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ON GENERALIZED SHIFT TRANSFORMATION SEMIGROUPS

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Abstract. In the following text we prove that for finite discrete X with at least two elements and infinite Γ, the generalized shift transformation semigroup $(\mathcal{S},X^\Gamma)$ is equicontinuous (resp. has at least an equicontinuous point, is not sensitive) if and only if for all $w\in\Gamma$, $\{\varphi(w):\sigma_\varphi\in\mathcal{S}\}$ is finite. We continue our study regarding distality and expansivity of $(\mathcal{S},X^\Gamma)$.

1. Introduction

The concept of generalized shifts has been introduced for the first time in [2] as a generalization of one-sided shift $\underset{(a_1,a_2,\cdots)\mapsto(a_2,a_3,\cdots)}{\{1,\ldots,k\}^{\mathbb{N}}\to\{1,\ldots,k\}^{\mathbb{N}}}$ and two-sided shift $\underset{(a_n)_{n\in\mathbb{Z}}\mapsto(a_{n+1})_{n\in\mathbb{Z}}}{\{1,\ldots,k\}^{\mathbb{Z}}\to\{1,\ldots,k\}^{\mathbb{Z}}}$, which are well-known in dynamical systems' approach and ergodic theory [9]. Suppose K is a nonempty set with at least two elements, Γ is a nonempty set, and $\varphi:\Gamma\to\Gamma$ is an arbitrary map, then $\sigma_\varphi:\underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_{\varphi(\alpha)})_{\alpha\in\Gamma}}{K^\Gamma\to K^\Gamma}$ denotes a generalized shift. It's evident that for topological space K, $\sigma_\varphi:K^\Gamma\to K^\Gamma$ is continuous, where K^Γ is equipped with product topology, moreover if K has a group structure, then $\sigma_\varphi:K^\Gamma\to K^\Gamma$ is a group homomorphism. Dynamical (and non-dynamical) properties of generalized shifts has been studied in several papers, like [3] and [8]. In the following text we study equicontinuity and distality in $(\mathcal{S},X^\Gamma)$, where X is a finite discrete space and $\mathcal{S}$ is a semigroup of generalized shifts on X^Γ.

2. Priliminaries

Background on uniform spaces. Let's recall that for arbitrary set Y we say $\mathcal{K}$ is a *uniformity* on Y if $\mathcal{K}$ is a collection of subsets of $Y\times Y$ such that:

- $\forall\alpha\in\mathcal{K}\quad(\Delta_Y\subseteq\alpha)$;
- $\forall\alpha,\beta\in\mathcal{K}\quad(\alpha\cap\beta\in\mathcal{K})$;
- $\forall\alpha\in\mathcal{K}\quad\forall\beta\subseteq Y\times Y\quad(\alpha\subseteq\beta\Rightarrow\beta\in\mathcal{K})$;
- $\forall\alpha\in\mathcal{K}\quad\exists\beta\in\mathcal{K}\quad(\beta\circ\beta^{-1}\subseteq\alpha)$;

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where $\Delta_Y = \{(y,y) : y \in Y\}$ and for $\alpha, \beta \subseteq Y \times Y$ we have $\alpha^{-1} = \{(y,x) : (x,y) \in \alpha\}$, $\alpha \circ \beta = \{(x,y) : \exists z \, ((x,z) \in \alpha \wedge (z,y) \in \beta)\}$. If $\mathcal{K}$ is a uniformity on Y we call $(Y, \mathcal{K})$ a *uniform space*, also for all $\alpha \in \mathcal{K}$ and $x \in Y$, let $\alpha[x] := \{y : (x,y) \in \alpha\}$; so $\tau := \{U \subseteq Y : \exists \alpha \in \mathcal{K} \, (\alpha[x] \subseteq U)\}$ is a topology on Y, we call it the *uniform topology induced by* $\mathcal{K}$ and equip $(Y, \mathcal{K})$ by τ. For topological space Z we say Z is *uniformzable* and uniformity $\mathcal{H}$ is a *compatible uniformity* on Z, if the uniform topology induced by $\mathcal{H}$ coincides with original topology of Z. Every compact Hausdorff topological space is uniformzable and has a unique compatible uniformity. For more details on uniform structures we refer the interested reader to [6].
If A is a collection of maps from uniform space $(Y, \mathcal{K}_Y)$ to uniform space $(Z, \mathcal{K}_Z)$, we say A is *equicontinuous* if for all $\alpha \in \mathcal{K}_Z$, there exists $\beta \in \mathcal{K}_Y$ with $A\beta := \{(f(x), f(y)) : f \in A, (x,y) \in \beta\} \subseteq \alpha$.

