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Superring of Polynomials over a Hyperring

Reza Ameri [1,*,†], Mansour Eyvazi [1,†] and Sarka Hoskova-Mayerova [2,†]

[1] School of Mathematics, Statistic and Computer Sciences, University of Tehran, 79416-55665 Tehran, Iran; mansoureyvazi@ut.ac.ir

[2] Department of Mathematics and Physics, University of Defence in Brno, Kounicova 65, 662 10 Brno, Czech Republic; sarka.mayerova@unob.cz

* Correspondence: rameri@ut.ac.ir

† These authors contributed equally to this work.

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Abstract: A Krasner hyperring (for short, a hyperring) is a generalization of a ring such that the addition is multivalued and the multiplication is as usual single valued and satisfies the usual ring properties. One of the important subjects in the theory of hyperrings is the study of polynomials over a hyperring. Recently, polynomials over hyperrings have been studied by Davvaz and Musavi, and they proved that polynomials over a hyperring constitute an additive-multiplicative hyperring that is a hyperstructure in which both addition and multiplication are multivalued and multiplication is distributive with respect to the addition. In this paper, we first show that the polynomials over a hyperring is not an additive-multiplicative hyperring, since the multiplication is not distributive with respect to addition; then, we study hyperideals of polynomials, such as prime and maximal hyperideals and prove that every principal hyperideal generated by an irreducible polynomial is maximal and Hilbert's basis theorem holds for polynomials over a hyperring.

Keywords: hyperring; Krasner hyperring; hyperfield; superring; polynomial; fundamental relation; hyperideal

1. Introduction

A well established branch of classical algebraic theory is the theory of algebraic hyperstructures respectively hyperalgebraic system. In 1934, Marty first defined hyperstructures and began examining their properties, particularly with respect to group applications, rational fractions, and the algebraic functions [1]. At first, the research of properties and relations continued slowly, but, since the end of the last century, it has been very popular with mathematicians. Corsini in his work [2,3] showed that the theory of hyperstructures has many applications in both pure and applied sciences, e.g., semi-hypergroups are the simplest algebraic hyperstructures having closure and associativity properties. Since then, the theory of hyperstructures has been widely studied by many mathematicians. Let us mention at least some of them: Ameri and his school studied hypergroups, hypermodules, multialgebras, hyperideals, etc., in [4–9]. A recent paper of Asadi and Ameri deals with categorical connection between categories (m,n)-hyperrings and (m,n)-rings via the fundamental relation [10]. Hoskova-Mayerova provided a deep analysis of topological properties of hypergroupoids in her paper [11]. Th. Vougiouklis studied the fundamental relation in hyperrings and the general hyperfield in his paper [12]. Extension of elliptic curves on Krasner hyperfields was studied in [13].

In 1956, Krasner introduced the notion of the hyperfield in order to define a certain approximation of a complete valued field by sequences of such fields [14]. Krasner's hyperfield is based on the generalization of the additive group in a field by the structure of a special hypergroup. Later on, this hypergroup was named by Mittas "*canonical hypergroup*" [15]. The hyperfield that appears in [14] was named by Krasner "*residual hyperfield*". Krasner also introduced the hyperring, which is related to the

hyperfield in the same way as the ring is related to the field. In 1973, Mittas introduced the superring as an outcome of his study on expressions with coefficients from a hyperring. He named these expressions *hyperpolynomials* because the hyperpolynomials become polynomials when the hyperring is a ring. C. G Massouros studied the theory of hyperrings and hyperfields in [16,17] and [18]. G. G. Massouros and Ch. G. Massouros also defined hyperringoids and applied them in a generalization of rings in [19].

Some examples and results on Krasner hyperrings that are a generalization of classical rings was also published Davvaz [20]. In what follows, we, for short use, sometimes only use a hyperring.

Contrary to classical algebra, in hyperstructure theory, there are various kinds of hyperrings. Hyperrings and hyperfields in the sense of Krasner are more interesting classes of hyperrings and, recently, the authors in [21–24] studied noncommutative geometry and algebraic geometry. In addition, hyperfield extension is one of the important topics in the theory of algebraic hyperstructures, which not only can be considered as a development of the classical field theory, but it is also an important tool to study non-commutative geometry and algebraic geometry [25].

As it is well known, polynomials are important tools to study hyperfield theory.[26,27]. For instance, to characterize hyperalgebraic extension or algebraic closure of a hyperfield, we need to use polynomials over a hyperfield.[28,29] However, contrary to polynomials over a ring (or a field) in classical algebra, the behaviour of polynomials over a hyperring or hyperfield is completely different and much more complicated, since the product of two polynomials is not only a polynomial, but it is also a set of polynomials. In addition, in this regard, we show that, for polynomials over a hyperring even over a hyperfield[30], the product is not distributive with respect to addition (Theorem 3.7); in fact, it has a weak distributive property, and it constitutes a hyperring, which is called a *superring*. We will proceed to study the hyperideals of this superring such as prime and maximal hyperideals. Finally, we prove that, for a Krasner hyperfield F, its superring $F[x]$ is a PHH (a principal hyperideal hyperdomain), and investigate some main properties of $F[x]$. In particular, it is shown that the Hilbert's Basis theorem holds for a Krasner hyperring R that is, if R is a Noetherian Krasner hyperring, so is the superring $R[x]$.

2. Preliminaries

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. A *hyperoperation* is a mapping from $\circ : H \times H \longrightarrow P^*(H)$, where $(x, y) \longmapsto x \circ y$, and $(H, \circ)$ is called a *hypergroupoid* or a *hyperstructure*. For subsets A and B of H, $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$. A hypergroupoid $(H, \circ)$, with an associative property, which is $\forall a, b, c \in H, (a \circ b) \circ c = a \circ (b \circ c)$, is called a *semihypergroup*. A *hypergroup* is a semihypergroup $(H, \circ)$ with a reproduction axiom, that is,

$$a \circ H = H \circ a = H, \forall a \in H.$$

Definition 1 ([25,31,32])**.** *A Krasner hyperring (or, for short, a hyperring) is an algebraic hyperstructure $(R, +, .)$, such that the following conditions are satisfied:*

1. *$(R, +)$ is a canonical hypergroup, i.e.:*

   (i) for every $x, y, z \in R, x + (y + z) = (x + y) + z$; (ii) for every $x, y \in R, x + y = y + x$; (iii) there exists $0 \in R$ such that $0 + x = x, \forall x \in R$; (iv) for every $x \in R$, there exists a unique element $x' \in R$ such that $0 \in x + x'$(we write $-x$ for x'); (v) $z \in x + y$ implies $y \in z - x$ and $x \in z - y$.
2. *$(R, .)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0.x = 0$.*
3. *the multiplication . is distributive with respect to the hyperoperation $+$, that is, for all a, b, c in R, the following hold:*

$$a.(b + c) = a.b + a.c \ \text{ and } \ (a + b).c = a.c + b.c.$$

Definition 2 ([25,32,33])**.** *A hyperfield is a hyperring in which $(R \setminus \{0\}, .)$ is a commutative group.*

Example 1. $\mathbf{K} = \{0, 1\}$ *is a hyperfield with hyperoperation and multiplication given as follows:*

