

SOME RESULTS ON HYPERVECTOR SPACES

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Abstract. The aim of this paper is to study hypervector spaces. In this regard at first some new nontrivial examples of hypervector spaces are introduced. Then the notions of linearly span, linearly independence, basis, ordered basis, coordinates and linear transformation are investigated and some related results are obtained. Especially, it is proved that for a linear transformation $T : V \rightarrow W$ between two hypervector spaces, $\dim \ker T + \dim T(V) = \dim V$, and under certain conditions $\dim L(V, W) = \dim V \times \dim W$.

Keywords: hypervector space, linearly span, linearly independent set, basis, linear transformation, coordinate.

1. Introduction

The theory of hyperstructures was born in 1934, when Marty [6] defined hypergroups. Since then many researches have worked on hyperalgebraic structures and developed this theory (for more see [3], [4], [5], [13]). In 1990, M. Scafati Tallini introduced the notion of hypervector spaces [8], and studied ba-

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sic properties of them (for more see [9], [10], [11]). Recently Ameri [1], [2], Vaezpoor [7] and Taghavi [12] in Iran have developed this concept. In this paper we follow [2] and study more properties of hypervector spaces. In this regard at first some new interesting nontrivial examples of hypervector spaces are introduced. Then their basic notions are investigated and some related results are obtained. Especially, it is proved that for a linear transformation $T : V \rightarrow W$, $\dim \ker T + \dim T(V) = \dim V$, and under certain conditions, $\dim L(V, W) = \dim V \times \dim W$.

2. Preliminaries

In this section we present some definitions and simple properties of hypervector spaces that we shall use in later.

A map $\circ : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or join operation, where $P_*(H)$ is the set of all non-empty subsets of H . The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a .

Definition 2.1. ([8]) Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K to be the quadruplet $(V, +, \circ, K)$, Where “ \circ ” is a mapping:

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H₁) $a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law,

(H₂) $(a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law,

(H₃) $a \circ (b \circ x) = (ab) \circ x$, associative law,

(H₄) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,

(H₅) $x \in 1 \circ x$.

Remark 2.2 ([8]). (i) In the right hand of (H₁) the sum is meant in the sense of Frobenius, i.e.

$$a \circ x + a \circ y = \{p + q : p \in a \circ x, q \in a \circ y\}.$$

Similarly we have in (H₂). Moreover, the left hand side of (H₃) means the set-theoretical union of all the sets $a \circ y$, where y runs over set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(ii) ([9]) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$\forall a, b \in K, \forall x \in V, (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive if equality holds. In a similar way we define anti-right distributive and strongly right distributive. V is called strongly distributive if it is both strongly left and right distributive.

(iii) The mapping “ \circ ” in Definition 2.1, is called external hyperoperation.

Example 2.3 ([8]). In $(\mathbb{R}^2, +)$ we define the product times a scalar in \mathbb{R} by setting:

$$a \circ x = \begin{cases} \text{line pass origin and point } x, & \text{if } x \neq \underline{0}, \\ \{\underline{0}\}, & \text{if } x = \underline{0}. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a hypervector space.

Example 2.4 ([12]). $(\mathbb{C}, +, \circ, \mathbb{R})$ is a hypervector space, where “ $+$ ” is the usual sum and the mapping “ $\circ : \mathbb{R} \times \mathbb{C} \longrightarrow P_*(\mathbb{C})$ ” is defined by the following:

$$a \circ z = \begin{cases} \{re^{i\theta} : 0 < r \leq |a||z|, \theta = \arg(az)\}, & \text{if } a \neq 0 \text{ and } z \neq \underline{0}, \\ \{\underline{0}\}, & \text{if } a = 0 \text{ or } z = \underline{0}. \end{cases}$$

Lemma 2.5 ([8]). Let $(V, +, \circ, K)$ be a hypervector space and $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. Then

1. If V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$.
2. If V is anti-left distributive, then for all $x \in V$ the set $0 \circ x$ is a subgroup of $(V, +)$.
3. If V is strongly left distributive, then $a \circ \underline{0} = \Omega_V = a \circ \Omega_V$, for all $a \in K$.

Proposition 2.6. Let $(V, +, \circ, K)$ be a strongly left distributive hypervector space such that $|1 \circ \underline{0}| = 1$ and for all $x \in V$, $-x \neq x$, unless $x = \underline{0}$. Then for any $x \in V$, the following holds:

$$x = \underline{0} \iff \forall 0 \neq a \in K; a \circ x + a \circ x = \Omega.$$

Proof. (\implies) Let $x = \underline{0}$ and $0 \neq a \in K$. Then by Lemma 2.5, it follows that:

$$a \circ x + a \circ x = a \circ \underline{0} + a \circ \underline{0} = \Omega + \Omega = \Omega.$$

(\impliedby) By Definition 2.1, $a \circ 2x = a \circ (x + x) \subseteq a \circ x + a \circ x = \Omega = a \circ \underline{0}$. So $a^{-1} \circ (a \circ 2x) \subseteq a^{-1} \circ (a \circ \underline{0})$. Thus $1 \circ 2x \subseteq 1 \circ \underline{0} = \{\underline{0}\}$. Hence $2x = \underline{0}$ and so $x = -x$. Therefore $x = \underline{0}$. \square

3. New examples of hypervector spaces

In this section we present some new nontrivial examples of hypervector spaces.

Example 3.1. $(\mathbb{Z}, +, \circ, \mathbb{Q})$ is a hypervector space, where “+” is the usual sum and the mapping “ \circ ” is defined by the following:

$$\begin{cases} \circ : \mathbb{Q} \times \mathbb{Z} \longrightarrow P_*(\mathbb{Z}) \\ \frac{r}{s} \circ n = \{m(rn) : m \in \mathbb{Z}\}. \end{cases}$$

Example 3.2. If the external hyperoperation “ $\circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2)$ ” for all $r, a, b \in \mathbb{R}$ is defined by each of the followings, then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a hypervector space:

- (i) $r \circ (a, b) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq ra, 0 \leq y \leq rb\},$
- (ii) $r \circ (a, b) =$ Environment of the rectangle bounded with lines $x = 0, x = ra, y = 0, y = rb,$
- (iii) $r \circ (a, b) = \{(x, y) \in \mathbb{R}^2 \mid -ra \leq x \leq ra, -rb \leq y \leq rb\},$
- (iv) $r \circ (a, b) =$ Environment of the rectangle bounded with lines $x = -ra, x = ra, y = -rb, y = rb,$
- (v) $r \circ (a, b) =$ Environment of the circle with origin $(0, 0)$ and radius $|r|\sqrt{a^2 + b^2}.$

