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# Fuzzy geometric spaces associated to fuzzy hyperrings

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**Abstract**. For given two families $\widetilde{B}_R$, $\widetilde{B}_C$ of fuzzy subsets of a fuzzy hyperring $R$, we obtain some sufficient conditions such that two fuzzy geometric spaces $(\Delta, \widetilde{B}_R)$, $(\Delta, \widetilde{B}_C)$ are strongly transitive and $\Delta$ be a nonzero fuzzy subset of $R$. Moreover, we show that the relation $\alpha$ and $\Gamma$ on a fuzzy hyperfield $R$ are transitive and obtain some related basic results.

Keywords: Fuzzy geometric spaces, fuzzy hyperring, fuzzy hyperfield, strongly transitive

## 1. Introduction

Fuzzy hyperstructure is an interesting research topic of fuzzy subsets. A hyperoperation assigns a subset of $H$ to every pair of elements of $H$, that is defined by Marty in [18] as a generalization of a group, while a fuzzy hyperoperation assigns a fuzzy subset of $H$ to every pair of elements of $H$. This idea was introduced by Corsini and Tofan [10] and studied by Serafimidis, Kehagias and Konstantinidou [22] and they obtained interesting properties in connections with an important hyperstructure, called an interesting paper concerning the join spaces. Recently, Sen, Ameri and Chowdhury introduced and analyzed fuzzy hypersemigroups in [21]. Afterwards these ideas extended to fuzzy hyperrings and fuzzy hypermodules by Leoreanu and Davvaz in [16, 17]. A geometric space is a pair $(S, B)$ such that $S$ is a nonempty set and $B$ is a nonempty family of subsets of $S$, that are called points and blocks. This concept was initiated by D. Freni in [12] and he investigated some results in order to connection between geometric spaces and its structures. Also D. Freni indicated that the relation $\beta$ defined by Koskas [15] (studied mainly by Corsini [8] and Vougiouklis [24]) is transitive in hypergroups. In [4] Anvariyeh and Davvaz introduced the concept of strongly transitive geometric spaces associated to hypermodules. In [19] Mirvakili and Davvaz studied on strongly transitive geometric spaces: applications to hyperrings. The concept of fuzzy geometric space was introduced by Ameri et.al. in [3] and they investigate some important structures and relationships between them. In this paper we follow [3], and introduce transitive and strongly transitive relations on a given fuzzy geometric space and obtain some related basic results. In particular, we prove that the fuzzy geometric space associated to a hyperfield is strongly transitive.

## 2. Preliminaries

A *hypergroupoid* $(H, \circ)$ is a non-empty set $H$ equipped with a hyperoperation $\circ$, that is a map $\circ : H \times H \to P^*(H)$, where $P^*(H)$ denotes the family of all non-empty subset of $H$. If $x, y \in H$, we will denote by $x \circ y$ the hyperproduct of $x$ and $y$.

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A hypergroupoid $(H, \circ)$ is said to be *semi-hypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$. A hypergroup is a semi hypergroup $(H, \circ)$ such that $x \circ H = H \circ x = H$, for all $x \in H$(this condition is called *reproducibility*).
A non-empty set$K$ of a hypergroup $H$ is a *subhypergroup* of $H$ if $x \circ K = K \circ x = K$, for every $x \in K$.

**Definition 2.1.** [21] Let $S$ be a non-empty set and $F(S)$ denotes the set of all fuzzy subset of $S$. A *fuzzy hyperoperation* on $S$ is the mapping $\oplus : S \times S \to F(S)$ written as $(a, b) \mapsto a \oplus b$. $S$ together with a fuzzy hyperoperation $\oplus$ is called *fuzzy hypergroupoid*. A fuzzy hypergroupoid $(S, \oplus)$ is called a *fuzzy hypersemigroup*, if for all $a, b, c$ of $S$ we have $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, where for any fuzzy subset $\mu$ of $F(S)$

$$(a \oplus \mu)(r) = \begin{cases} \bigvee_{t \in S}((a \oplus t)(r) \wedge \mu(t)), & if\ \mu \neq 0 \\ 0, & otherwise \end{cases}$$

and

$$(\mu \oplus a)(r) = \begin{cases} \bigvee_{t \in S}(\mu(t) \wedge (t \oplus a)(r)), & if\ \mu \neq 0 \\ 0, & otherwise \end{cases}$$

for all $r \in S$.