$+$	**0**	**1**
0	0	1
1	1	**K**

$\cdot$	**0**	**1**
0	0	0
1	0	1

Example 2. *(Sign hyperfield)* $\mathbf{S} = \{-1, 0, 1\}$ *is a hyperfield with hyperoperation and multiplication given by as follows:*

$+$	**−1**	**0**	**1**
−1	−1	−1	**S**
0	−1	0	1
1	**S**	1	1

$\cdot$	**−1**	**0**	**1**
−1	1	0	−1
0	0	0	0
1	−1	0	1

Example 3 ([14])**.** *Let* $(R, +, \cdot)$ *be a ring with identity and* G *be a normal subgroup of semigroup* $(R^{\times}, \cdot)$*. Take* $\bar{R} = R/G = \{aG | a \in R\}$ *with the hyperaddition and multiplication given by:*

$$\begin{cases} aG \oplus bG = \{cG | c \in aG + bG\}, \\ aG \odot bG = abG, \end{cases}$$

then $(R, \oplus, \odot)$ *is a hyperring, which is called a quotient hyperring. Moreover, if* R *is a field, then* $(R, \oplus, \odot)$ *is a hyperfield.*

Remark 1. *Note that, in the above example, the normal condition for* G *is not necessary, since Massouross in [16] generalized this construction using it for no normal multiplicative subgroups, since he proved that, in a ring, there exist multiplicative subgroups* G *of multiplicative semigroup* $(R, .)$*, which satisfy the property* $xGyG = xyG$*, even though they are not normal.*

Example 1 is the trivial case of monogene hyperfields introduced by Massouross in [17] with self-opposite elements. The construction of this monogene hyperfield is as follows:

Let K be the union of a multiplicative group $(G, .)$ with a bilaterally absorbing element 0. In K, the following hypercomposition $+$ is introduced:

$x + y = \{x, y\}$, for all x, y in G with $x \neq y$,

$x + 0 = 0 + x = x$, for all x in K,

$x + x = K$, for all x in G.

Then, $(K, +, .)$ is a hyperfield. If $G = 1$, then K is the hyperfield of Example 1.

Similarly, Example 2 is the trivial case of monogene hyperfields with no self-opposite elements, which is constructed over the multiplicative group $G = \{-1, 1\}$.

Both Examples 1 and 2 are quotient hyperfields, since Example 1 is the quotient of a field by its multiplicative group, while Example 2 is, for example, the quotient of the field of real numbers by the multiplicative subgroup of the positive real numbers. The question of whether all monogene hyperfields are quotient hyperfields is a hitherto open question [17].

In this step, we recall one of the important relations on a hyperring $(R, +, \circ)$. Let $\mathcal{U}$ denotes the set of all finite sums of finite products of elements of R. Note that an element $u \in \mathcal{U}$ may be the sum of only one element. Define a relation Γ on R as follows:

$$a \Gamma b \iff \exists u \in \mathcal{U} : \{a, b\} \subseteq u$$

.

In fact, there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$, such that $u = \Sigma_{i=1}^{n} \Pi_{j=1}^{k_i} x_{ij}$. Clearly, Γ is reflexive and symmetric relation on R. Let Γ^* denote the *transitive closure* of Γ. Consider the quotient R/Γ^*. Define hyperoperations $\oplus$ and operation $\odot$ on R/Γ^* as follows:

$$\Gamma^*(a) \oplus \Gamma^*(b) = \Gamma^*(c), \forall c \in \Gamma^*(a) + \Gamma^*(b),$$

$$\Gamma^*(a) \odot \Gamma^*(b) = \Gamma^*(d), \forall d \in \Gamma^*(a) \circ \Gamma^*(b).$$

Then, Γ^* is the smallest equivalence relation on R, such that the quotient space R/Γ^* is a ring, and it is called the *fundamental relation* of R and R/Γ^* is called fundamental ring of R (for more details, see [12]).

Definition 3. *(Homomorphism of hyperrings) Let R and S be two hyperrings. A map $f: R \longrightarrow S$ is called a (resp. good) homomorphism if the following holds:*

1. $f(a+b) \subseteq f(a) + f(b) (resp.\ f(a+b) = f(a) + f(b)), \forall a, b \in R.$
2. $\forall a, b \in R,\ f(ab) = f(a)f(b).$

Definition 4. *A map f is said to be an isomorphism if it is a bijective good homomorphism.*

3. Polynomials over Krasner Hyperrings

In this section, we discuss on polynomials over a hyperring(hyperfield) and prove that they constitute a *superring*.

Definition 5. *A hyperstructure $(S, +, .)$ is said to be a superring if $+$ and $.$ are both hyperoperations on S such that the following statements are satisfied:*

(i) $(S, +)$ *is a canonical hypergroup;*
(ii) $(S, .)$ *is a semihypergroup having zero as a bilaterally absorbing element, i.e., $x.0 = 0 = 0.x = 0$;*
(iii) *multiplication hyperoperation . is distributive from left and right with respect to hyperaddition $+$ that is*

$$a.(b+c) \subseteq a.b + a.c, (b+c).a \subseteq b.a + c.a;$$

(iv) *for all $a, b \in S$, $-(a.b) = (-a).b = a.(-b)$.*

Remark 2.

(i) *If in (iii) of the above definition the equality holds, then R is called an strongly distributive superring.*
(ii) *Every strongly distributive superring R is in fact an additive-multiplicative hyperring in the sense [34].*

Example 4. *Let $S = \{0, a, b, c\}$ be a set with two hyperoperations "+" and "$\cdot$" defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0,a\}$	c	$\{b,c\}$
b	b	c	$\{0,b\}$	$\{a,c\}$
c	c	$\{b,c\}$	$\{a,c\}$	S

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	$\{0,a\}$	0	$\{0,a\}$
b	0	0	$\{0,b\}$	$\{0,b\}$
c	0	$\{0,a\}$	$\{0,b\}$	$\{0,c\}$

Then, $(S, +, .)$ is a superring.

Definition 6. *A non-empty subset A of a superring S is a left(resp. right) hyperideal if,*

1. *for every $a, b \in A$ implies $a - b \subseteq A$;*
2. *for every $a \in A, r \in S$ implies $r.a \subseteq A(resp.\ a.r \subseteq A)$.*

Let X be a subset of a superring S. Let $\{A_i | i \in J\}$ be the family of all hyperideals in S which contain X. Then, $\bigcap\limits_{i \in J} A_i$ is called the hyperideal generated by X. This hyperideal is denoted by $< X >$. If $X = \{x_1, x_2, ..., x_n\}$, then the hyperideal $< X >$ is denoted by $< x_1, x_2, ..., x_n >$.