Proposition 3.3. If $(V, +, \circ, K)$ is a hypervector space such that $-x \in 1 \circ x$ for all $x \in V$, then (V, \oplus, \odot, K) construct a hypervector space with the following mappings:

$$\begin{cases} \oplus : V \times V \longrightarrow V \\ x \oplus y = x + y, \end{cases} \quad \text{and} \quad \begin{cases} \odot : K \times V \longrightarrow P_*(V) \\ a \odot x = -a \circ x. \end{cases}$$

Proof. Straightforward. □

Proposition 3.4. Let $(V, +, \circ, K)$ be a hypervector space. Suppose

$$V^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in V \right\}.$$

Then $(V^n, \oplus_n, \odot_n, K)$ is a hypervector space, where “ \oplus_n ” and “ \odot_n ” are defined by the followings:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \left\{ \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} : x'_i \in a \circ x_i, 1 \leq i \leq n \right\}.$$

Proof. Let $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in V^n$ and $a \in K$. Then

$$\begin{aligned}
a \odot_n \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) &= a \odot_n \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \\
&= \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i \in a \circ (x_i + y_i) \right\} \\
&\subseteq \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i \in a \circ x_i + a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} : z_i = \acute{x}_i + \acute{y}_i, \acute{x} \in a \circ x_i, \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 + \acute{y}_1 \\ \vdots \\ \acute{x}_n + \acute{y}_n \end{bmatrix} : \acute{x} \in a \circ x_i, \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 \\ \vdots \\ \acute{x}_n \end{bmatrix} \oplus_n \begin{bmatrix} \acute{y}_1 \\ \vdots \\ \acute{y}_n \end{bmatrix} : \acute{x} \in a \circ x_i, \acute{y} \in a \circ y_i \right\} \\
&= \left\{ \begin{bmatrix} \acute{x}_1 \\ \vdots \\ \acute{x}_n \end{bmatrix} : \acute{x} \in a \circ x_i \right\} \oplus_n \left\{ \begin{bmatrix} \acute{y}_1 \\ \vdots \\ \acute{y}_n \end{bmatrix} : \acute{y} \in a \circ y_i \right\} \\
&= \left(a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \oplus_n \left(a \odot_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right).
\end{aligned}$$

Thus

$$a \odot_n \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \oplus_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) \subseteq \left(a \odot_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \oplus_n \left(a \odot_n \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right).$$

In a similar way, it is easy to verify the other parts of Definition 2.1. \square

Remark 3.5. In Proposition 3.4, if V is strongly left, Strongly right or strongly distributive, then V^n is strongly left, Strongly right or strongly distributive, respectively.

Proposition 3.6. *Let $(V, +, \circ, K)$ be a hypervector space. Suppose*

$$M_{m \times n}^V = \left\{ \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} : x_{ij} \in V, 1 \leq i \leq m, 1 \leq j \leq n \right\}.$$

Then $(M_{m \times n}^V, \oplus, \odot, K)$ is a hypervector space, where \oplus is the usual sum of matrices and external hyperoperation $\odot : K \times M_{m \times n}^V \rightarrow P_(M_{m \times n}^V)$ is defined by the following:*

$$a \odot \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} = \left\{ \begin{bmatrix} x'_{11} & \cdots & x'_{1n} \\ \vdots & & \vdots \\ x'_{m1} & \cdots & x'_{mn} \end{bmatrix} : x'_{ij} \in a \circ x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n \right\}.$$

Corollary 3.7. *Let $(V, +, \circ, K)$ be a hypervector space. Then the set*

$$\left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in V \right\}$$

together operation \oplus and external hyperoperation \odot is a hypervector space over the field K , where \oplus is the usual sum of matrices and \odot is defined by the following:

$$a \odot \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \left\{ \begin{bmatrix} w & -z \\ z & w \end{bmatrix} : w \in a \circ x, z \in a \circ y \right\}.$$

Definition 3.8 ([9]). A nonempty subset W of V is called a subhyperspace of V , if W is itself a hypervector space with the external hyperoperation on V , i.e.

$$\begin{cases} W \neq \phi, \\ \forall x, y \in W \implies x - y \in W, \\ \forall a \in K, \forall x \in W \implies a \circ x \subseteq W. \end{cases}$$

In this case we write $W \leq V$.

Example 3.9. (i) Let $L = \{(a, 0) \mid a \in \mathbb{R}\}$ and $R = \{(0, b) \mid b \in \mathbb{R}\}$. Then L and R are subhyperspaces of the hypervector space $(\mathbb{R}^2, +, \circ, \mathbb{R})$ in Example 2.3.

(ii) In Example 3.2, the hypervector spaces are defined in parts (ii) and (iv) are subhyperspaces of the hypervector spaces are defined in parts (i) and (iii), respectively.

(iii) The hypervector space is defined in part (v) of Example 3.2, is a subhyperspace of the hypervector space in Example 2.4.

Proposition 3.10. *If V is either strongly left or right distributive hypervector space, then Ω_V is the smallest subhyperspace of V .*

Proof. By Lemma 2.5, Ω_V is a subgroup of V and $a \circ \Omega_V = \Omega_V$, for all $a \in K$. Thus Ω_V is a subhyperspace of V . Now if H is a subhyperspace of V , then for any $x \in \Omega_V$, we have:

$$x \in 0 \circ \underline{0} \subseteq 0 \circ H \subseteq H.$$

So $\Omega_V \subseteq H$. □

Proposition 3.11. *If V is a hypervector space over the field K and $H \leq V$, then for any $x, x_1, \dots, x_n \in H$ and $a, a_1, \dots, a_n \in K$, the following conditions hold:*

1. $(a_1 + \dots + a_n) \circ x \subseteq a_1 \circ x + \dots + a_n \circ x$,
2. $a \circ (-x) \subseteq H$,
3. $a_1 \circ x_1 + \dots + a_n \circ x_n \subseteq H$.

Proof. Straightforward. □

4. Basis and dimension

In the sequel of this note, unless otherwise specified, we assume that $V = (V, +, \circ, K)$ is a hypervector space over the field K .

Definition 4.1 ([2]). Let S be a subset of V . Then the linear span of S is the smallest subhyperspace of V containing S and denoted by $SP(S)$. If S is a nonempty subset of V , then

$$\begin{aligned} SP(S) &= \{x \in V : x \in \sum_{i=1}^n a_i \circ y_i, a_i \in K, y_i \in S, 1 \leq i \leq n, n \in \mathbb{N}\} \\ &= \{x_1 + \dots + x_n : x_i \in a_i \circ y_i, a_i \in K, y_i \in S, 1 \leq i \leq n, n \in \mathbb{N}\}. \end{aligned}$$

We say that S spans V , if $SP(S) = V$. It is easy to verify that if A and B are subsets of V such that $A \subseteq B$, then $SP(A) \subseteq SP(B)$.