Let $\mu, \nu$ be two fuzzy subset of a fuzzy hypergroupoid $(S, \oplus)$, then we define $(\mu \oplus \nu)(t) = \bigvee_{p,q \in S}(\mu(p) \wedge (p \oplus q)(t) \wedge \nu(q))$, for all $t \in S$.
A fuzzy hypersemigroup $(S, \oplus)$ is called a *fuzzy hypergroup*, if $x \oplus S = S \oplus x = \chi_S$, for all $x \in S$, that is called *reproducibility axiom*.

**Definition 2.2.** [21] If $(S, \oplus)$ be a fuzzy hypergroup, by reproducibility axiom, for every $x \in S$ there exists a pair $(a, b)$ of elements of $S$ such that $(a \oplus b)(x) > 0$.

**Definition 2.3.** [16] A fuzzy hyperring is a multi-valued system $(R, \oplus, \otimes)$ which satisfies the following axioms:

(1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, for all $a, b, c \in R$,
(2) $x \oplus R = R \oplus x = \chi_R$, for all $x \in R$,
(3) $a \oplus b = b \oplus a$, for all $a, b \in R$,
(4) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, for all $a, b, c \in R$,
(5) The multiplication is distributive with respect to the fuzzy hyperoperation $\oplus$. i.e., $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, for all $a, b, c \in R$.

**Definition 2.4.** [17] Let $(R, \oplus, \otimes)$ be a fuzzy hyperring and $(M, \oplus)$ be a commutative fuzzy hypergroup. $M$ is said to be a fuzzy hypermodule over a fuzzy hyperring $R$, if there exists:

$$\odot : R \times M \to F^*(M);\ (a, m) \mapsto a \odot m,$$

such that for all $a, b \in M$ and $m_1, m_2, m \in M$, we have

(1) $a \odot (m_1 \oplus m_2) = (a \odot m_1) \oplus (a \odot m_2)$
(2) $(a \oplus b) \odot m = (a \odot m) \oplus (b \odot m)$,
(3) $(a \otimes b) \odot m = a \odot (b \odot m)$.

**Definition 2.5.** [21] Let $\rho$ be an equivalence relation on a fuzzy hypersemigroup $(S, \oplus)$ and let $\mu, \nu$ be two fuzzy subset on $(S, \oplus)$. If $\mu(a) > 0$ implies there exists $b \in S$ such that $\nu(b) > 0$ and $a\rho b$, and if $\nu(x) > 0$ implies there exists $y \in S$ such that $\mu(y) > 0$ and $x\rho y$, then we say that $\mu\overline{\rho}\nu$. Also, $\mu\overline{\overline{\rho}}\nu$, if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, we have $x\rho y$.

**Definition 2.6.** [3] Let $S$ is a nonempty set. A *fuzzy geometric space* is a pair $(\Delta, B)$ such that $\Delta$ is a nonzero fuzzy subset of $S$ and $B$ is a nonempty family of fuzzy subsets of $S$ such that $\nu \leq \Delta$, for all $\nu \in B$, whose elements we called fuzzy blocks.