Next, lemma is a superring version of Lemma 3.1 in [34]

Lemma 1. *Let S be a superring and $X \subset S$. Then, for $a \in S$, the following statements are satisfied:*

1. *The principal hyperideal $< a >$ is equal to*

$$\{t| \ t \in ra + as + na + k(a-a) + \sum_{i=1}^{m} r_i a s_i, r, s, r_i, s_i \in S, m \in \mathbb{N}, n, k \in \mathbb{Z}\}.$$

2. *If S has a unit element, then*

$$< a >= \{t| \ t \in k(a-a) + \sum_{i=1}^{m} r_i a s_i, r_i, s_i \in S, m, k \in \mathbb{N}\}.$$

3. *If a is in the center of S, then*

$$< a >= \{t| t \in na + k(a-a) + \sum_{i=1}^{m} r_i a, r_i \in S, m, n, k \in \mathbb{N}\},$$

where the center of S is the set $\{x \in S| \ xy = yx, \ \forall y \in S\}$.

4. *$Sa = \{\sum\limits_{i=1}^{m} r_i a | r_i \in S, m, k \in \mathbb{N}\}$ is a left hyperideal in S and $aS = \{\sum\limits_{i=1}^{m} a r_i | r_i \in S, m, k \in \mathbb{N}\}$ is a right hyperideal in S.*
5. *If S has a unit element and a is in the center of S, then $Sa =< a >= aS$.*
6. *If S has a unit element and X is included in the center of S, then*

$$< X >= \{t| \ t \in \sum_{i=1}^{m} r_i x_i, r_i \in S, x_i \in X, m \in \mathbb{N}\}.$$

Proof. The proof is similar to the proof for Krasner hyperrings in [20] by some manipulations. □

Definition 7. *A commutative hyperring R with identity is said to be Noetherian if every hyperideal of R is finitely generated, i.e., if I is a hyperideal of R, then $I =< a_1, a_2, ..., a_n >$ for some $n \in \mathbb{N}$ and $a_i \in I$, $i \in \{1, 2, ..., n\}$.*

Let R be a Krasner hyperring and $R[x]$ be the hyperring of polynomials introduced in [35]. Recall that hyperaddition and hypermultiplication on $R[x]$ for $f(x) = a_0 + a_1 x + ... + a_n x^n$, and $g(x) = b_0 + b_1 x + ... + b_n x^m$ are defined as follows:

$$f(x) \oplus g(x) = \{\sum_{i=0}^{M} c_i x^i | c_i \in a_i + b_i\},$$

where

$$M = \max\{\deg f(x), \deg g(x)\}$$

and

$$f(x) \odot g(x) = \{\sum_{k=0}^{m+n} c_k x^k \ | \ c_k \in \sum_{i+j=k} a_i b_j\}.$$

In [34], the authors stated and proved Theorem 3.2 as follows:

Theorem 1. $(R[x], \oplus, \odot)$ *is an additive-multiplication hyperring.*

In the following, by some counterexamples, we will show that the Theorem 3.2 in [34] is not true because, in the hyperstructure of polynomials over a Krasner hyperring, the hypermultiplication is not strongly distributive with respect to the hyperaddition, even if we replace a hyperring with a hyperfield. In the following, we will show that the polynomial over a hyperring(or a hyperfield) constitutes a *superring*, which is called the superring of polynomials. For instance, we prove that, for hyperfield $\mathbf{K}$ of order 2 and signs hyperfield, $\mathbf{S}$ of order 3, their polynomials hyperrings $\mathbf{K}[x]$ and $\mathbf{S}[x]$, the hypermultiplication is not distributive with respect to the sum of hyperaddition.

Example 5. *The polynomial hyperring* $\mathbf{K}[x]$ *is not an additive-multiplication hyperring because:*

$$(1+x^2)(1\oplus 1) = (1+x^2)\{0,1\} = \{0, 1+x^2\}.$$

On the other hand, one has

$$(1+x^2)\oplus(1+x^2) = (1\oplus 1)\oplus(x^2\oplus x^2) = \{0,1\}\oplus\{0,x^2\} = \{0,1,x^2,1+x^2\}.$$

Thus, the distributivity does not hold in $\mathbf{K}[x]$ *as an additive-multiplicative hyperring. In fact,* $\mathbf{K}[x]$ *is a superring, which is not an additive-multiplication hyperring.*

Example 6. *The polynomial hyperring* $\mathbf{S}[x]$ *is not an additive-multiplication hyperring because:*

$$(1+x)(1\ominus 1) = \{0, 1+x, -1-x\}$$

and

$$(1+x)\ominus(1+x) = (1\ominus 1)+(1\ominus 1)x = \{0,1,-1,x,-x,1+x,1-x,-1+x,-1-x\}.$$

In fact, for polynomials over a hyperring, even over a hyperfield, but only the left-hand side weak distributivity holds, which is

$$f(g+h)\subseteq fg+fh, \quad f,g,h\in F[x].$$

Thus, we issue the modified version of above mentioned Theorem 1.

Theorem 2. $(R[x],\oplus,\odot)$ *is a superring.*

Proof. Here, we just verify the weak distributivity. The proof of other properties is the as same as Theorem 1. Suppose that $M=\max\{\deg g(x), \deg h(x)\}$ and $n=\deg f(x)$. Since R is a hyperring, then, for every $0\le i\le n$ and $0\le j\le M$, $a_i(b_j+c_j)=a_ib_j+a_ic_j$. Thus, $f(x)\odot(g(x)\oplus h(x)) = \{\sum\limits_{l=0}^{n+M} d_lx^l \mid d_l\in \sum\limits_{i+j=l} a_i(b_j+c_j)\} = \{\sum\limits_{l=0}^{n+M} d_lx^l \mid d_l\in \sum\limits_{i+j=l} a_ib_j + \sum\limits_{i+j=l} a_ic_j\} = \{\sum\limits_{l_1=l_2=l=0}^{n+M} d_lx^l \mid d_l\in \sum\limits_{i+j=l_1} a_ib_j + \sum\limits_{i+j=l_2} a_ic_j\} \subseteq \{\sum\limits_{l_1=0}^{n+M} d_{l_1}x^{l_1} \mid d_{l_1}\in \sum\limits_{i+j=l_1} a_ib_j\}\oplus\{\sum\limits_{l_2=0}^{n+M} d_{l_2}x^{l_2} \mid d_{l_2}\in \sum\limits_{i+j=l_2} a_ic_j\}$. □

Theorem 3. *(Hyper-version of Hilbert's Basis Theorem) If R is a Noetherian Krasner hyperring, so is the superring $R[x]$.*

Proof. The sketch of proof is extracted from the proof of Theorem 21, *Ch*.9 in [36].

Let I be a hyperideal in $R[x]$ and L be the set of all leading coefficients of elements in I. We first prove that L is an hyperideal in R. Since $0\in I$, then $0\in L$. Let $f=ax^d+\ldots$ and $g=bx^e+\ldots$ be polynomials in I of degrees d,e and $a,b\in R$ are leading coefficients of f,g, respectively. Then, for any $r\in R$, $ra-b$ is the leading coefficients of some elements of rx^ef-x^dg. Since polynomials are in I, we have $ra-b\subseteq L$, which shows L is a hyperideal of R. Since R is a Noetherian hyperring, L is finitely

generated (considering R as a R-hypermodule and, according to proposition 9.2. in [37]), denoted by $a_1, ..., a_n \in R$. For each $i = 1, ..., n$, let f_i be an element of I with leading coefficient a_i. Let us denote e_i the degree of f_i and let N be the maximum of elements $e_1, ..., e_n$. For each $d \in \{0, 1, ..., N-1\}$, let L_d be the set of all leading coefficients of polynomials in I of degree d and L_d also contains 0. A similar argument as that for L proves that each L_d is a hyperideal of R, again finitely generated since R is Noetherian. For each hyperideal L_d, let $b_{d,1}, ..., b_{d,n_d} \in R$ be a set of generators for L_d and let $f_{d,i}$ be a polynomial of degree d in I with leading coefficients $b_{d,i}$.