Background on transformation semigroups. By a *(topological) transformation semigroup* (S, Y, π) or simply (S, Y) we mean a compact Hausdorff topological space Y (*phase space*), discrete topological semigroup S (*phase semigroup*) with identity e and continuous map $\pi : S \times Y \to Y$ ($\pi(s,x) = sx$ for $s \in S$ and $x \in Y$) such that for all $x \in Y$ and $s, t \in S$ we have $ex = x$ and $(st)x = s(tx)$. Consider the transformation semigroup (S, Y) with unique compatible uniformity $\mathcal{H}$ on Y, we say:

- (S, Y) is *equicontinuous* if for all $\alpha \in \mathcal{H}$ there exists $\beta \in \mathcal{H}$ with $S\beta := \{(sx, sy) : s \in S, (x,y) \in \beta\} \subseteq \alpha$;
- $x \in Y$ is an *equicontinuous point of* (S, Y) if for all $\alpha \in \mathcal{H}$ there exists open neighbourhood U of x with $\{(sx, sy) : s \in S, , y \in U\} \subseteq \alpha$;
- (S, Y) is *expansive* if there exists $\alpha \in \mathcal{H}$ such that for all distinct $x, y \in Y$ there exists $s \in S$ with $(sx, sy) \notin \alpha$;
- (S, Y) is *sensitive* if there exists $\alpha \in \mathcal{H}$ such that for all $x \in Y$ and open neighbourhood U of x there exists $s \in S$ and $y \in U$ with $(sx, sy) \notin \alpha$. So if (S, Y) is sensitive, then it has a non-equicontinuous point and it is not equicontinuous.

Moreover in the transformation semigroup (S, Y) for all $s \in S$ the map $\pi^s : Y \to Y$ with $\pi^s(y) = sy$ (for $y \in Y$) is continuous, sometimes we denote π^s simply by s. We call the closure of $\{\pi^s : s \in S\}$ in Y^Y with pointwise convergence (or product) topology, the *enveloping semigroup* and denote it by $E(S, Y)$ or simply by $E(Y)$. If (S, Y) is an equicontinuous transformation semigroup, then $E(S, Y)$ is an equicontinuous family on Y [7] and [4], in particular all elements of $E(S, Y)$ are continuous maps on Y. The enveloping semigroup has a semigroup operation under the composition of maps [7]. In the transformation semigroup (S, Y) we say:

- (S, Y) is *distal* if it satisfies each of the following equivalent conditions (hence, in distal transformation semigroup (S, Y) all elements of $E(S, Y)$ (and in particular S) are bijections from Y to Y) [7]:
  - $E(S, Y)$ is a group;
  - for all $x, y, z \in Y$ and net $\{s_\lambda\}_{\lambda \in \Lambda}$ in S with $\lim\limits_{\lambda \in \Lambda} s_\lambda x = z = \lim\limits_{\lambda \in \Lambda} s_\lambda y$ we have $x = y$;
- (S, Y) is *weakly almost periodic*, if all elements of $E(S, Y)$ are continuous maps on Y (so, it's evident that all equicontinuous transformation semigroups are weakly almost periodic) [5].

Convention 2.1. *In the following text suppose X is a finite discrete space with at least two elements, Γ is an infinite set, $\mathcal{S}$ is a semigroup of generalized shifts on X^Γ (equip X^Γ with product topology) containing the identity map on X^Γ, $\mathcal{T} = \{\varphi \in \Gamma^\Gamma : \sigma_\varphi \in \mathcal{S}\}$. Thus we may consider the transformation semigroup $(\mathcal{S}, X^\Gamma)$ where the elements of $\mathcal{S}$ acts in a natural way on X^Γ. For $H \subseteq \Gamma$, let:*

$$\alpha_H := \{((x_w)_{w\in\Gamma}, (y_w)_{w\in\Gamma}) \in X^\Gamma \times X^\Gamma : \forall w \in H \quad (x_w = y_w)\} .$$

Then

$$\mathcal{F} := \{\alpha \subseteq X^\Gamma \times X^\Gamma : \text{ there exists a finite subset } H \text{ of } \Gamma \text{ with } \alpha_H \subseteq \alpha\}$$

is the unique compatible uniformity on X^Γ.
On the other hand one may verify that for all $\varphi, \eta : \Gamma \to \Gamma$, we have

- $\sigma_\varphi \circ \sigma_\eta = \sigma_{\eta\circ\varphi}$;
- $\sigma_\varphi = \sigma_\eta$ *if and only if* $\varphi = \eta$;
- $\sigma_{\mathrm{id}_\Gamma} = \mathrm{id}_{X^\Gamma}$ *(where by $\mathrm{id}_Z : Z \to Z$ we mean $\mathrm{id}_Z(w) = w (w \in Z)$).*