**Remark 2.7.** [3] $B$ is a covering of $\Delta$ if
$\Delta \leq \bigvee_{\nu \in B} \nu$.

**Definition 2.8.** [3] If $\nu_1, \nu_2, \ldots, \nu_n$ are fuzzy blocks of a fuzzy geometric space $(\Delta, B)$ such that
$\nu_i \wedge \nu_{i+1} > 0$, for any $i \in \{1, 2, \ldots, n-1\}$, then the $n$-tuple $(\nu_1, \nu_2, \ldots, \nu_n)$ is called a fuzzy polygonal of $(\Delta, B)$. The concept of fuzzy polygonal allows us to define on $S$ the following relation:

$$x \approx y \Leftrightarrow x = y \ or \ \exists(\nu_1, \nu_2, \ldots, \nu_n);\ \nu_1(x) > 0,\ \nu_n(y) > 0,$$

where $(\nu_1, \nu_2, \ldots, \nu_n)$ is a fuzzy polygonal. The relation $\approx$ is an equivalence and it is coincides with the transitive closure of the following relation:

$$x \sim y \Leftrightarrow x = y \ or \ \exists \nu \in B;\ \nu(x) > 0,\ \nu(y) > 0,$$

so $\approx$ is equal to $\approx = \bigcup_{n \geq 1} \sim^n$, where $\sim^n = \sim \circ \sim \circ \ldots \circ \sim$ $n$ times.
If $B$ is a covering of $\Delta$, the relation $\sim$ and $\approx$ is defined in the following simpler way:

$$x \sim y \Leftrightarrow \exists \nu \in B;\ \nu(x) > 0,\ \nu(y) > 0,$$

$x \approx y \Leftrightarrow \exists(\nu_1, \nu_2, \ldots, \nu_n);\ \nu_1(x) > 0,\ \nu_n(y) > 0.$

**Theorem 2.9.** *[3] A fuzzy geometric space $(\Delta, B)$ is strongly transitive, if the family $B$ is a fuzzy cover of $\Delta$, and the following condition is satisfied.*
*For every pair $(\mu, \nu)$ of fuzzy blocks of a fuzzy geometric space $(\Delta, B)$ and for any $n \in N$ :*

$$\mu \wedge \nu > 0,\ \nu(x) > 0 \Longrightarrow \exists \eta \in B, 0 < \alpha \leq 1; (\mu \vee x_\alpha) \leq \eta.$$

**Theorem 2.10.** *[3] If $(\Delta, B)$ is a strongly transitive fuzzy geometric space, then the relation $\sim$ on $S$ is transitive, thus $\sim = \approx$ .*

## 3. Strongly transitive fuzzy geometric spaces

Let $(R, \oplus, \otimes)$ be a fuzzy hyperring and $x_{ij} \in R$ be elements of $R$. If $\sigma \in S_n$, then the fuzzy hypersums and hyperproducts of the elements $x_{ij}$ respecting is denoted $\widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{k_i} x_{ij}$. Denote by $S_n$ is the symmetric group of all permutations of the set $1, 2, \ldots, n$ in this order. Using this notations we define for every $n \in N \cup \{0\}$, $k_i \in N \cup \{0\}$ where $i = 1, 2, \ldots, n$ and $j = 1, \ldots, k_i$, we set:

$$\widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{k_i} x_{ij}$$

We can consider the fuzzy geometric space $(\Delta, \widetilde{B}_R)$ whose $\Delta$ is a nonzero fuzzy subset of $R$ and fuzzy blocks are the fuzzy hypersums of hyperproducts of elements of $R$. Also, we can consider another fuzzy geometric space $(\Delta, \widetilde{B}_C)$ whose $\Delta$ is a nonzero fuzzy subset of $R$ and fuzzy blocks are the supremum of all fuzzy hypersums of hyperproducts obtained by permuting in the following possible ways:
for every $n \in N \cup \{0\}$, $k_i \in N \cup \{0\}$ and $x_{ij} \in R$ where $i = 1, 2, \ldots, n$ and $j = 1, \ldots, k_i$, we set:

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \bigvee \{\widetilde{\sum}_{i=1}^{n} \widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} \\ |\sigma \in S_n, \sigma_i \in S_{k_i}\}.$$

We can consider the fuzzy geometric space $(\Delta, FP_\star(R))$. By $FP_\star(R)$ we mean the set of all fuzzy finite hypersums of hyperproducts of elements of $R$, that is a typical elements of $FP_\star(R)$ is the form $(x_{11} \otimes \ldots \otimes x_{1k_1}) \oplus \ldots \oplus (x_{n1} \otimes \ldots \otimes x_{nk_n})$, $n \in N$.