We prove that the polynomials $f_1, ..., f_n$ cooperating polynomials $f_{d,i}$ are a set of generators for I, i.e.,

$$I =< \{f_1, ..., f_n\} \cup \{f_{d,i} | 0 \leq d \leq N-1, 1 \leq i \leq n_d >\}.$$

By construction of hyperideal, I', the right-hand side of the above, is contained in I. If $I' \neq I$, there exists a non-zero polynomial $f \in I$ with a minimum degree with $f \notin I'$. Let d be deg f and let a be the leading coefficient of f. Suppose first that $d \geq N$. As $a \in L$, we can write it as an element of R-linear combination of the generators of L, which is as $a \in r_1 a_1 + ... + r_n a_n$. Then, there exists $g \in r_1 x^{d-e_1} f_1 + ... + r_n x^{d-e_n} f_n$ an element of I' with the same degree d and the same leading coefficient a as f. Then, $f - g$ contains a polynomial in I of smaller degree than f. By the minimality of f, we must have $0 \in f - g$(really $0 = f - g$ or there exists a non-zero element in $f - g$, which, by minimality of f, has the same degree as the degree of f and f is a monomial), a contradiction.

Suppose next that $d < N$. In this case, $a \in L_d$ for some $d < N$, and so we can write $a \in r_1 b_{d,1} + ... + r_{n_d} b_{d,n_d}$ for some $r_i \in R$. Then, there exists $g \in r_1 f_{d,1} + ... + r_{n_d} f_{d,n_d}$ inI' with the same degree d and the same leading coefficient a as f and we have a contradiction as before. It follows that $I = I'$ is finitely generated and, since I was an arbitrary choice, the proof is complete. □

At the following, we present some more examples of superrings:

Example 7. *(Superring of matrices) Let R be a hyperring and $M_{n\times n}(R)$ denotes the set of all $n \times n$ matrices over R. Then, $M_{n\times n}(R)$ by usual matrix addition and multiplication is a superring, which is not an additive-multiplication hyperring.*

Example 8. *(Superring of formal power series) Define the set $R[[x]]$ of formal power series in the indeterminate x with coefficients from R to be all formal infinite sums*

$$\sum_{i=1}^{n} a_n x_n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Define hyperaddition and hypermultiplication as classical operations for classical formal power series. Then, $R[[x]]$ is a superring, which is not an additive-multiplication hyperring.

Definition 8. *A Krasner hyperring R with identity, which is zero-divisor free i.e., $0 \in ab \Rightarrow a = 0$ or $b = 0$ for $a, b \in R$, is called a hyperdomain.*

Equivalently, one can define superdomain as a zero divisor free superring with identity element.

Theorem 4. *If D is a hyperdomain, then $D[x]$ is a superdomain.*

Proof. Since $D[x]$ is a superring with identity, it is enough that we show that $D[x]$ is also zero divisor free. Suppose that $0 \in f(x)g(x)$ for $f(x) = \sum\limits_{i=0}^{n} a_i x^i, g(x) = \sum\limits_{j=0}^{m} b_j x^j$ in $D[x]$. As D is a hyperdomain, by Theorem 2 $degf(x) + degg(x) = 0$, and so $f(x) = a_0$ and $g(x) = b_0$. Thus, $0 \in a_0 b_0$. Thus, $a_0 = 0$ or $b_0 = 0$, since D is superdomain, and hence $f(x) = 0$ or $g(x) = 0$. □

Theorem 5. *Let $(D, +, .)$ be a hyperdomain and $f(x), g(x) \in D[x]$.*

1. *If $h(x) \in f(x) \odot g(x)$, then $deg\, h(x) = deg\, f(x) + deg\, g(x)$.*
2. *If $t(x) \in f(x) + g(x)$, then $deg\, t(x) = max\{deg\, f(x), deg\, g(x)\}$.*

Theorem 6. *Let F be a hyperfield. Then, $F[x]$ is a PHH (Principal Hyperideal Hyperdomain).*

Proof. We prove that all hyperideals of $F[x]$ are to form $< x^m >, m \neq 0$. Suppose that I is a hyperideal and $f(x) \in I$. Thus, $f(x) = \sum\limits_{j=0}^{n} a_j x^j$, and hence $f(x) \ominus f(x) = \sum\limits_{j=0}^{n} (a_j - a_j)x^j$. Then, $a_0 \neq 0$ implies that $< f(x) >= F[x]$. Otherwise, suppose that $a_m, m \in \mathbb{N}$, being the smallest first non-zero coefficient of all elements of I. Then, $I \subseteq < x^m > \subseteq I$. Thus, $I =< x^m >$. □

Definition 9. *An element $\alpha \in F$ is a root of $f(x) \in F[x]$ if $0 \in F(\alpha) = \sum\limits_{i=0}^{n} a_i \alpha^i$.*

Theorem 7. *If α is a root of $f(x) \in F[x]$, then there exists $g(x) \in F[x]$, such that $f(x) \in (x - \alpha) \odot g(x)$.*

Proof. We prove by induction on the degree of $f(x)$. If $f(x) = a_0 + a_1 x$, it is trivial. For degree two without loss of generality, we suppose that $f(x) = a_0 + a_1 x + x^2$. If $\alpha = 0$, then the result is obvious. Now, suppose that $\alpha \neq 0$ and $0 \in f(\alpha) = a_0 + a_1\alpha + \alpha^2$. Multiplying each side of the inclusion by α^{-1} one has $0 \in a_0\alpha^{-1} + a_1 + \alpha$, then, by reversibility, we have

$$-a_1 \in a_0\alpha^{-1} + \alpha. \tag{1}$$

Now, $(x - \alpha) \odot (x - a_0\alpha^{-1}) = a_0 - (a_0\alpha^{-1} + \alpha)x + x^2$. Thus, by Equation (1), we have $f(x) = a_0 + a_1 x + x^2 \in (x - \alpha) \odot (x - a_0\alpha^{-1})$. Suppose that the claim holds for every polynomial of degree $n - 1$. For $0 \in f(\alpha)$ and $deg f(x) = n$, $0 \in a_0 + a_1\alpha + ... + \alpha^n$ (without loss of generality, it is supposed that $f(x)$ is monic). Again, multiplying each side of the inclusion by α^{-1}, one has $0 \in a_0\alpha^{-1} + a_1 + a_2\alpha + ... + \alpha^{n-1}$. $\exists a'_0 \in a_0\alpha^{-1} + a_1$ such that $0 \in a'_0 + a_2\alpha + ... + \alpha^{n-1}$. Put $f'(x) = a'_0 + a_2 x + ... + x^{n-1}$. Thus, $0 \in f'(\alpha)$. By hypothesis, $\exists g(x) \in F[x]$ that $f'(x) \in (x - \alpha) \odot g(x)$.