Proposition 4.2. *Let V be strongly left distributive and $x_1, \dots, x_n, y_1, \dots, y_m \in V$, such that $\{y_1, \dots, y_m\}$ spans V and $y_1, \dots, y_m \in SP(x_1, \dots, x_n)$. Then $\{x_1, \dots, x_n\}$ spans V .*

Proof. Let $x \in V$. Then $x \in b_1 \circ y_1 + \dots + b_m \circ y_m$, for some $b_i \in K, 1 \leq i \leq m$. On the other hand,

$$\begin{cases} y_1 \in a_{11} \circ x_1 + \dots + a_{1n} \circ x_n \\ y_2 \in a_{21} \circ x_1 + \dots + a_{2n} \circ x_n \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ y_m \in a_{m1} \circ x_1 + \dots + a_{mn} \circ x_n \end{cases}$$

for some $a_{ij} \in K, 1 \leq i \leq m, 1 \leq j \leq n$. Thus

$$\begin{aligned}
x &\in b_1 \circ (a_{11} \circ x_1 + \cdots + a_{1n} \circ x_n) + \cdots + b_m \circ (a_{m1} \circ x_1 + \cdots + a_{mn} \circ x_n) \\
&\subseteq b_1 \circ (a_{11} \circ x_1) + \cdots + b_1 \circ (a_{1n} \circ x_n) + \cdots \\
&\quad + b_m \circ (a_{m1} \circ x_1) + \cdots + b_m \circ (a_{mn} \circ x_n) \\
&= (b_1 a_{11}) \circ x_1 + \cdots + (b_1 a_{1n}) \circ x_n + \cdots + (b_m a_{m1}) \circ x_1 + \cdots + (b_m a_{mn}) \circ x_n \\
&= (b_1 a_{11} + \cdots + b_m a_{m1}) \circ x_1 + \cdots + (b_1 a_{1n} + \cdots + b_m a_{mn}) \circ x_n \\
&\subseteq SP(x_1, \dots, x_n).
\end{aligned}$$

Hence $SP(x_1, \dots, x_n) = V$. \square

Definition 4.3. (i) A hypervector space V over the field K is said to be K -weak invertible or shortly weak invertible if and only if

$$\forall a \in K, \forall u, v \in V, u \in a \circ v \text{ implies that } v \in a' \circ u, \text{ for some } a' \in K.$$

(ii) ([2]) A hypervector space V over the field K is said to be K -invertible or shortly invertible if and only if

$$\forall a \neq 0 \in K, \forall u, v \in V, u \in a \circ v \text{ implies that } v \in a^{-1} \circ u.$$

Proposition 4.4. Let V be weak invertible and $\{x_1, x_2, y_1, y_2\} \subseteq V$, such that $x_1 \in b_1 \circ y_1$ and $x_2 \in b_2 \circ y_2$, for some $b_1, b_2 \in K$. Then $SP(x_1, x_2) = SP(y_1, y_2)$.

Proof. Let $z \in SP(x_1, x_2)$. Then $z \in a_1 \circ x_1 + a_2 \circ x_2$, for some $a_1, a_2 \in K$. So

$$\begin{aligned}
z &\in a_1 \circ (b_1 \circ y_1) + a_2 \circ (b_2 \circ y_2) \\
&= (a_1 b_1) \circ y_1 + (a_2 b_2) \circ y_2 \\
&\subseteq SP(y_1, y_2).
\end{aligned}$$

Thus $SP(x_1, x_2) \subseteq SP(y_1, y_2)$. On the other hand, $y_1 \in b'_1 \circ x_1$ and $y_2 \in b'_2 \circ x_2$, for some $b'_1, b'_2 \in K$. Now let $w \in SP(y_1, y_2)$. Then $w \in c_1 \circ y_1 + c_2 \circ y_2$, for some $c_1, c_2 \in K$. So

$$\begin{aligned}
w &\in c_1 \circ (b'_1 \circ x_1) + c_2 \circ (b'_2 \circ x_2) \\
&= (c_1 b'_1) \circ x_1 + (c_2 b'_2) \circ x_2 \\
&\subseteq SP(x_1, x_2).
\end{aligned}$$

Thus $SP(y_1, y_2) \subseteq SP(x_1, x_2)$. Therefore $SP(x_1, x_2) = SP(y_1, y_2)$. \square

Proposition 4.5. Let V be strongly left distributive and $x_1, \dots, x_n, y \in V$. Then $SP(x_1, \dots, x_n, y) = SP(x_1, \dots, x_n)$ if and only if $y \in SP(x_1, \dots, x_n)$.

Proof. (\Leftarrow) Let $y \in SP(x_1, \dots, x_n)$. Then $y \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. Now if $x \in SP(x_1, \dots, x_n, y)$, then $x \in a'_1 \circ x_1 + \dots + a'_n \circ x_n + b \circ y$, for some $a'_1, \dots, a'_n, b \in K$. Thus

$$\begin{aligned} x &\in a'_1 \circ x_1 + \dots + a'_n \circ x_n + b \circ (a_1 \circ x_1 + \dots + a_n \circ x_n) \\ &\subseteq a'_1 \circ x_1 + \dots + a'_n \circ x_n + b \circ (a_1 \circ x_1) + \dots + b \circ (a_n \circ x_n) \\ &= a'_1 \circ x_1 + \dots + a'_n \circ x_n + (ba_1) \circ x_1 + \dots + (ba_n) \circ x_n \\ &= (a'_1 + ba_1) \circ x_1 + \dots + (a'_n + ba_n) \circ x_n \\ &\subseteq SP(x_1, \dots, x_n). \end{aligned}$$

Consequently, $SP(x_1, \dots, x_n, y) \subseteq SP(x_1, \dots, x_n)$. On the other hand, by Definition 4.1, $SP(x_1, \dots, x_n) \subseteq SP(x_1, \dots, x_n, y)$, and so the equality holds.

(\Rightarrow) It is clear that $y \in SP(x_1, \dots, x_n, y)$. Thus $y \in SP(x_1, \dots, x_n)$. \square

Definition 4.6 ([2]). A subset S of V is called linearly independent if for every vectors $x_1, \dots, x_n \in S$ and $c_1, \dots, c_n \in K$, $\underline{0} \in c_1 \circ x_1 + \dots + c_n \circ x_n$ implies that $c_1 = \dots = c_n = 0$. Note that some hypervector spaces V (some set W of vectors) may not have any collection of linearly independent vectors. Such hypervector space (set) is called independentless. S is called linearly dependent if it is not linearly independent. A basis for V is a linearly independent subset of V such that spans V . We say that V has finite dimensional if it has a finite basis. If V has a basis with n vectors, then every basis for V has n vectors. In this case the number n is called the dimension of V , denoted by $\dim V = n$.