Therefore, if $\mathcal{M}$ is a family of generalized shift on X^Γ, $\mathcal{M}$ is a semigroup (containing the identity map on X^Γ) if and only if $\{\varphi \in \Gamma^\Gamma : \sigma_\varphi \in \mathcal{M}\}$ is a semigroup (containing the identity map on Γ), in addition $\{\varphi \in \Gamma^\Gamma : \sigma_\varphi \in \mathcal{M}\} \underset{\eta\mapsto\sigma_\eta}{\to} \mathcal{M}$ is bijective. In particular, $\mathcal{T}$ is a semigroup (under the operation of composition of maps) containing id_Γ and $\mathcal{T} \underset{\eta\mapsto\sigma_\eta}{\to} \mathcal{S}$ is bijective.

Note 2.2. *The set $\{\sigma_\varphi : \varphi \in \Gamma^\Gamma\}$ is a closed subset of $C(X^\Gamma, X^\Gamma)$ (the collection of all continuous maps from X^Γ to X^Γ) with pointwise convergence (product) topology* [1], *so all continuous elements of $E(\mathcal{S}, X^\Gamma)$ has the form σ_φ for some $\varphi : \Gamma \to \Gamma$. In particular, $(\mathcal{S}, X^\Gamma)$ is weakly almost periodic if and only if $E(\mathcal{S}, X^\Gamma) \subseteq \{\sigma_\varphi : \varphi \in \Gamma^\Gamma\}$.*
We call $a \in A$ a quasi-periodic point (resp. periodic point) of $h : A \to A$, if there exist $n > m \geq 1$ such that $h^n(a) = h^m(a)$ (resp. $h^m(a) = a$). For $f : A \to B$ and $C \subseteq A$, by $f \restriction_C : C \to B$ ($f \restriction_C (x) = f(x) (x \in C)$) we mean the restriction of f to C.

3. When is $(\mathcal{S}, X^\Gamma)$ equicontinuous?

In this section we prove that $(\mathcal{S}, X^\Gamma)$ is equicontinuous if and only if for all $w \in \Gamma$, $\mathcal{T}w$ is finite.

Lemma 3.1. *If $(\mathcal{S}, X^\Gamma)$ has an equicontinuous point, then for all $w \in \Gamma$, $\mathcal{T}w$ is finite.*

Proof. Suppose $(q_w)_{w\in\Gamma}$ is an equicontinuous point of $(\mathcal{S}, X^\Gamma)$. Choose $v \in \Gamma$, then there exists open neighbourhood U of $(q_w)_{w\in\Gamma}$ with $\mathcal{S}\{(z, (q_w)_{w\in\Gamma}) : z \in U\} \subseteq \alpha_{\{v\}}$. There exists finite subset H of Γ with $\alpha_H[(q_w)_{w\in\Gamma}] \subseteq U$, hence for all $\varphi \in \mathcal{T}$ we have

$$\{(\sigma_\varphi((z_w)_{w\in\Gamma}), \sigma_\varphi((q_w)_{w\in\Gamma})) : (z_w)_{w\in\Gamma} \in \alpha_H[(q_w)_{w\in\Gamma}]\} \subseteq \alpha_{\{v\}} .$$

Choose distinct $p, q \in X$ and let:

$$x_w := \begin{cases} q_w & w \in H\,, \\ p & w \in \Gamma \setminus H\,, \end{cases} \qquad \text{and} \qquad y_w := \begin{cases} q_w & w \in H\,, \\ qp & w \in \Gamma \setminus H\,. \end{cases}$$

Then $(x_w)_{w\in\Gamma},(y_w)_{w\in\Gamma}\in\alpha_H[(q_w)_{w\in\Gamma}]$, therefore

$$(\sigma_\varphi((x_w)_{w\in\Gamma}),\sigma_\varphi((q_w)_{w\in\Gamma})),(\sigma_\varphi((y_w)_{w\in\Gamma}),\sigma_\varphi((q_w)_{w\in\Gamma}))\in\alpha_{\{v\}}\,.$$

Thus $x_{\varphi(v)}=q_{\varphi(v)}=y_{\varphi(v)}$ which leads to $\varphi(v)\in H$, φ is an arbitrary element of $\mathcal{T}$, we have $\mathcal{T}v\subseteq H$. Since H is finite, $\mathcal{T}v$ is finite too. $\square$

Lemma 3.2. *If for all $w\in\Gamma$, $\mathcal{T}w$ is finite, then $(\mathcal{S},X^\Gamma)$ is equicontinuous.*