$f(x) \in x \odot (f'(x) \ominus \alpha^{-1}a_0) \oplus a_0 = x \odot ((x - \alpha) \odot g(x) \ominus \alpha^{-1}a_0) \oplus a_0 = (x - \alpha) \odot x \odot g(x) \ominus \alpha^{-1}a_0 x \oplus a_0 = (x - \alpha) \odot x \odot g(x) \ominus (x - \alpha) \odot \alpha^{-1}a_0 = (x - \alpha) \odot (x \odot g(x) - a_0\alpha^{-1})$. Thus, $f(x) \in (x - \alpha) \odot (x \odot g(x) \ominus a_0\alpha^{-1})$. There exists a $g'(x) \in x \odot g(x) \ominus a_0\alpha^{-1}$ such that $f(x) \in (x - \alpha) \odot g'(x)$. □

Remark 3. *Note that, contrary to classical ring theory, for superring of polynomials over a hyperring, a polynomial of degree n may have more than n roots. For example, consider the hyperfield of order 5 defined by the following tables:*

$+$	**0**	**1**	***a***	***b***	***c***
0	0	1	a	b	c
1	1	$\{1, a, b\}$	$\{1, a, c\}$	$\{0, 1, a, b, c\}$	$\{1, b, c\}$
a	a	$\{1, a, c\}$	$\{a, b, c\}$	$\{1, a, b\}$	$\{0, 1, a, b, c\}$
b	b	$\{0, 1, a, b, c\}$	$\{1, a, b\}$	$\{1, b, c\}$	$\{a, b, c\}$
c	c	$\{1, b, c\}$	$\{0, 1, a, b, c\}$	$\{a, b, c\}$	$\{1, a, c\}$

$\cdot$	**0**	**1**	***a***	***b***	***c***
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	b	c	1
b	0	b	c	1	a
c	0	c	1	a	b

For $f(x) = 1 + x + x^2$, one can verify that $0 \in f(1) \cap f(a) \cap f(b) \cap f(c)$.

Let R be a hyperring and $\alpha \notin R$. Define $R[\alpha] = \{c_0 + c_1\alpha + ... + c_{n-1}\alpha^{n-1} | c_i \in R,\ n \in \mathbb{N}\}$. One can define addition and multiplication hyperoperations on $R[\alpha]$ as follows:

$$\sum_{i=0}^{n-1} a_i\alpha^i \oplus \sum_{i=0}^{n-1} b_i\alpha^i = \sum_{i=0}^{n-1} (a_i + b_i)\alpha^i \text{ for } a_i, b_i \in R,$$

$$\sum_{i=0}^{n-1} a_i\alpha^i \odot \sum_{j=0}^{m-1} b_j\alpha^j = \{\sum_{k=0}^{m+n-2} c_k\alpha^k | c_k \in \sum_{i+j=k} a_ib_j\},$$

for $a_i, b_i \in R$, such that

$$\alpha^n = -(c'_0 + c'_1\alpha + ... + c'_{n-1}\alpha^{n-1}),$$

for some $c'_i \in R$, $i \in \{0, ..., n-1\}$.

It is easy to verify that $(R[\alpha], \oplus, \odot)$ is a superring and it is an extension of R. In fact, this is a method for constructing a superring via a hyperring.

Theorem 8. *Let F be a commutative hyperfield and $\alpha^2 = c \in F$, $\alpha \notin F$. Then, $f(x) = x^2 - c$ has no root in F if and only if $F[\alpha]$ is a superfield extension of F.*

Proof. $(\Rightarrow)$ Suppose that $0 \in (a_0 + a_1x)(b_0 + b_1x)$ and $\{a_0, a_1\} \neq \{0\}$. Then,

$(a_0 + a_1x)(b_0 + b_1x) = a_0b_0 + (a_0b_1 + a_1b_0)x + a_1b_1x^2 = a_0b_0 + (a_0b_1 + a_1b_0)x + a_1b_1c = (a_0b_0 + a_1b_1c) + (a_0b_1 + a_1b_0)x$. Thus, $0 \in a_0b_0 + a_1b_1c$ and $0 \in a_0b_1 + a_1b_0$, and hence $b_0 = -a_0^{-1}a_1cb_1$. Thus, $0 \in a_0b_1 - a_1a_0^{-1}a_1cb_1$. Thus, $0 \in (a_0^2 - ca_1^2)b_1$. Thus, it has to be $0 \in a_0^2 - ca_1^2$ or $b_1 = 0$. If the first case happens, it leads to $0 \in a_0^2a_1^{-2} - c = (a_0a_1^{-1})^2 - c$. Thus, $a_0a_1^{-1} \in F$ is a root of $x^2 - c$. Therefore, $x^2 - c$ has a root in F which contradicts the hypothesis. Thus, the second case happens i.e., $b_1 = 0$. Since $b_0 = -a_0^{-1}a_1cb_1$, we also have $b_0 = 0$. Thus, $F[\alpha]$ is a superdomain.

Now, for non-zero $a_0 \in F$, consider $(a_0 + a_1\alpha)(b_0 + b_1\alpha)$. Let $b_0 = a_0^{-1}$ and $b_1 = -a_0^{-2}a_1$. In this case, it is obvious that $1 \in (a_0 + a_1\alpha)(b_0 + b_1\alpha)$. If $a_0 = 0$, then $a_1\alpha a_1^{-1}\alpha c^{-1} = 1$. Therefore, $F[\alpha]$ is a superfield.

$(\Leftarrow)$ Suppose that $u \in F$ is a root of $f(x) = x^2 - c$. By Theorem 7, we have $f(x) \in (x - u)(x + u)$. Hence, $0 \in f(\alpha) \subseteq (\alpha - u)(\alpha + u)$, which means that $F[\alpha]$ is not a superdomain and naturally is not a superfield. □

Example 9. *Let $S = \{0, 1, \alpha, 1 + \alpha\}$ be a set with two hyperoperations as follows:*

$+$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{\alpha}$	$\mathbf{1+\alpha}$
0	0	1	α	$1+\alpha$
1	1	$\{0,1\}$	$1+\alpha$	$\{\alpha, 1+\alpha\}$
α	α	$1+\alpha$	$\{0,\alpha\}$	$\{1, 1+\alpha\}$
$1+\alpha$	$1+\alpha$	$\{\alpha, 1+\alpha\}$	$\{1, 1+\alpha\}$	S

$\cdot$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{\alpha}$	$\mathbf{1+\alpha}$
0	0	0	0	0
1	0	1	α	$1+\alpha$
α	0	α	1	$1+\alpha$
$1+\alpha$	0	$1+\alpha$	$1+\alpha$	S

for which $\alpha \neq 1$ is a root of $f(x) = 1 + x^2$. It is easy to check that $(S, +, .)$ is a superring, which is not strongly distributive.