Proposition 4.7. *If V is strongly distributive, then for any linearly independent subset $\{x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ of V , the set*

$$S = \left\{ \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots \underline{0} \end{bmatrix}, \dots, \begin{bmatrix} \underline{0} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix} \right\}$$

is a basis for the hypervector space $(M_{m \times n}^V, \oplus, \odot, \mathbb{R})$ is defined in Proposition 3.6, and so $\dim M_{m \times n}^V = mn$.

Proof. Let $\begin{bmatrix} x_{11} \cdots x_{1n} \\ \vdots \\ x_{m1} \cdots x_{mn} \end{bmatrix} \in M_{m \times n}^V$. Then

$$\begin{aligned} \begin{bmatrix} x_{11} \cdots x_{1n} \\ \vdots \\ x_{m1} \cdots x_{mn} \end{bmatrix} &= \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots \underline{0} \end{bmatrix} + \dots + \begin{bmatrix} \underline{0} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix} \\ &\in 1 \odot \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots \underline{0} \end{bmatrix} + \dots + 1 \odot \begin{bmatrix} \underline{0} \cdots \underline{0} \\ \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix}. \end{aligned}$$

Thus S spans $M_{m \times n}^V$. Now we show that S is linearly independent. For this, let

$$\begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} \in a_{11} \circ \begin{bmatrix} x_{11} \cdots \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} + a_{12} \circ \begin{bmatrix} \underline{0} x_{12} \cdots \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} + \cdots + a_{mn} \circ \begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} \cdots x_{mn} \end{bmatrix},$$

for some $a_{11}, a_{12}, \dots, a_{mn} \in K$. Then

$$\begin{aligned} \begin{bmatrix} \underline{0} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots \\ \underline{0} & \cdots & \underline{0} \end{bmatrix} &= \left\{ \begin{bmatrix} x_{11}^{11} \cdots x_{1n}^{11} \\ \vdots & \vdots & \vdots \\ x_{m1}^{11} \cdots x_{mn}^{11} \end{bmatrix} : x_{11}^{11} \in a_{11} \circ x_{11}, x_{ij}^{11} \in a_{11} \circ \underline{0}, ij \neq 11 \right\} + \cdots + \\ &+ \left\{ \begin{bmatrix} x_{11}^{mn} \cdots x_{1n}^{mn} \\ \vdots & \vdots & \vdots \\ x_{m1}^{mn} \cdots x_{mn}^{mn} \end{bmatrix} : x_{mn}^{mn} \in a_{mn} \circ x_{mn}, x_{ij}^{mn} \in a_{mn} \circ \underline{0}, ij \neq mn \right\} \\ &= \left\{ \begin{bmatrix} \sum_{i=1}^{mn} x_{11}^i \cdots \sum_{i=1}^{mn} x_{1n}^i \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{mn} x_{m1}^i \cdots \sum_{i=1}^{mn} x_{mn}^i \end{bmatrix} : x_{ij}^{rs} \in \begin{cases} a_{ij} \circ x_{ij} & rs = ij \\ a_{ij} \circ \underline{0} & rs \neq ij \end{cases} \right\}. \end{aligned}$$

Thus $\underline{0} \in \sum_{i=1}^{mn} x_{11}^i, \dots, \underline{0} \in \sum_{i=1}^{mn} x_{1n}^i, \dots, \underline{0} \in \sum_{i=1}^{mn} x_{m1}^i, \dots, \underline{0} \in \sum_{i=1}^{mn} x_{mn}^i$. Hence

$$\begin{cases} \underline{0} \in a_{11} \circ x_{11} + \sum_{i=1}^{mn} a_i \circ \underline{0} \implies \underline{0} \in a_{11} \circ x_{11} \\ \underline{0} \in a_{12} \circ x_{12} + \sum_{i=1}^{mn} a_i \circ \underline{0} \implies \underline{0} \in a_{12} \circ x_{12} \\ \vdots \\ \underline{0} \in \sum_{i=1}^{mn} a_i \circ \underline{0} + a_{mn} \circ x_{mn} \implies \underline{0} \in a_{mn} \circ x_{mn} \end{cases}$$

Consequently, $\underline{0} \in a_{11} \circ x_{11} + a_{12} \circ x_{12} + \cdots + a_{mn} \circ x_{mn}$ and so $a_1 = a_2 = \cdots = a_{mn} = 0$. Therefore S is linearly independent and forms a basis for $M_{m \times n}^V$. \square

Lemma 4.8. *Let V be strongly left distributive and $\{x, y, z\}$ be a linearly independent subset of V . Then $\{x + y, y + z, x + z\}$ is a linearly independent subset of V .*

Proof. Let $\underline{0} \in a_1 \circ (x + y) + a_2 \circ (y + z) + a_3 \circ (x + z)$, for some $a_1, a_2, a_3 \in K$. Then

$$\begin{aligned} \underline{0} &\in a_1 \circ x + a_1 \circ y + a_2 \circ y + a_2 \circ z + a_3 \circ x + a_3 \circ z \\ &= (a_1 + a_3) \circ x + (a_1 + a_2) \circ y + (a_2 + a_3) \circ z. \end{aligned}$$

Hence $a_1 + a_3 = 0, a_1 + a_2 = 0, a_2 + a_3 = 0$, and so $a_1 = a_2 = a_3 = 0$. \square

Proposition 4.9. *Let V be strongly left distributive and x, y be linearly independent vectors of V . If $a_1, a_2, b_1, b_2 \in K$, such that $b_1 \neq 0, b_2 \neq 0$ and $a_1 b_2 - b_1 a_2 \neq 0$, then for all $t_1 \in a_1 \circ x, t_2 \in a_2 \circ y, s_1 \in b_1 \circ x$ and $s_2 \in b_2 \circ y$, the vectors $t_1 + t_2$ and $s_1 + s_2$ are linearly independent in V .*

Proof. Let $\underline{0} \in c \circ (t_1 + t_2) + d \circ (s_1 + s_2)$, for some $c, d \in K$. Then $\underline{0} \in c \circ (a_1 \circ x + a_2 \circ y) + d \circ (b_1 \circ x + b_2 \circ y) \subseteq c \circ (a_1 \circ x) + c \circ (a_2 \circ y) + d \circ (b_1 \circ x) + d \circ (b_2 \circ y) = (ca_1) \circ x + (ca_2) \circ y + (db_1) \circ x + (db_2) \circ y = (ca_1 + db_1) \circ x + (ca_2 + db_2) \circ y$. Thus $ca_1 + db_1 = 0, ca_2 + db_2 = 0$. Hence $c = d = 0$. Therefore $t_1 + t_2$ and $s_1 + s_2$ are linearly independent. \square