Proof. Suppose for all $w\in\Gamma$, $\mathcal{T}w$ is finite and $\alpha\in\mathcal{F}$. There exist $\beta_1,\dots,\beta_m\in\Gamma$ with $\alpha_{\{\beta_1,\dots,\beta_m\}}\subseteq\alpha$. Thus $H:=\mathcal{T}\{\beta_1,\dots,\beta_m\}=\{\varphi(\beta_i):\varphi\in\mathcal{T},1\le i\le m\}$ is a finite subset of Γ and $\alpha_H\in\mathcal{F}$. We show $\mathcal{S}\alpha_H\subseteq\alpha$, for this aim suppose $((x_w)_{w\in\Gamma},(y_w)_{w\in\Gamma})\in\alpha_H$ and consider:

$$\begin{aligned}
&((x_w)_{w\in\Gamma},(y_w)_{w\in\Gamma})\in\alpha_H\\
&\qquad\Rightarrow\quad(\forall w\in H\quad(x_w=y_w))\\
&\qquad\Rightarrow\quad(\forall\varphi\in\mathcal{T}\quad\forall i\in\{1,\dots,m\}\quad(x_{\varphi(\beta_i)}=y_{\varphi(\beta_i)}))\\
&\qquad\Rightarrow\quad(\forall\varphi\in\mathcal{T}\quad(((x_{\varphi(w)})_{w\in\Gamma},(y_{\varphi(w)})_{w\in\Gamma})\in\alpha_{\{\beta_1,\dots,\beta_m\}})\\
&\qquad\Rightarrow\quad(\forall\varphi\in\mathcal{T}\quad((\sigma_\varphi((x_w)_{w\in\Gamma}),\sigma_\varphi((y_w)_{w\in\Gamma}))\in\alpha_{\{\beta_1,\dots,\beta_m\}})\\
&\qquad\Rightarrow\quad(\forall\sigma_\varphi\in\mathcal{S}\quad((\sigma_\varphi((x_w)_{w\in\Gamma}),\sigma_\varphi((y_w)_{w\in\Gamma}))\in\alpha_{\{\beta_1,\dots,\beta_m\}})\\
&\qquad\Rightarrow\quad(\forall\sigma_\varphi\in\mathcal{S}\quad((\sigma_\varphi((x_w)_{w\in\Gamma}),\sigma_\varphi((y_w)_{w\in\Gamma}))\in\alpha)
\end{aligned}$$

which leads us to the desired result. $\square$

Lemma 3.3. *If $(\mathcal{S},X^\Gamma)$ is weakly almost periodic, then for all $w\in\Gamma$, $\mathcal{T}w$ is finite.*

Proof. Suppose $(\mathcal{S},X^\Gamma)$ is weakly almost periodic and consider $w_0\in\Gamma$ such that $\mathcal{T}w_0$ is infinite, choose $\beta_1,\beta_2,\ldots\in\mathcal{T}$ such that $\{\beta_n(w_0)\}_{n\ge1}$ is a one–to–one sequence. We may suppose X has a finite cyclic group structure with identity u. Choose $v\in X\setminus\{u\}$. For $i\ge1$ choose $y_i=(y^i_w)_{w\in\Gamma}\in X^\Gamma$ and $y=(y_w)_{w\in\Gamma}\in X^\Gamma$ such that:

$$y^i_w:=\begin{cases} v & w=\beta_i(w_0)\,,\\ u & \text{otherwise}\,,\end{cases}\qquad\text{and}\qquad y_w:=\begin{cases} v & w=\beta_1(w_0),\beta_2(w_0),\dots\,,\\ u & \text{otherwise}\,.\end{cases}$$

Then $\lim\limits_{i\to\infty}(y_1+y_2+\cdots+y_i)=y$. The sequence $\{\sigma_{\beta_i}\}_{i\ge1}$ has a convergent subnet $\{\sigma_{\beta_{i_\lambda}}\}_{\lambda\in\Lambda}$ in $E(\mathcal{S},X^\Gamma)$ to $p\in E(\mathcal{S},X^\Gamma)$. Since $(\mathcal{S},X^\Gamma)$ is weakly almost periodic, by Note 2.2, there exists $\psi:\Gamma\to\Gamma$ with $p=\sigma_\psi$. Let $(z_w)_{w\in\Gamma}:=py=(y_{\psi(w)})_{w\in\Gamma}$ and $(z^n_w)_{w\in\Gamma}:=py_n=(y^n_{\psi(w)})_{w\in\Gamma}$ $(n\ge1)$. Using the continuity of p and $\lim\limits_{i\to\infty}(y_1+y_2+\cdots+y_i)=y$ we have:

$$\lim_{i\to\infty}(py_1+\cdots+py_i)=\lim_{i\to\infty}p(y_1+\cdots+y_i)=py\,,$$

which leads to:

