h)(g', h') \in E(G[H])$. By adding all these edges to $E(T)$, we can obtain an Steiner tree, say T . The total number of edges adding to T is $k - l$. So in this case, the contribution to $USW_k(G[H])$ is

$$\sum_{l=2}^k (k-l) \mu_{UG[H]} + \binom{n}{l} (l-1) x_{lG} \mu_{UG} + l \left[\binom{n}{l} - x_{lG} \right] \mu_{UG} \quad (13)$$

Combining (11), (12) and (13), we arrive at Eq.(10).

Example 2.1: For example assume m -polar fuzzy graph $G = (\sigma_1, \mu_1)$ a path P_6 and m -polar fuzzy graph $H = (\sigma_2, \mu_2)$ a path P_2 , then the Unify Steiner weiner index for $k = 2$ and for composition of two m -polar fuzzy graphs G and H by above theorem is calculable: $n = 6$, $n' = 2$, $x_2 G = 5$, $x_2 H = 1$, $\mu_{UH} = (\frac{1}{8}, \frac{1}{6}, \frac{1}{9})$ and, $\mu_{UG} = (\frac{1}{6}, \frac{1}{7}, \frac{1}{8})$.

$$USW_2(P_6[P_2]) = 6 \times 2 \binom{2}{2} \left(\frac{1}{8}, \frac{1}{6}, \frac{1}{9} \right) - 6 \times 1 \left(\frac{1}{8}, \frac{1}{6}, \frac{1}{9} \right)$$

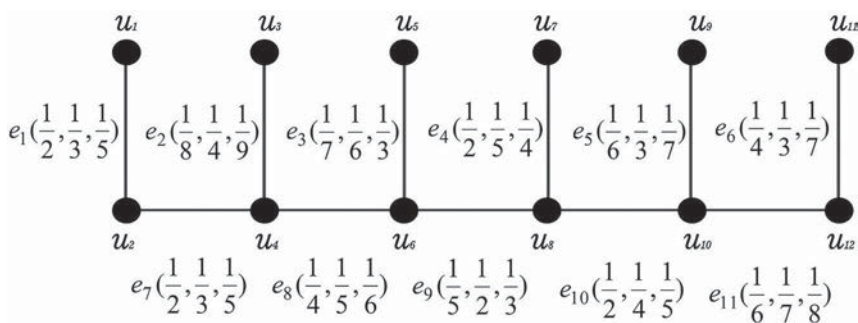


Figure 3. P6[P2].

$$\begin{aligned} &+ \binom{2}{2} \left[(2-1)5 \left(\frac{1}{6}, \frac{1}{7}, \frac{1}{8} \right) + \left[\binom{6}{2} - 5 \right] \left(\frac{1}{6}, \frac{1}{7}, \frac{1}{8} \right) \right] \\ &+ \binom{5}{2} (2-1)5 \left(\frac{1}{6}, \frac{1}{7}, \frac{1}{8} \right) + 2 \left[\binom{5}{2} - 5 \right] \left(\frac{1}{6}, \frac{1}{7}, \frac{1}{8} \right) \\ &= \left(\frac{81}{6}, \frac{81}{7}, \frac{81}{8} \right) \end{aligned}$$

3. Application

Fuzzy graphs of the 1-polar type are nothing more than the most familiar fuzzy graphs and have many applications for cluster analysis and solving fuzzy intersection equations, database theory, problems concerning group structure, and so on. The further possible applications of *m*-polar fuzzy graphs in real world problems can be viewed in the case of bipolar fuzzy graphs, i.e. 2 polar fuzzy graphs. Bipolar fuzzy graphs have many applications in social networks, engineering, computer science, database theory, expert systems, neural networks, artificial intelligence, signal processing, pattern recognition, robotics, computer networks, and medical diagnosis and so on. Additionally, *m*-polar fuzzy graphs (*m* > 2) are very useful in many decision making situations.[22] Nowadays fuzzy sets are playing a substantial role in chemistry. For *k* = 2, the above defined Steiner Wiener index coincides with the ordinary Wiener index $W(G) = \sum_{u,v \in V(G)} d(u,v)$. [23,24] We define one of the applications

of Steiner Wiener index for *k* = 2 by definition of Wiener index. Harold Wiener reported the existence of correlations between the new index and a large number of physico-chemical

Table 1. Correlation between the wiener number (*W*) of isomeric alkanes and Pitzers acentric factor (*W*), and between *W* and the van der Waal's constant *a*; experimental data for!

Isomer set	Correlationw	<i>W</i>	Correlation	<i>W</i>
	Correlation Coe.	No. of points	Correlation Coe.	No. of points
C5-alkanes	0:998	3	0:997	3
C6-alkanes	0:982	5	0:988	5
C7-alkanes	0:973	9	0:973	9
C8-alkanes	0:968	18	0:963	18
C9-alkanes	0:986	14	0:952	11
C10-alkanes	0:986	6	00:966	4