Proposition 4.10. *If V is strongly left distributive, then any subset of V containing zero is linearly dependent.*

Proof. Let $H = \{\underline{0}, x_1, x_2, \dots, x_n\} \subseteq V$. Then $\underline{0} \in 1 \circ \underline{0} + 0 \circ x_1 + 0 \circ x_2 + \dots + 0 \circ x_n$. Thus H is linearly dependent. \square

Theorem 4.11. *Let V be anti-left distributive. Then $\beta = \{x_1, \dots, x_n\}$ is a basis for V if and only if every element $x \in V$ belongs to a unique sum in the form $c_1 \circ x_1 + \dots + c_n \circ x_n$, with $c_i \in K$.*

Proof. (\implies) [2, Lemma 3.4.]

(\impliedby) By hypothesis β spans V . Now let $\underline{0} \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. By Lemma 2.5, $0 \circ x_i \leq V, 1 \leq i \leq n$. So $\underline{0} \in 0 \circ x_i, 1 \leq i \leq n$. Hence $\underline{0} \in 0 \circ x_1 + \dots + 0 \circ x_n$. From uniqueness it follows that $a_i = 0, 1 \leq i \leq n$. Therefore β is linearly independent and consequently it is a basis for V . \square

Proposition 4.12. *If $(V, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space over the field \mathbb{R} and $\{x, y, z\}$ is a basis for V , then the set $\{x + y, y + z, x + z\}$ is another basis for V .*

Proof. By Lemma 4.8, the set $\{x + y, y + z, x + z\}$ is linearly independent. Now let $w \in V$. Then there exist $a, b, c \in \mathbb{R}$, such that $w \in a \circ x + b \circ y + c \circ z$. Suppose

$$a' = \frac{a - b + c}{2}, b' = \frac{c - a + b}{2}, c' = \frac{b - c + a}{2}.$$

Then it is easy to verify that $w \in a' \circ (x + y) + b' \circ (y + z) + c' \circ (x + z)$. Therefore $\{x + y, y + z, x + z\}$ spans V and so it is a basis for V . \square

Theorem 4.13. *Let V be invertible and H be a subhyperspace of V with basis β . Then $\beta \cup \{y\}$ is linearly independent, for all $y \in V \setminus H$, such that $0 \circ y = \{\underline{0}\}$.*

Proof. Let $\beta = \{x_1, \dots, x_n\}$ and $\beta \cup \{y\}$ be linearly dependent. Then $\underline{0} \in a'_1 \circ x_1 + \dots + a'_n \circ x_n + b \circ y$, for some $a'_1, \dots, a'_n, b \in K$, such that at least one of the coefficients is nonzero. Thus $\underline{0} = t_1 + \dots + t_n + c$, for some $t_i \in a_i \circ x_i, c \in b \circ y$. Now if $b \neq 0$, then $y \in b^{-1} \circ c$. Hence $y \in b^{-1} \circ (-t_1 - \dots - t_n) \subseteq b^{-1} \circ (-t_1) + \dots + b^{-1} \circ (-t_n) \subseteq b^{-1} \circ (-a'_1 \circ x_1) + \dots + b^{-1} \circ (-a'_n \circ x_n) = (-b^{-1}a'_1) \circ x_1 + \dots + (-b^{-1}a'_n) \circ x_n \subseteq H$. Which is contradiction. Also if $b = 0$, then $\underline{0} \in a'_1 \circ x_1 + \dots + a'_n \circ x_n + 0 \circ y$, such that at least one of a'_i 's is nonzero. Thus $\underline{0} \in a'_1 \circ x_1 + \dots + a'_n \circ x_n$, Which is a contradiction, too. Therefore $\beta \cup \{y\}$ is linearly independent. \square

Theorem 4.14 ([2]). *Let V be strongly left distributive and invertible. If V has a finite basis with n elements, then every linearly independent subset of V has no more than n elements.*

Theorem 4.15. *Let V be strongly left distributive and invertible such that $\dim V = n$ and $0 \circ y = \{\underline{0}\}$, for all $y \in V$. Then any linearly independent subset S of V with n vectors is a basis for V .*

Proof. Let $S = \{x_1, \dots, x_n\}$ and $H = SP(S)$. If $H \neq V$, then there exists $y \in V \setminus H$. Thus by Theorem 4.13, $\{x_1, \dots, x_n, y\}$ is linearly independent with $n + 1$ elements, which is in contradiction with the Theorem 4.14. Therefore $H = V$ and so $\{x_1, \dots, x_n\}$ is a basis for V . \square

Theorem 4.16. *Let X be a finite spanning set for V . Then $X \cup \{y\}$ is linearly dependent, for any $y \in V \setminus X$.*

Proof. Let $X = \{x_1, \dots, x_n\}$ and $y \in V \setminus X$. Then $y \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. Thus $y = t_1 + \dots + t_n$, for some $t_i \in a_i \circ x_i$, $1 \leq i \leq n$. So $\underline{0} = y - t_1 - \dots - t_n \in 1 \circ y - a_1 \circ x_1 - \dots - a_n \circ x_n$. Therefore $\{x_1, \dots, x_n, y\}$ is linearly dependent. \square

Proposition 4.17. *Let V be invertible and $0 \circ y = \{\underline{0}\}$, for all $y \in V$. If the set $\{x_1, \dots, x_n\}$ is linearly independent in V , such that $\{x_1, \dots, x_n, x_{n+1}, \dots, x_m\}$ is linearly dependent, for any $m > n$, then $\{x_1, \dots, x_n\}$ is a basis for V .*

Proof. Let $H = SP(x_1, \dots, x_n)$ and $V \neq H$. Then there exists $y \in V \setminus H$ such that by Theorem 4.13, the set $\{x_1, \dots, x_n, y\}$ is linearly independent, which is a contradiction. Therefore $V = H$. \square

Proposition 4.18. *Let W_1 and W_2 be strongly left distributive and invertible subspaces of V such that $W_1 \subseteq W_2$ and $\dim W_1 = \dim W_2$. Then $W_1 = W_2$.*

Proof. Let $\{x_1, \dots, x_n\}$ be a basis for W_1 and $W_1 \neq W_2$. Then there exists $y \in W_2 \setminus W_1$. Thus by Theorem 4.13, $\{x_1, \dots, x_n, y\}$ is a linearly independent subset of W_2 with $n + 1$ vectors, which is in contradiction with the Theorem 4.14. Therefore $W_1 = W_2$. \square