Example 10. *Let $K = \{0, 1, -1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Define two hyperoperations on K by the following tables:*

$+$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{-1}$	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_4}$	$\boldsymbol{\alpha_5}$	$\boldsymbol{\alpha_6}$
0	0	1	-1	α_1	α_2	α_3	α_4	α_5	α_6
1	1	1	$-1,0,1$	α_3	α_4	α_3	α_4	$\alpha_5,\alpha_1,\alpha_3$	$\alpha_6,\alpha_2,\alpha_4$
-1	-1	$-1,0,1$	-1	α_4	α_6	$\alpha_5,\alpha_1,\alpha_3$	$\alpha_6,\alpha_2,\alpha_4$	α_5	α_6
α_1	α_1	α_3	α_4	α_1	$\alpha_2,0,\alpha_1$	α_3	$\alpha_4,1,\alpha_3$	α_5	$\alpha_6,-1,\alpha_5$
α_2	α_2	α_4	α_6	$\alpha_2,0,\alpha_1$	α_2	$\alpha_4,1,\alpha_3$	α_4	$\alpha_6,-1,\alpha_5$	α_6
α_3	α_3	α_3	$\alpha_5,\alpha_1,\alpha_3$	α_3	$\alpha_4,1,\alpha_3$	α_3	$\alpha_4,1,\alpha_3$	$\alpha_5,\alpha_1,\alpha_3$	K
α_4	α_4	α_4	$\alpha_6,\alpha_2,\alpha_4$	$\alpha_4,1,\alpha_3$	α_4	$\alpha_4,1,\alpha_3$	α_4	K	$\alpha_6,\alpha_2,\alpha_4$
α_5	α_5	$\alpha_5,\alpha_1,\alpha_3$	α_5	α_5	$\alpha_6,-1,\alpha_5$	$\alpha_5,\alpha_1,\alpha_3$	K	α_5	$\alpha_6,-1,\alpha_5$
α_6	α_6	$\alpha_6,\alpha_2,\alpha_4$	α_6	$\alpha_6,-1,\alpha_5$	α_6	K	$\alpha_6,\alpha_2,\alpha_4$	$\alpha_6,-1,\alpha_5$	α_6

$\cdot$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{-1}$	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_4}$	$\boldsymbol{\alpha_5}$	$\boldsymbol{\alpha_6}$
0	0	0	0	0	0	0	0	0	0
1	0	1	-1	α_1	α_2	α_3	α_4	α_5	α_6
-1	0	-1	1	α_2	α_1	α_6	α_5	α_4	α_3
α_1	0	α_1	α_2	-1	1	α_5	α_3	α_6	α_4
α_2	0	α_2	α_1	1	-1	α_4	α_6	α_3	α_5
α_3	0	α_3	α_6	α_5	α_4	$\alpha_5, \alpha_1, \alpha_3$	$\alpha_4, 1, \alpha_3$	$\alpha_6, -1, \alpha_5$	$\alpha_6, \alpha_2, \alpha_4$
α_4	0	α_4	α_5	α_3	α_6	$\alpha_4, 1, \alpha_3$	$\alpha_6, \alpha_2, \alpha_4$	$\alpha_5, \alpha_1, \alpha_3$	$\alpha_6, -1, \alpha_5$
α_5	0	α_5	α_4	α_6	α_3	$\alpha_6, -1, \alpha_5$	$\alpha_5, \alpha_1, \alpha_3$	$\alpha_6, \alpha_2, \alpha_4$	$\alpha_4, 1, \alpha_3$
α_6	0	α_6	α_3	α_4	α_5	$\alpha_6, \alpha_2, \alpha_4$	$\alpha_6, -1, \alpha_5$	$\alpha_4, 1, \alpha_3$	$\alpha_5, \alpha_1, \alpha_3$

It is easy to check that $K \supseteq \mathbf{S}$ is a superring. Then, by Theorem 10 for $c = -1$, $f(x) = x^2 + 1$ has no root in $\mathbf{S}$ and since $K = \mathbf{S}[\alpha]$ in which $\alpha_1 = \alpha, \alpha_2 = -\alpha, \alpha_3 = 1 + \alpha, \alpha_4 = 1 - \alpha, \alpha_5 = -1 + \alpha, \alpha_6 = -1 - \alpha$, is a superfield extension of $\mathbf{S}$.

Remark 4. *Consider Example 10 for which K is a superfield in which distributivity is weak. For instance, it is easy to verify that $\alpha_4(\alpha_2 + \alpha_3) \subsetneq \alpha_4\alpha_2 + \alpha_4\alpha_3$. Thus, the distributivity could be weak even though the superring is really a superfield.*

Theorem 9. *(Division algorithm) Let F be a Krasner hyperfield with unit element 1, $(F[x], \oplus, \odot)$ is the polynomial hyperring of F. If $a(x)$ and $b(x) \in F[x]$ and $b(x) \neq 0$, then there exists a pair of polynomials $q(x)$ and $r(x)$ such that*

$$a(x) \in q(x) \odot b(x) \oplus r(x), deg\, r(x) < deg\, b(x).$$

Proof. The proof is just the proof of Theorem 3.4 in [34] since $q_1(x) \odot b(x) \oplus r_1(x) \oplus a_n b_m^{-1} x^{n-m} \odot b(x) = q_1(x) \odot b(x) \oplus a_n b_m^{-1} x^{n-m} \odot b(x) \oplus r_1(x) = (q_1(x) \oplus a_n b_m^{-1} x^{n-m}) \odot b(x) \oplus r_1(x)$ as degree $q_1(x) < n - m$ in superring $F[x]$. $\square$

Definition 10. *Let R be a hyperring and $f, d \in R[x]$; then, we say that d divides(counts) f if and only if $f \in < d >$; we denote it by $d|f$.*

Definition 11. *Let R be a hyperring. An element $d \in R[x]$ such that $d|f, d|g$ and, for every $e \in R[x], e|f, e|g$, implies that $e|d$ is said to be a great common divisor of f, g and denote it by $d \in (f, g)$.*

Proposition 1. *Let F be a hyperfield. Then, there exists a great common divisor for every two elements in $F[x]$.*

Proof. Let $f, g \in F[x]$. By a division algorithm, one has:

$$\begin{aligned} f &\in p_0 g + r_0, \\ g &\in p_1 r_0 + r_1, \\ r_0 &\in p_2 r_1 + r_2, \\ r_1 &\in p_3 r_2 + r_3. \\ \vdots\ &\ \vdots\ \ \vdots \\ r_n &\in p_{n+2} r_{n+1}, \end{aligned}$$

and hence $r_{n+1}|f, g$. Thus, there exists a common divisor for every two elements in $F[x]$. Define $\mathcal{C} = \{d \in (f, g)\} \cup \{f, g\}$. One can define a partial relation on non-unit elements of $\mathcal{C}$ as $r \leq s \Leftrightarrow r|s$. We make a totally ordered ascending chain of these elements with upper bound f. By Zorn's lemma,

this ascending chain has a maximal in $\mathcal{C}$. If all common divisors are unit, we define a great common divisor of the two elements by 1. □

Proposition 2. *Let F be a hyperfield and $f, g, d \in F[x]$ and $d \in gcd(f, g)$. Then, $< f, g >=< d >$.*

Theorem 10. *Let F be a hyperfield. Then, $F[x]/ < x >\cong F$.*

Proof. By a division algorithm, the residue class is isomorphic to F. □

Definition 12. *Let R be a superring. Then, $f \in R$ is called a unit element if $< f >= R$.*

Definition 13. *Let R be a superring. An element $f \in R$ is said to be irreducible if $f \in< u >, u \in R$; then, $< f >=< u >$.*

Theorem 11. *Let R be a Krasner hyperring and I be a hyperideal of $R[x]$ and $(R[x]/I, \oplus, \odot)$ be the quotient of $R[x]$ by I with hyperoperations $\oplus, \odot$ defined as follows:*

$$\bar{x} \oplus \bar{y} = (x + I) \oplus (y + I) = \{z + I | z \in x + y\},$$
$$\bar{x} \odot \bar{y} = (x + I) \odot (y + I) = \{w + I | w \in xy\}.$$

Then, $(R[x]/I, \oplus, \odot)$ is a superring.