$$\lim_{i\to\infty}(z^1_{w_0}+z^2_{w_0}+\cdots+z^i_{w_0})=z_{w_0}\,. \tag{*}$$

For $n\ge1$ and all $i\ge n$ we have $y^n_{\beta_i(w_0)}=u$, so $\lim\limits_{i\to\infty}y^n_{\beta_i(w_0)}=u$, which leads to $z^n_{w_0}=\lim\limits_{\lambda\in\Lambda}y^n_{\beta_{i_\lambda}(w_0)}=u$. Hence:

$$\forall n\ge1\quad(z^1_{w_0}+z^2_{w_0}+\cdots+z^n_{w_0}=u) \tag{**}$$

Using (*) and (**) we have $z_{w_0}=u$. On the other hand for all $i\ge1$ we have $y_{\beta_i(w_0)}=v$, therefore $v=\lim\limits_{i\to\infty}y_{\beta_i(w_0)}=\lim\limits_{\lambda\in\Lambda}y_{\beta_{i_\lambda}(w_0)}=z_{w_0}$, which is a contradiction by $u\ne v$. $\square$

Lemma 3.4. *If $(\mathcal{S}, X^\Gamma)$ is not equicontinuous, if and only if it is sensitive.*

Proof. Suppose $(\mathcal{S}, X^\Gamma)$ is not equicontinuous, by Lemma 3.2 there exists $v \in \Gamma$ such that $\mathcal{T}v$ is infinite. Consider $x = (x_w)_{w\in\Gamma} \in X^\Gamma$ and open neighbourhood U of x, there exists $\beta_1, \ldots, \beta_n \in \Gamma$ such that

$$\{(y_w)_{w\in\Gamma} \in X^\Gamma : y_{\beta_1} = x_{\beta_1}, \ldots, y_{\beta_n} = x_{\beta_n}\} \subseteq U\ .$$

Choose $\beta \in \mathcal{T}v \setminus \{\beta_1, \ldots, \beta_n\}$ and $p \in X \setminus \{x_\beta\}$ also let:

$$y_w := \begin{cases} x_w & w \neq \beta\ , \\ p & w = \beta\ , \end{cases}$$

then $y := (y_w)_{w\in\Gamma} \in U$. There exists $\varphi \in \mathcal{T}$ with $\beta = \varphi(v)$, so for $(u_w)_{w\in\Gamma} := \sigma_\varphi(x)$ and $(t_w)_{w\in\Gamma} := \sigma_\varphi(y)$ we have $u_v = x_\beta \neq p = y_\beta = t_v$, thus

$$(\sigma_\varphi(x), \sigma_\varphi(y)) = ((u_w)_{w\in\Gamma}, (t_w)_{w\in\Gamma} \notin \alpha_{\{v\}}\ .$$

Hence for all $x \in X^\Gamma$ and open neighbourhood U of x there exists $y \in U$ and $\sigma_\varphi \in \mathcal{S}$ with $(\sigma_\varphi(x), \sigma_\varphi(y)) \notin \alpha_{\{v\}}$ which completes the proof. □

Theorem 3.5 (main)**.** *In the transformation semigroup $(\mathcal{S}, X^\Gamma)$, the following statements are equivalent:*

1. *the transformation semigroup $(\mathcal{S}, X^\Gamma)$ is equicontinuous;*
2. *transformation semigroup $(\mathcal{S}, X^\Gamma)$ has at least an equicontinuous point;*
3. *the transformation semigroup $(\mathcal{S}, X^\Gamma)$ is not sensitive;*
4. *all of the elements of $E(\mathcal{S}, X^\Gamma)$ are continuous maps on X^Γ;*
5. *for all $w \in \Gamma$, $\{\varphi(w) : \sigma_\varphi \in \mathcal{S}\}$ is finite;*
6. *for all $w \in \Gamma$, $\{\varphi(w) : \sigma_\varphi \in E(\mathcal{S}, X^\Gamma)\}$ is finite;*
7. *$E(\mathcal{S}, X^\Gamma) \subseteq \{\sigma_\varphi : \varphi \in \Gamma^\Gamma\}$.*

Proof. (1,2,3,4,5) are equivalent: First note that if (S, Z) is an equicontinuous transformation semigroup, then all of its points are equicontinuous points, it is not sensitive, and all of the elements of $E(S, Z)$ are continuous, so (1) implies (2), (3) and (4). By Lemmas 3.1 and 3.3, (2) and (4) imply (5). By Lemma 3.2, (5) implies (1). By Lemma 3.4, (3) implies (1).
(4) and (7) are equivalent by Note 2.2.
Its evident that (6) implies (5).
Finally, note that in equicontinuous transformation semigroup (S, Z), one may consider the equicontinuous transformation semigroup $(E(S, Z), Z)$, now using the equivalency of (1) and (5), two items (1) and (7) imply (6). □

4. When is $(\mathcal{S}, X^\Gamma)$ distal?

In this section we prove $(\mathcal{S}, X^\Gamma)$ is distal if and only if it is equicontinuous, and $\mathcal{T}$ is a collection of permutations on Γ. Lets recall that $\varphi : \Gamma \to \Gamma$ is injective (resp. surjective) if and only if $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is injective (resp. surjective) [2].