Proposition 4.19. *Let V be strongly left distributive and invertible and $\{x_1, \dots, x_n\}$ be a linearly independent subset of V . If $x \in V$ such that $0 \circ x = \{\underline{0}\}$ and $x \notin SP(x_1, \dots, x_n)$, then $\{x_1 + x, \dots, x_n + x\}$ is linearly independent in V .*

Proof. Let $\underline{0} \in a_1 \circ (x_1 + x) + \dots + a_n \circ (x_n + x)$, for some $a_1, \dots, a_n \in K$. Then $\underline{0} \in a_1 \circ x_1 + \dots + a_n \circ x_n + (a_1 + \dots + a_n) \circ x$. So $\underline{0} = t_1 + \dots + t_n + b$, for some $t_i \in a_i \circ x_i$, $1 \leq i \leq n$, $b \in (a_1 + \dots + a_n) \circ x$. Now if $a_1 + \dots + a_n \neq 0$, then $x \in (a_1 + \dots + a_n)^{-1} \circ b = (a_1 + \dots + a_n)^{-1} \circ (-t_1 - \dots - t_n) \subseteq (a_1 + \dots + a_n)^{-1} \circ (-a_1 \circ x_1 - \dots - a_n \circ x_n) \subseteq (-a_1(a_1 + \dots + a_n)^{-1}) \circ x_1 + \dots + (-a_n(a_1 + \dots + a_n)^{-1}) \circ x_n \subseteq SP(x_1, \dots, x_n)$. Which is a contradiction. Also

if $a_1 + \cdots + a_n = 0$, and $a_j \neq 0$ for some $1 \leq j \leq n$, then $0 \in a_1 \circ x_1 + \cdots + a_j \circ x_j + \cdots + a_n \circ x_n + 0 \circ x = a_1 \circ x_1 + \cdots + a_j \circ x_j + \cdots + a_n \circ x_n$. Which is a contradiction, too. Thus $a_1 + \cdots + a_n = 0$ and $a_i = 0$ for all $1 \leq i \leq n$, which it means that $\{x_1 + x, \dots, x_n + x\}$ is linearly independent in V . \square

Definition 4.20. Let V be anti-left distributive and finite dimensional with order basis $\beta = \{x_1, \dots, x_n\}$. (The basis β is called ordered basis, if the order of it's vectors is important). Then by Theorem 4.11, every vector $x \in V$ belongs to a unique sum in the form $a_1 \circ x_1 + \cdots + a_n \circ x_n$, with $a_i \in K$. The scalars a_1, \dots, a_n are called the coordinates of x relative to the basis β . The coordinate matrix (or coordinate vector) of x relative to β is the column matrix in K^n whose components are the coordinates of x , i.e.

$$[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

It is clear that the coordinate matrix of x relative to β is unique.

Theorem 4.21. Let V be strongly left distributive and finite dimensional. Let $\beta = \{x_1, \dots, x_n\}$ and $\beta' = \{x'_1, \dots, x'_n\}$ be two ordered basis for V , such that

$$\forall 1 \leq i \leq n, [x'_i]_\beta = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix}.$$

Suppose $P = [a_{ij}]_{n \times n}$. Then $[x]_\beta = P[x]_{\beta'}$, for all $x \in V$.

Proof. Let $x \in V$ and

$$[x]_{\beta'} = \begin{bmatrix} b'_1 \\ \vdots \\ b'_n \end{bmatrix}.$$

Then

$$x \in \sum_{i=1}^n b'_i \circ x'_i \subseteq \sum_{i=1}^n b'_i \circ \left(\sum_{j=1}^n a_{ij} \circ x_j \right) \subseteq \sum_{i=1}^n \sum_{j=1}^n a_{ij} b'_i \circ x_j.$$

Thus

$$[x]_\beta = \begin{bmatrix} \sum_{j=1}^n a_{1j} b'_j \\ \vdots \\ \sum_{j=1}^n a_{nj} b'_j \end{bmatrix} = P \begin{bmatrix} b'_1 \\ \vdots \\ b'_n \end{bmatrix} = P[x]_{\beta'}.$$

\square

Remark 4.22. The matrix P in Theorem 4.21, is called the transitive matrix from basis β' to basis β .

5. Linear transformations between hypervector spaces

In this section we investigate the concept of linear transformation between two hypervector spaces. Let V and W be hypervector spaces over the field K . Then a function $T : V \longrightarrow W$ is called

1. linear transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) \subseteq a \circ T(x), \forall x, y \in V, a \in K$,
2. good linear transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) = a \circ T(x), \forall x, y \in V, a \in K$.

The kernel of linear transformation $T : V \longrightarrow W$ is denoted by $\ker T$ and defined by $\ker T = \{x \in V : T(x) \in \Omega_V\}$.

Proposition 5.1. *Let $T : V \longrightarrow W$ be a good linear transformation. Then*

$$T\left(\sum_{i=1}^n a_i \circ x_i\right) = \sum_{i=1}^n a_i \circ T(x_i).$$

Proof. Straightforward. □

Proposition 5.2. *Let $(V^m, \oplus_m, \odot_m, K)$ and $(V^n, \oplus_n, \odot_n, K)$ be two strongly distributive hypervector spaces is defined in Proposition 3.4, such that $m < n$. Then $T : V^m \longrightarrow V^n$ with the following rule is a good linear transformation.*

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ \vdots \\ x_{m-1} + x_m \\ x_1 + x_2 \\ \vdots \\ x_1 + x_2 \end{bmatrix}$$

Proof. Straightforward. □

Example 5.3 ([2]). Let $(V, +, \cdot, K)$ be a classical vector space, P be a subspace of V and the external hyperoperation $\circ : K \times V \longrightarrow P_*(V)$ is defined by $a \circ x = a \cdot x + P$, for all $a \in K, x \in V$. Then $(V, +, \circ, K)$ is a strongly distributive hypervector space.

Example 5.4. Let $(\mathbb{R}^2, +, \cdot, \mathbb{R})$ be the classical vector space and $(\mathbb{R}^2, +, \circ, \mathbb{R})$ be the strongly distributive hypervector space is constructed in Example 5.3, with $P = \mathbb{R} \times \{0\}$. Then

$$\left[\begin{array}{l} T : \left((\mathbb{R}^2)^2, \oplus_2, \odot_2, \mathbb{R} \right) \longrightarrow \left((\mathbb{R}^2)^3, \oplus_3, \odot_3, \mathbb{R} \right) \\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \\ x + y \end{bmatrix} \end{array} \right]$$

is a linear transformation, where $((\mathbb{R}^2)^2, \oplus_2, \odot_2, \mathbb{R})$ and $((\mathbb{R}^2)^3, \oplus_3, \odot_3, \mathbb{R})$ are the hypervector spaces defined in Proposition 3.4.