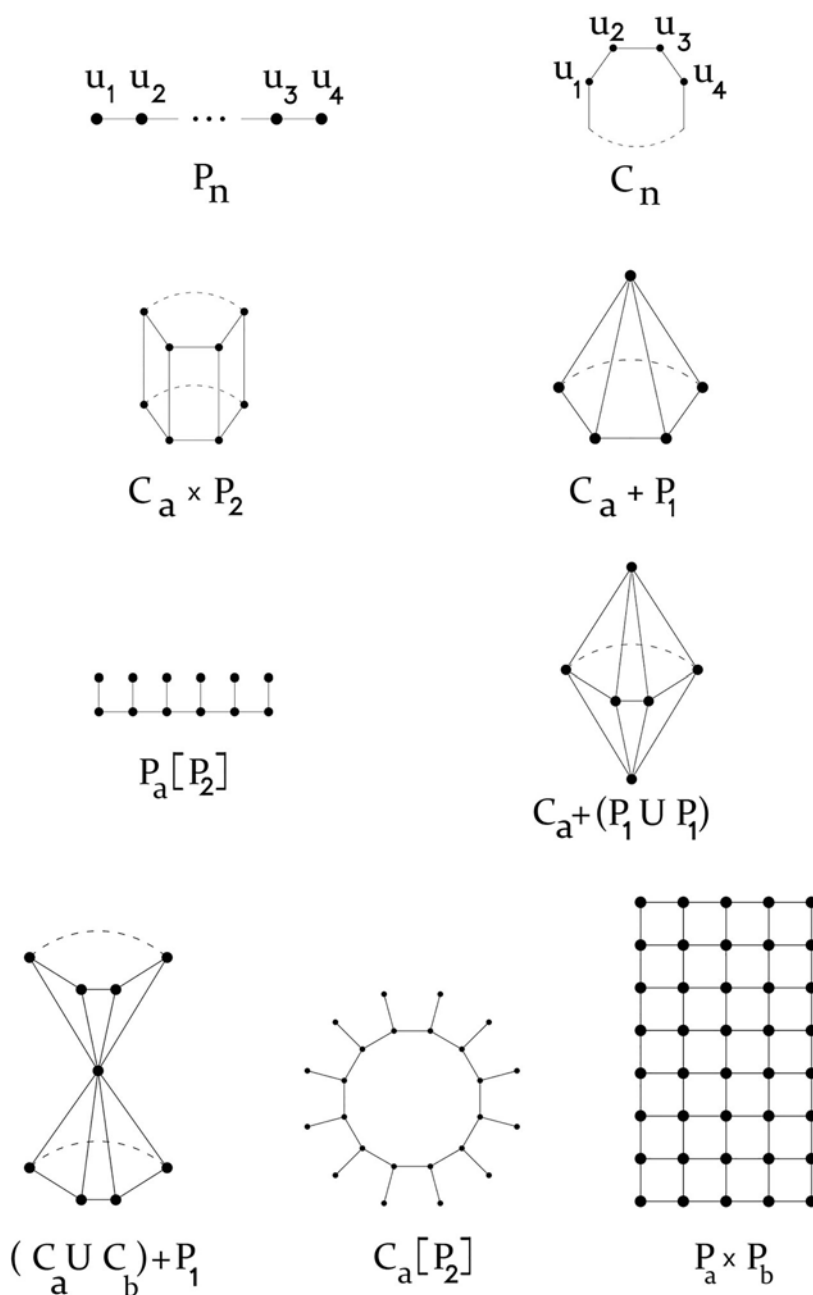


Figure 4. The path (P_n), the circuit (C_n) and some composite graphs of chemical interest that can be represented as products of path and circuits; their Wiener numbers are obtained using above theorems.

properties of alkanes. Wiener defined W only for alkanes as the number of carbon carbon bonds between all pairs of carbon atoms. There is a relation between W and the distances in the molecular graph. In particular, pointed out that W is equal to the half of the sum of all elements of the distance matrix of the respective molecular graph. By this, the concept

of Wiener number could be extended to cyclic molecules also. The applications of graphs in chemistry are because there are relation between structural formula and a graph. The first application of the Wiener number was for predicting the boiling points of alkanes on the formula: $b:p = \alpha w + \beta w(3)$ where, α and β are empirical constants and $w(3)$ is the so called path number, namely, the number of pairs of vertices whose distance is equal to 3. Wiener index used to estimate boiling points, molar volumes, refractive indices, heats of isomerisation and heats of vaporisation of alkanes. In Figure 3 and Table 1, it was shown some relation between Steiner wiener index for $k = 2$ (Wiener index) and chemical parameters. The Wiener number found several noteworthy applications in polymer chemistry. The melting points and other physical properties of polymers were predicted on the basis of their W – values. The average electron energies and the energy gaps in conjugated polymers were also shown to depend on W . Of the newest applications of the Wiener number, we may mention its use in the rationalisation of the mechanism of electro education of chlorobenzene derivatives and for distinguishing between fullerene isomers, as well as its role in the recent approaches towards the quantitation of molecular similarity. The fact that W is correlated with so many physico-chemical properties of non-polar organic substances leads to the conclusion that it must be a rough measure of the intermolecular forces. Curiously, however, in spite of the extensive research on the Wiener number in the last years, this fundamental feature was directly tested only quite recently.

4. Conclusions

Graph theory is an extremely useful tool for solving combinatorial problems in different areas, including algebra, number theory, geometry, topology, operations research, optimisation and computer science. Because research on or modelling of real world problems often involve multi-agent, multi-attribute, multi-object, multi-index, multi-polar information, uncertainty, and/or process limits, m -polar fuzzy graphs are very useful. The m -polar fuzzy models give increasing precision, flexibility, and comparability to the system compared to the classical, fuzzy and bipolar fuzzy models. On the other hand, the recent synthesis of macromolecules with highly branched skeletons (e.g. dendrimers), increase the need for estimating Wiener index via pertinent approximate formulae. So given that for $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index, therefore, in this paper, we introduced and studied k -Unify Steiner Weiner for m -polar fuzzy graphs and obtained this title for Join, Composition and Cartesian Product of two m -polar fuzzy graphs that can be a molecule graph. We plan to extend our research work on m -polar fuzzy graphs and its applications in support system (Figure 4).

Disclosure statement

No potential conflict of interest was reported by the author(s).

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