Lemma 4.1. *If $(\mathcal{S}, X^\Gamma)$ is distal, then for all $w \in \Gamma$ the set $\mathcal{T}^{-1}w(= \{t \in \Gamma : \exists\varphi \in \Gamma\ (\varphi(t) = w)\})$ is finite and for all $\varphi \in \mathcal{T}$ the map $\varphi : \Gamma \to \Gamma$ is bijective.*

Proof. If $(\mathcal{S}, X^\Gamma)$ is distal, then for all $\varphi \in \mathcal{T}$ the map $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is bijective, so $\varphi : \Gamma \to \Gamma$ is bijective too. Consider $v \in \Gamma$. If $\mathcal{T}^{-1}v$ is infinite, then choose the one to one sequence $\{v_n\}_{n\geq 1}$ in $\mathcal{T}^{-1}v$ and sequence $\{\varphi_n\}_{n\geq 1}$ in $\mathcal{T}$ such that for all

$n \geq 1$ we have $\varphi_n(v_n) = v$ (thus $\{\varphi_n\}_{n\geq 1}$ is a one-to-one sequence too). Choose distinct $p, q \in X$ and let:

$$z_w := \begin{cases} p & w = v\,, \\ q & w \in \Gamma \setminus \{v\}\,. \end{cases}$$

It's evident that $\lim\limits_{n\to\infty} \sigma_{\varphi_n}((q)_{w\in\Gamma}) = \lim\limits_{n\to\infty} (q)_{w\in\Gamma} = (q)_{w\in\Gamma}$. We prove $\lim\limits_{n\to\infty} \sigma_{\varphi_n}((z_w)_{w\in\Gamma}) = (q)_{w\in\Gamma}$. So we should prove $\lim\limits_{n\to\infty} z_{\varphi_n(w)} = q$ for all $w \in \Gamma$. Consider $w \in \Gamma$ there exists $N \geq 1$ such that $w \neq v_n$ for all $n \geq N$, thus $\varphi_n(w) \neq \varphi_n(v_n)$ which leads to $\varphi_n(w) \neq v$ and $z_{\varphi_n(w)} = q$ for all $n \geq N$, hence $\lim\limits_{n\to\infty} z_{\varphi_n(w)} = q$ which leads to the desired result. Since $\lim\limits_{n\to\infty} \sigma_{\varphi_n}((z_w)_{w\in\Gamma}) = \lim\limits_{n\to\infty} \sigma_{\varphi_n}((q)_{w\in\Gamma})$ and $(z_w)_{w\in\Gamma} \neq (q)_{w\in\Gamma}$, $(\mathcal{S}, X^\Gamma)$ is not distal, which is in contradiction with our assumption and completes the proof. $\square$

Lemma 4.2. *Suppose $\mathcal{T}$ is a semigroup of permutations on Γ and $w \in \Gamma$. The following statements are equivalents:*

1. *$\mathcal{T}w$ is finite;*
2. *$\mathcal{T}^{-1}w$ is finite;*
3. *$\mathcal{T}w = \mathcal{T}^{-1}w$ is finite.*

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{T}w$ is finite and consider $\varphi \in \mathcal{T}$, since $\{\varphi^n(w) : n \geq 1\} \subseteq \mathcal{T}w$, the set $\{\varphi^n(w) : n \geq 1\}$ is finite too, so there exists $n > m$ with $\varphi^n(w) = \varphi^m(w)$, hence $\varphi^{n-m}(w) = w$ and w is a periodic point of φ, however $\varphi^{2(n-m)-1} \in \mathcal{T}$ which leads to $\varphi^{-1}(w) = \varphi^{2(n-m)-1}(w) \in \mathcal{T}w$. Thus $\mathcal{T}^{-1}w \subseteq \mathcal{T}w$ and $\mathcal{T}^{-1}w$ is finite.
(2) $\Rightarrow$ (1): Suppose $\mathcal{T}^{-1}w$ is finite, then $\mathcal{T}' := \{\varphi^{-1} : \varphi \in \mathcal{T}\}(= \mathcal{T}^{-1})$ is a semigroup of permutations on Γ and $\mathcal{T}'w$ is finite, now using "(1) $\Rightarrow$ (2)", $\mathcal{T}'^{\,-1}w = \mathcal{T}w$ is finite.
Using the proof of "(1) $\Rightarrow$ (2)", and the above proof its clear that "(1,2) $\Rightarrow$ (3)". $\square$