Proposition 5.5. *Let $T : V \longrightarrow W$ be a linear transformation and $x_1, \dots, x_n \in V$ such that $T(x_1), \dots, T(x_n)$ be linearly independent in W . Then x_1, \dots, x_n are linearly independent in V .*

Proof. Let $\underline{0} \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_i \in K$. Then $\underline{0} = T(\underline{0}) \in T(a_1 \circ x_1 + \dots + a_n \circ x_n) \subseteq a_1 \circ T(x_1) + \dots + a_n \circ T(x_n)$. Thus $a_1 = \dots = a_n = 0$, and so x_1, \dots, x_n are linearly independent. \square

Theorem 5.6. *Let W be a hypervector space such that $\Omega_W = \{\underline{0}_W\}$ and $T : V \longrightarrow W$ be a linear transformation. Then T is injective if and only if $\ker T = \{\underline{0}_V\}$.*

Proof. Let T be injective and $x \in \ker T$. Then $T(x) \in \Omega_W = \{\underline{0}\}$. Thus $T(x) = \underline{0} = T(\underline{0})$, and so $x = \underline{0}$. Hence $\ker T \subseteq \{\underline{0}\}$. On the other hand, $\underline{0}_V \in \ker T$, because $T(\underline{0}_V) = \underline{0}_W \in \Omega_W$. Therefore $\ker T = \{\underline{0}_V\}$. Conversely, let $\ker T = \{\underline{0}\}$ and $x_1, x_2 \in V$, such that $T(x_1) = T(x_2)$. Then $T(x_1 - x_2) = T(x_1) - T(x_2) = \underline{0} \in \Omega_w$. Thus $x_1 - x_2 \in \ker T$, and so $x_1 = x_2$. Consequently T is injective. \square

Theorem 5.7 ([2]). *Let V be strongly left distributive, invertible and finite dimensional. If W is a subhyperspace of V , then $\dim W \leq \dim V$ and $\dim V/W = \dim V - \dim W$, where the external hyperoperation $\star : K \times V/W \longrightarrow P_*(V/W)$ is defined by $a \star (v + W) = a \circ v + W$.*

Theorem 5.8 ([2]). *Let V and W be strongly left distributive hypervector spaces over the field K , and $T : V \longrightarrow W$ be a linear transformation. Then*

$$\frac{V}{\ker T} \cong \frac{T(V)}{\Omega_W}.$$

Proposition 5.9 ([10]). (i) *If V is strongly left distributive, then $\dim \Omega_V = 0$ and $SP(\emptyset) = \Omega_V$.*

(ii) *If $W \leq V$ such that $\dim W = 0$, then $SP(\emptyset) = W$.*

Theorem 5.10. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces. If $T : V \longrightarrow W$ is a linear transformation, then*

$$\dim \ker T + \dim T(V) = \dim V.$$

Proof. By Theorem 5.8, $\frac{V}{\ker T} \cong \frac{T(V)}{\Omega}$. Thus $\dim \frac{V}{\ker T} = \dim \frac{T(V)}{\Omega}$. Hence by Theorem 5.7, $\dim V - \dim \ker T = \dim T(V) - \dim \Omega$. Then by Proposition 5.9, $\dim V - \dim \ker T = \dim T(V)$. Therefore $\dim \ker T + \dim T(V) = \dim V$. \square

Corollary 5.11. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces such that $\dim W < \dim V$. If $T : V \longrightarrow W$ is a linear transformation, then $\dim \ker T > 0$.*

Proof. It is clear that $T(V) \leq W$, so by Theorem 5.7, $\dim T(V) \leq \dim W$. Hence $\dim T(V) < \dim V$. Thus by Theorem 5.10, it follows that $\dim \ker T = \dim V - \dim T(V) > 0$. \square

Corollary 5.12. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces, such that $\dim V = \dim W$, $\Omega_V = \{\underline{0}_V\}$ and $\Omega_W = \{\underline{0}_W\}$. If $T : V \longrightarrow W$ is a linear transformation, then T is injective if and only if T is surjective.*

Proof. Let T be surjective. Then $T(V) = W$ and so by Theorem 5.10, $\dim \ker T = \dim V - \dim T(V) = \dim V - \dim W = 0$. Thus by Proposition 5.9, $\ker T = SP(\emptyset) = \Omega_V = \{\underline{0}_V\}$. Hence by Theorem 5.6, T is injective. Conversely, if T is injective, then by Theorem 5.6, $\ker T = \{\underline{0}_V\}$. Thus by Proposition 5.9, $\ker T = \Omega_V = SP(\emptyset)$ and so $\dim \ker T = 0$. Thus by Theorem 5.10, $\dim T(V) = \dim V = \dim W$. Hence by Proposition 4.18, $T(V) = W$. \square

Corollary 5.13. *Let V and W be strongly left distributive, invertible and finite dimensional hypervector spaces, such that $\dim V = \dim W$, $\Omega_V = \{\underline{0}_V\}$ and $\Omega_W = \{\underline{0}_W\}$. If $T : V \longrightarrow W$ is a linear transformation, then the followings are equivalent:*

1. T is isomorphism;
2. T is injective;
3. T is surjective.

Theorem 5.14. *Let V and W be hypervector spaces and $T : V \longrightarrow W$ be an isomorphism (i.e. T is a good linear transformation, which is injective and surjective). Then the followings hold:*

1. If $\{x_1, \dots, x_n\}$ is linearly independent in V , then $\{T(x_1), \dots, T(x_n)\}$ is linearly independent in W .
2. If $\{x_1, \dots, x_n\}$ spans V , then $\{T(x_1), \dots, T(x_n)\}$ spans W .
3. If $\{x_1, \dots, x_n\}$ is a basis for V , then $\{T(x_1), \dots, T(x_n)\}$ is a basis for W .
4. $\dim V = \dim W$.

Proof. 1. Let $\underline{0} \in a_1 \circ T(x_1) + \dots + a_n \circ T(x_n)$, for some $a_i \in K$. Then $\underline{0} = T(\underline{0}) \in T(a_1 \circ x_1 + \dots + a_n \circ x_n)$ and so $\underline{0} \in a_1 \circ x_1 + \dots + a_n \circ x_n$. Hence $a_i = 0$, $1 \leq i \leq n$. Therefore $\{T(x_1), \dots, T(x_n)\}$ is linearly independent.