Proof. Let $x, y \in R[x]$. We define $x \equiv y \ (mod\ I)$ if and only if $x \in (y + I)$. This relation is an equivalence relation. Thus, the set $\{x + I | x \in R[x]\}$ is a partition of $R[x]$. Now, we prove that the hyperoperations defined on $R[x]/I$ are well defined. Let $x' \in \bar{x}, y' \in \bar{y}$; then, there are $u, v \in I$, such that $x' \in x + u$ and $y' \in y + v$. $\bar{x'} \oplus \bar{y'} = (x' + I) \oplus (y' + I) \subseteq (x + u + I) \oplus (y + v + I) = (x + I) \oplus (y + I) = \bar{x} \oplus \bar{y}$. Thus, $\bar{x'} \oplus \bar{y'} \subseteq \bar{x} \oplus \bar{y}$. $x \in x' - u$ and $y \in y' - v$. Therefore, we have $\bar{x} \oplus \bar{y} \subseteq \bar{x'} \oplus \bar{y'}$. Thus, $\bar{x'} \oplus \bar{y'} = \bar{x} \oplus \bar{y}$.

$\bar{x'} \odot \bar{y'} = (x' + I) \odot (y' + I) \subseteq (x + u + I) \odot (y + v + I) = (x + I) \odot (y + I) = \bar{x} \odot \bar{y}$. Similarly, we have $\bar{x} \odot \bar{y} \subseteq \bar{x'} \odot \bar{y'}$. It is routine to verify the other conditions of the superring. □

Theorem 12. *Let R be a Krasner hyperring, $f \in R[x]$ be monic and $degf \leq 2$. If $< f >$ is a maximal hyperideal in $R[x]$, then, provided that $R[x]/ < f >$ is finite, $R[x]/ < f >$ is a hyperfield.*

Proof. Since $< f >$ is maximal, the quotient structure is zero-divisor free superring. We prove that, for $\bar{a} \neq 0$,

$$\bar{a} \odot \bar{b} \cap \bar{a} \odot \bar{c} = \emptyset.$$

Because if it is not true, then

$$0 \in \bar{a} \odot \bar{b} - \bar{a} \odot \bar{c} = (a+ < f >) \odot (b+ < f >) - (a+ < f >) \odot (c+ < f >)$$

$$= (ab+ < f >) - (ac+ < f >) = (ab - ac)+ < f > .$$

Thus, it is enough to prove that, for $a \neq 0, \exists u \in< f >$, such that $u \in ab - ac \Rightarrow \bar{b} = \bar{c}$. By hypothesis $a = a_0 + a_1 x, b = b_0 + b_1 x$, $c = c_0 + c_1 x$ and $u = u_0 + u_1 x + u_2 x^2$, then

$ab - ac = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_2 b_2 - a_0 c_0 + (a_0 c_1 + a_1 c_0)x + a_2 c_2 = a_0(b_0 - c_0) + [a_0(b_1 - c_1) + a_1(b_0 - c_0)]x + a_1(b_1 - c_1)x^2$. Since $u \in< f >, u - u = (u_0 - u_0) + (u_1 - u_1)x + (u_2 - u_2)x^2 \subseteq< f >$. Thus, $u_0(1 - 1) \subseteq< f >, u_1 x(1 - 1) \subseteq< f >$ and $u_2 x^2(1 - 1) \subseteq< f >$.

Suppose that, for some $u_i, i \in \{0, 1, 2\}$, $u_i \notin< f >$. Then, $x^i \in< f >$ or $(1 - 1) \subseteq< f >$. If $x^i \in< f >$, then, because $< f >$ is a maximal hyperideal in $R[x]$, we have $x \in< f >$. Thus, $R[x]/ < f >$ is a Krasner hyperring. In addition, since it does not have any zero divisor element, then it is a hyperdomain and, since it is finite, thus it is a hyperfield (see [37] Corollary 5.2). If $(1 - 1) \subseteq< f >$, then $\bar{1} - \bar{1} = \bar{0}$. Consequently, $\bar{a} - \bar{a} = \bar{0}$, and then $\bar{a} \subseteq \bar{a} + \bar{b} - \bar{b} = \bar{a}$. Consider $\bar{c}, \bar{d} \in \bar{a} + \bar{b}$. Thus,

$\bar{c} - \bar{b} = \bar{a}$ and $\bar{d} - \bar{b} = \bar{a}$. Therefore, $\bar{c} - \bar{d} = \bar{0}$, and hence $\bar{c} = \bar{d}$. Thus, hyperaddition is single valued. $\bar{a} \odot \bar{b} = (ab + < f >) = (a_0 b_0 + [a_0 b_1 + a_1 b_0]x + a_1 b_1 x^2 + < f >) = (c_0 + [c_1 + c_1']x + c_2 x^2 + < f >) = \bar{c_0} \oplus \bar{c_1} \oplus \bar{c_1'} \oplus \bar{c_2} = \bar{d}$. Thus, the multiplication is also single valued. Therefore, $R[x]/ < f >$ is a ring. Since it is zero divisor free, it is an integral domain and, since it is finite, then it is a field, and consequently a hyperfield.

Suppose that all coefficients of polynomial u belongs to $< f >$. Therefore,

$$\begin{cases} u_0 \in a_0(b_0 - c_0) \Rightarrow a_0 \in < f > \textbf{ or } \bar{b_0} = \bar{c_0}, \\ u_1 \in [a_0(b_1 - c_1) + a_1(b_0 - c_0)]x, \\ u_2 \in a_1(b_1 - c_1)x^2 \Rightarrow a_1 \in < f > \textbf{ or } \bar{b_1} = \bar{c_1} \textbf{ or } x^2 \in < f > . \end{cases}$$

If $a_0 \notin < f >$ and $a_1 \notin < f >$, then one obtains that $\bar{b_0} = \bar{c_0}$ and $\bar{b_1} = \bar{c_1}$, which means $\bar{b} = \bar{c}$ (for $x^2 \in < f >$, and hence the quotient superring is a hyperfield). If $a_0 \in < f >$, then $a_1 \notin < f >$. Therefore, $\bar{b_1} = \bar{c_1}$. Since $a_0 \in < f >$, $u_0 \in a_1(b_0 - c_0)$, then $\bar{b_0} = \bar{c_0}$ and we have $\bar{b} = \bar{c}$. For the case $a_0 \notin < f >$ and $a_1 \in < f >$, similarly, we come to the conclusion that $\bar{b} = \bar{c}$. Thus, for non-zero $\bar{a} \in R[x]/ < f >$, $\bar{a} \odot R[x]/ < f >$ is a partition of $R[x]/ < f >$. It is possible provided that the multiplication is single valued. Thus, the quotient space is a multiring. Noticing that every element has an inverse, it is easy to verify that distributivity of multiplication with respect to addition is strong. In [38] Section 4.3, it has been proved that every multifield is a commutative hyperfield. The commutativity is dispensable; since the result again holds, $ab + ac = a^{-1}a(ab + ac) = aa^{-1}(ab + ac) \subseteq a(b + c)$. Thus, this multiring is a hyperfield and this completes the proof. □

Theorem 13. *Let R be a hyperring. Then,* $R[x]/\bar{\Gamma}^* \cong R/\Gamma^*[x]$.