Lemma 4.3. *If for all $\varphi \in \mathcal{T}$, $\varphi : \Gamma \to \Gamma$ is bijective and for all $w \in \Gamma$, $\mathcal{T}w = \mathcal{T}^{-1}w$ is finite, then $(\mathcal{S}, X^\Gamma)$ is distal.*

Proof. Suppose for all $\varphi \in \mathcal{T}$, $\varphi : \Gamma \to \Gamma$ is bijective and for all $w \in \Gamma$, $\mathcal{T}w = \mathcal{T}^{-1}w$ is finite. Consider the net $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{T}$ and $(x_w)_{w\in\Gamma}, (y_w)_{w\in\Gamma}, (z_w)_{w\in\Gamma} \in X^\Gamma$ with $\lim\limits_{\lambda\in\Lambda} \sigma_{\varphi_\lambda}((x_w)_{w\in\Gamma}) = (z_w)_{w\in\Gamma} = \lim\limits_{\lambda\in\Lambda} \sigma_{\varphi_\lambda}((y_w)_{w\in\Gamma})$. Choose $v \in \Gamma$, the map $\varphi \restriction_{\mathcal{T}v} : \mathcal{T}v \to \mathcal{T}v$ is bijective since it is one to one and $\mathcal{T}v$ is finite. We have $\lim\limits_{\lambda\in\Lambda} \sigma_{\varphi_\lambda} \restriction_{X^{\mathcal{T}v}} ((x_w)_{w\in\mathcal{T}v}) = (z_w)_{w\in\mathcal{T}v} = \lim\limits_{\lambda\in\Lambda} \sigma_{\varphi_\lambda} \restriction_{X^{\mathcal{T}v}} ((y_w)_{w\in\mathcal{T}v})$, note to the fact that $\{\sigma_{\varphi_\lambda} \restriction_{X^{\mathcal{T}v}} : \lambda \in \Lambda\}$ is a set of permutations on $X^{\mathcal{T}v}$ and it is finite (since $X^{\mathcal{T}v}$ is finite), leads us to $(x_w)_{w\in\mathcal{T}v} = (y_w)_{w\in\mathcal{T}v}$ and in particular $x_v = y_v$. Hence $(x_w)_{w\in\Gamma} = (y_w)_{w\in\Gamma}$ and $(\mathcal{S}, X^\Gamma)$ is distal. $\square$

Using Theorem 3.5, Lemmas 4.1, 4.2 and 4.3 we have the following theorem.

Theorem 4.4. *In the transformation semigroup $(\mathcal{S}, X^\Gamma)$, the following statements are equivalent:*

- *the transformation semigroup $(\mathcal{S}, X^\Gamma)$ is equicontinuous and for all $\varphi \in \mathcal{T}$, $\varphi : \Gamma \to \Gamma$ is bijective;*
- *the transformation semigroup $(\mathcal{S}, X^\Gamma)$ is distal;*
- *for all $w \in \Gamma$, $\{\varphi(w) : \sigma_\varphi \in \mathcal{S}\}$ is finite and for all $\varphi \in \mathcal{T}$, $\varphi : \Gamma \to \Gamma$ is bijective;*

- *for all* $w \in \Gamma$, $\{v \in \Gamma : \exists \sigma_\varphi \in \mathcal{S} \, (\varphi(v) = w)\}$ *is finite and for all* $\varphi \in \mathcal{T}$, $\varphi : \Gamma \to \Gamma$ *is bijective.*

5. Expansive generalized shift transformation semigroup $(\mathcal{S}, X^\Gamma)$

In this section we prove that the transformation semigroup $(\mathcal{S}, X^\Gamma)$ is expansive, if and only if there exists finite subset H of Γ such that $\Gamma = \mathcal{T}H$, in particular using Theorem3.5 one may verify directly that if $(\mathcal{S}, X^\Gamma)$ is expansive, then it is sensitive.

Theorem 5.1. *The transformation semigroup* $(\mathcal{S}, X^\Gamma)$ *is expansive, if and only if there exists finite subset* H *of* Γ *such that* $\Gamma = \mathcal{T}H$.