2. Let $y \in W$. Then $y = T(x)$, for some $x \in V$. Thus $x \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_i \in K$. Hence $y = T(x) \in T(a_1 \circ x_1 + \dots + a_n \circ x_n) = a_1 \circ T(x_1) + \dots + a_n \circ T(x_n)$. Therefore $\{T(x_1), \dots, T(x_n)\}$ spans W .

3. It is obtained from (1) and (2).

4. It is obtained from (3). \square

Theorem 5.15. *Let V and W be finite dimensional hypervector spaces with ordered bases $\beta = \{x_1, \dots, x_n\}$ and $\beta' = \{y_1, \dots, y_m\}$, respectively. If $T : V \rightarrow W$ is a linear transformation such that*

$$\forall 1 \leq j \leq n, [T(x_j)]_{\beta'} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Then the $m \times n$ matrix $A_T = [a_{ij}]$ is such that $\forall x \in V$, $[T(x)]_{\beta'} = A_T [x]_{\beta}$.

Proof. Proof is similar to the proof of Theorem 4.21. \square

Remark 5.16. The matrix A_T in Theorem 5.15, is called the matrix of T relative to the bases β and β' .

Proposition 5.17 ([7]). *Let V and W be hypervector spaces over the field \mathbb{R} . Assume that $L(V, W)$ denotes the set of all good linear transformations from V to W . For every $T, S \in L(V, W)$, $a \in \mathbb{R}$, and $x \in V$, suppose that:*

- i $(T + S)(x) = T(x) + S(x)$,
- ii $a \odot T = \{T' \in L(V, W) : T'(x) \in T(a \circ x), \text{ for every } x \in V\}$.

Then $(L(V, W), +, \odot, \mathbb{R})$ is a hypervector space over the field \mathbb{R} . If V and W are strongly left distributive, then $L(V, W)$ is strongly left distributive, too.

Theorem 5.18. *Let V and W be hypervector spaces over the field \mathbb{R} , with bases $\beta = \{x_1, \dots, x_n\}$ and $\beta' = \{y_1, \dots, y_m\}$, respectively. Let W be strongly left distributive, $a \circ \underline{0} = \{\underline{0}_W\}$, for all $a \in \mathbb{R}$ and $0 \circ y = \{\underline{0}_W\}$, for all $y \in W$. Then the mapping*

$$\begin{bmatrix} G : L(V, W) \longrightarrow M_{m \times n} \\ T \longmapsto A_T \end{bmatrix}$$

is an injective and good linear transformation, where $M_{m \times n} = (M_{m \times n}, +, \cdot, \mathbb{R})$ is the ordinary vector space of matrices and A_T is the matrix of T relative to the bases β and β' .

Proof. Let $T, S \in L(V, W)$ and

$$\forall 1 \leq j \leq n, [T(x_j)]_{\beta'} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, [S(x_j)]_{\beta'} = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix}.$$

Then $G(T) = A_T = [a_{ij}]_{mn}$ and $G(S) = A_S = [b_{ij}]_{mn}$. Also,

$$(T + S)(x_j) = T(x_j) + S(x_j) \in \sum_{i=1}^m a_{ij} \circ y_i + \sum_{i=1}^m b_{ij} \circ y_i = \sum_{i=1}^m (a_{ij} + b_{ij}) \circ y_i.$$

Hence

$$G(T+S) = A_{T+S} = [a_{ij} + b_{ij}]_{mn} = [a_{ij}]_{mn} + [b_{ij}]_{mn} = A_T + A_S = G(T) + G(S).$$

Therefore $G(T+S) = G(T) + G(S)$. Now if $a \in K$ and $T \in L(V, W)$, then

$$\begin{aligned} G(a \odot T) &= \{G(S) : S \in a \odot T\} = \{G(S) : \forall x \in V, S(x) \in a \odot T(x)\} \\ &= \{A_S : \forall x \in V, S(x) \in a \odot T(x)\}, \end{aligned}$$

such that for all $A_S \in G(a \odot T)$ and $1 \leq j \leq n$,

$$\begin{aligned} S(x_j) &\in a \odot T(x_j) \subseteq a \odot (a_{1j} \odot y_1 + \cdots + a_{mj} \odot y_m) \\ &\subseteq aa_{1j} \odot y_1 + \cdots + aa_{mj} \odot y_m, \end{aligned}$$

so

$$[S(x_j)]_{\beta} = \begin{bmatrix} aa_{1j} \\ \vdots \\ aa_{mj} \end{bmatrix}.$$

Thus $A_S = [aa_{ij}]_{mn} = a \cdot [a_{ij}]_{mn} = a \cdot A_T = a \cdot G(T)$. Therefore $G(a \odot T) = a \cdot G(T)$. Consequently G is a good linear transformation. Now let T be a good linear transformation such that $G(T) = A_T = [0]_{mn}$. Then $T(x_j) \in 0 \odot y_1 + \cdots + 0 \odot y_n = \{0\}$, for all $1 \leq j \leq n$. Thus $T = 0$, because $a \odot 0 = \{0_W\}$, for all $a \in K$. Therefore G is injective. \square

Corollary 5.19. *Let V and W be hypervector spaces over the field \mathbb{R} , with bases $\beta = \{x_1, \dots, x_n\}$ and $\beta = \{y_1, \dots, y_m\}$, respectively. Let W be strongly left distributive, $a \odot 0 = \{0_W\}$, for all $a \in \mathbb{R}$ and $0 \odot y = \{0_W\}$, for all $y \in W$. Then $L(V, W) \cong M_{m \times n}$.*

Proof. Let G be the good linear transformation defined in Theorem 5.18. Then By Theorem 5.8,

$$\frac{L(V, W)}{\ker G} \cong \frac{M_{m \times n}}{\Omega_{M_{m \times n}}}.$$

It is easy to verify that $\Omega_{M_{m \times n}} = \{0_{M_{m \times n}}\}$. Also by Theorem 5.6, $\ker G = \{0\}$. Therefore $L(V, W) \cong M_{m \times n}$. \square

Corollary 5.20. *Let V and W be hypervector spaces over the field \mathbb{R} , with bases $\beta = \{x_1, \dots, x_n\}$ and $\beta = \{y_1, \dots, y_m\}$, respectively. Let W be strongly left distributive, $a \odot 0 = \{0_W\}$, for all $a \in \mathbb{R}$ and $0 \odot y = \{0_W\}$, for all $y \in W$. Then $\dim L(V, W) = \dim V \times \dim W$.*

Proof. By Corollary 5.19, $L(V, W) \cong M_{m \times n}$. Thus by Theorem 5.14,

$$\dim L(V, W) = \dim M_{m \times n} = m \times n = \dim V \times \dim W.$$

\square

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