Proof. Define

$$\varphi : R[x] \longrightarrow R/\Gamma^*[x],$$

$$\sum_{i=0}^{n} a_i x^i \longmapsto \sum_{i=0}^{n} \Gamma^*(a_i)x^i.$$

1. At first, we show that φ is a good homomorphism.

   Let $f, g \in R[x], f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j$. Then, $\varphi(f + g) = \varphi(\sum_{k=0}^{Max\{m,n\}} (a_k + b_k)x^k) = \{\varphi(\sum_{k=0}^{Max\{m,n\}} c_{ki}x^k) | c_{ki} \in a_k + b_k\} = \{\sum_{k=0}^{Max\{m,n\}} \Gamma^*(c_{ki})x^k | c_{ki} \in a_k + b_k\} = \sum_{k=0}^{Max\{m,n\}} \Gamma^*(a_k + b_k)x^k = \sum_{k=0}^{Max\{m,n\}} [\Gamma^*(a_k) \oplus \Gamma^*(b_k)]x^k = \sum_{k=0}^{Max\{m,n\}} \Gamma^*(a_k)x^k \oplus \sum_{k=0}^{Max\{m,n\}} \Gamma^*(b_k)x^k = \sum_{k=0}^{n} \Gamma^*(a_i)x^i \oplus \sum_{k=0}^{m} \Gamma^*(b_j)x^j = \varphi(f) \oplus \varphi(g)$.
2. φ is one to one:

   $\varphi(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \Gamma^*(a_i)x^i$.

   $Ker\varphi = \{\sum_{i=0}^{n} a_i x^i | \sum_{i=0}^{n} \Gamma^*(a_i)x^i = 0\}$.

   $\sum_{i=0}^{n} \Gamma^*(a_i)x^i = 0 \ \Rightarrow \ \forall a_i, \Gamma^*(a_i) = \bar{0} \Rightarrow \Gamma^*(a_i) = \Gamma^*(0)$.

   $(f\Gamma g \Leftrightarrow \{f, g\} \subseteq \sum_{j} \prod_{I_j} f_{ji})$.

Note that $\bar{\Gamma}^*(a_0 + a_1x + ... + a_nx^n) = \bar{\Gamma}^*(a_0) \oplus \bar{\Gamma}^*(a_1)x \oplus ... \oplus \bar{\Gamma}^*(a_n)x^n$. $\bar{0} = \sum_{i=0}^{n} \Gamma^*(a_i)x^i = \bar{\Gamma}^*(\sum_{i=0}^{n} a_ix^i) \Rightarrow p \in ker\varphi \Leftrightarrow \bar{\Gamma}^*(p) = \bar{0}$. □

Theorem 14. *Let R be a hyperring. Then, $M_n(R)/\bar{\Gamma}^* \cong M_n(R/\Gamma^*)$.*

Proof. We define the map

$$\begin{aligned} \psi : M_n(R) &\longrightarrow M_n(R/\Gamma^*) \\ M_n(a_{ij}) &\longmapsto M_n(\Gamma^*(a_{ij})). \end{aligned}$$

1. Analogous to the proof of Theorem 13, we verify ψ to be a good homomorphism.
   Let $M_n(a_{ij}), M_n(b_{ij}) \in M_n(R)$. Then, $\psi(M_n(a_{ij}) + M_n(b_{ij})) = \psi(M_n(a_{ij} + b_{ij})) = \{\psi(M_n(c_{ij}))|c_{ij} \in a_{ij} + b_{ij}\} = \{M_n(\Gamma^*(c_{ij}))|c_{ij} \in a_{ij} + b_{ij}\} = M_n(\Gamma^*(a_{ij} + b_{ij}) = M_n(\Gamma^*(a_{ij}) \oplus \Gamma^*(b_{ij})) = M_n(\Gamma^*(a_{ij})) \oplus M_n(\Gamma^*(b_{ij})) = \psi(M_n(a_{ij})) \oplus \psi(M_n(b_{ij}))$.
2. ψ is one to one since $\psi(M_n(a_{ij})) = M_n(\Gamma^*(a_{ij}))$.
   $ker\psi = \{M_n(a_{ij})|\psi(M_n(a_{ij}) = 0\}$.
   Thus, $M_n(\Gamma^*(a_{ij})) = 0 \Rightarrow \forall a_{ij}, \Gamma^*(a_{ij}) = \bar{0} \Rightarrow \bar{\Gamma}^*(M_n(a_{ij})) = \bar{\Gamma}^*(0)$.
   Therefore, $\bar{\Gamma}^*(M_n(a_{ij})) = \bar{0} \Leftrightarrow M_n(\Gamma^*(a_{ij})) = 0$.
   Thus, $M_n(R)/\bar{\Gamma}^* \cong M_n(R/\Gamma^*)$. □

Theorem 15. *Let R be a hyperring. Then, the following are satisfied:*

(i) $\Gamma^* = \Gamma$.
(ii) *Let R be a hyperring and Γ^* be its fundamental relation. If there exists a unit element $u \in x - x$ for some $x \in R$, then $R/\Gamma^* \cong 0$.*

Proof.

(i) It is obvious since multiplication is single valued and $(R, \oplus)$ is a hypergroup.
(ii) Since $u \in x - x$ is a unit, then $1 \in u^{-1}x - u^{-1}x$. Thus, for every element $y \in R$, one has $\{0, y\} \subseteq yu^{-1}x - yu^{-1}x$. Therefore, $\Gamma^*(0) = R$ and $R/\Gamma^* = 0$.

□

Corollary 1.

1. *Let F be a non-trivial hyperfield. Then, $F/\Gamma \cong 0$.*
2. *Let R be a hyperring extension of* **K** *or* **S**. *Then, $R/\Gamma^* \cong 0$.*

Corollary 2. *Let F be a non-trivial hyperfield. Then, $F[x]/\bar{\Gamma}^* \cong 0$.*

Proof. It is an immediate consequence of Theorem 13 and Corollary 1, item 2. □

4. Conclusions

We proved that the polynomials over a Krasner hyperring constitute a superring, which is not an additive-multiplicative hyperring. In addition, hyperideals of a superring of polynomials, such as prime and maximal hyperideals, were studied and it was proved that every principal hyperideal generated by an irreducible polynomial is maximal, and Hilbert's Basis theorem holds for polynomials over a Krasner.

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