Proof. Suppose $(\mathcal{S}, X^\Gamma)$ is expansive, there exists $\alpha \in \mathcal{F}$ such that for all distinct $x, y \in X^\Gamma$ there exists $\varphi \in \mathcal{T}$ with $(\sigma_\varphi(x), \sigma_\varphi(y)) \notin \alpha$. There exists finite subset H of Γ with $\alpha_H \subseteq \alpha$. Choose $v \in \Gamma$ and distinct $p, q \in X$ and let:

$$z_w := \begin{cases} p & w = v\,, \\ q & w \in \Gamma \setminus \{v\}\,. \end{cases}$$

There exists $\varphi \in \mathcal{T}$ with $((z_{\varphi(w)})_{w\in\Gamma}, (q)_{w\in\Gamma}) = (\sigma_\varphi(z_w)_{w\in\Gamma}), \sigma_\varphi((q)_{w\in\Gamma})) \notin \alpha$ which leads to $((z_{\varphi(w)})_{w\in\Gamma}, (q)_{w\in\Gamma}) \notin \alpha_H$, thus there exists $w \in H$ with $z_{\varphi(w)} \neq q$, and $\varphi(w) = v$, hence $v \in \varphi(H)$ and $\Gamma = \mathcal{T}H$.
Now conversely, suppose there exists finite subset H of Γ such that $\Gamma = \mathcal{T}H$ also consider distinct $(x_v)_{v\in\Gamma}, (y_v)_{v\in\Gamma} \in X^\Gamma$. There exists $w \in \Gamma$ with $x_w \neq y_w$. Also there exist $\varphi \in \mathcal{T}$ and $h \in H$ with $\varphi(h) = w$, thus $x_{\varphi(h)} \neq y_{\varphi(h)}$ and $((x_{\varphi(v)})_{v\in\Gamma}, (y_{\varphi(v)})_{v\in\Gamma}) = (\sigma_\varphi((x_v)_{v\in\Gamma}), \sigma_\varphi((y_v)_{v\in\Gamma})) \notin \alpha_H$. □

6. A diagram

Now we are ready to compare equicontinuity, distality, sensitivity and expansivity in generalized shift transformation semigroups, via diagrams and examples. For this aim suppose $\mathcal{C}$ is the collection of all transformation semigroups like $(\mathcal{E}, Y^\Lambda)$ where Y is a finite discrete space with at least two elements, Λ is a nonempty set and $\mathcal{E}$ is a subsemigroup of generalized shifts on Y^Λ (i.e., $\mathcal{E}$ is a subsemigroup of $\{\sigma_\varphi : \varphi \in \Lambda^\Lambda\}$), then we have the following diagram:

Equicontinuous elements of $\mathcal{C}$ 6.1	Sensitive elements of $\mathcal{C}$ 6.3
Distal elements of $\mathcal{C}$ 6.2	Expansive elements of $\mathcal{C}$ 6.4

Example 6.1. *Consider* $\varphi : \mathbb{Z} \to \mathbb{Z}$ *with* $\varphi(n) = |n|$ *(for* $n \in \mathbb{Z}$*), then* $(\{\sigma_\varphi, \sigma_{\mathrm{id}_Z}\}, \{0,1\}^\mathbb{Z})$ *is equicontinuous and it is not distal (note that* $\sigma_\varphi : \{0,1\}^\mathbb{Z} \to \{0,1\}^\mathbb{Z}$ *is not surjective).*

Example 6.2. *Consider* $\varphi : \mathbb{Z} \to \mathbb{Z}$ *with* $\varphi(n) = -n$ *(for* $n \in \mathbb{Z}$*), then* $(\{\sigma_\varphi, \sigma_{\mathrm{id}_Z}\}, \{0,1\}^\mathbb{Z})$ *is distal.*

Example 6.3. *Consider* $\varphi : \mathbb{Z} \to \mathbb{Z}$ *with* $\varphi(n) = n^2$ *(for* $n \in \mathbb{Z}$*), then* $(\{\sigma_{\varphi^n} : n \geq 0\}, \{0,1\}^\mathbb{Z})$ *is sensitive and it is not expansive (since* $\{\varphi^n(2) : n \geq 0\} = \{2^{2^n} : n \geq 0\}$ *is infinite,* $\{\varphi^n(1) : n \geq 0\} = \{1\}$ *is finite, and for all finite*

subset A of $\mathbb{Z}$ the set $\{\varphi^n(i) : i \in A, n \geq 0\} = \{i^{2^n} : i \in A, n \geq 0\}$ is a proper subset of $\mathbb{Z}$).

Example 6.4. *Consider $\varphi : \mathbb{Z} \to \mathbb{Z}$ with:*

$$\varphi(n) := \begin{cases} n+1 & n \geq 1\,, \\ 0 & n = 0\,, \\ n-1 & n \leq -1\,, \end{cases}$$

then $(\{\sigma_{\varphi^n} : n \geq 0\}, \{0,1\}^{\mathbb{Z}})$ is expansive.

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