**Notation 3.1.** For $x_{ij} \in R$, $\mu \in FP_\star(R)$ we note $x_{ij} \in \mu \Leftrightarrow \chi_{x_{ij}} \leq \mu \Leftrightarrow \mu(x_{ij}) = 1$.

**Lemma 3.2.** *Let $(R, \oplus, \otimes)$ be a fuzzy hyperring. Then*

(1) $\widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \leq \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])$.

(2) Moreover, we have

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \bigvee \{\widetilde{B}_R([x_{\sigma(1)\sigma_{\sigma(1)}(1)}^{\sigma(1)\sigma_{\sigma(1)}(k_{\sigma(1)})}], \ldots, [x_{\sigma(n)\sigma_{\sigma(n)}(1)}^{\sigma(n)\sigma_{\sigma(n)}(k_{\sigma(n)})}]|\sigma \in S_n, \sigma_i \in S_{k_i}\}.$$

**Lemma 3.3.** *Let $(R, \oplus, \otimes)$ be a fuzzy hyperring. Then for every $y \in R$ we have*

(1) $\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \oplus y = \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}], [y])$.

(2) $y \oplus \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \widetilde{B}_C([y], [x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])$.

(3) $\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \otimes y = \widetilde{B}_C([x_{11}^{1k_1}, y], \ldots, [x_{n1}^{nk_n}, y])$.

(4) $y \otimes \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \widetilde{B}_C([y, x_{11}^{1k_1}], \ldots, [y, x_{n1}^{nk_n}])$.

(5) $\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \oplus \widetilde{B}_C([y_{11}^{1k_1}], \ldots, [y_{m1}^{ml_m}]) = \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}], [y_{11}^{1k_1}], \ldots, [y_{m1}^{ml_m}])$.

(6) $\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \otimes \widetilde{B}_C([y_{11}^{1k_1}], \ldots, [y_{m1}^{ml_m}]) = \widetilde{B}_C([x_{11}^{1k_1}, y_{11}^{1l_1}], \ldots, [x_{11}^{1k_1}, y_{m1}^{ml_m}], \ldots, [x_{n1}^{nk_n}, y_{11}^{1l_1}], \ldots, [x_{n1}^{nk_n}, y_{m1}^{ml_m}])$.

**Proof.** It is straightforward.

**Lemma 3.4.** *Lemma 3.3 is true for the fuzzy geometric space* $(\Delta, \widetilde{B}_R)$.

**Lemma 3.5.** *Let* $(R, \oplus, \otimes)$ *be a fuzzy hyperring. Then for every* $\sigma \in S_n$ *and* $\sigma_i \in S_{k_i}$ *we have*

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) = \widetilde{B}_C([x_{\sigma(1)\sigma_{\sigma(1)}(1)}^{\sigma(1)\sigma_{\sigma(1)}(k_{\sigma(1)})}], \ldots, [x_{\sigma(n)\sigma_{\sigma(n)}(1)}^{\sigma(n)\sigma_{\sigma(n)}(k_{\sigma(n)})}])$$

Moreover, if $(R, \oplus, \otimes)$ is a commutative fuzzy hyperring then two fuzzy geometric spaces $(\Delta, \widetilde{B}_R)$ and $(\Delta, \widetilde{B}_C)$ are equal.

**Proof.** It obtain from definition of fuzzy geometric spaces $(\Delta, \widetilde{B}_R)$ and $(\Delta, \widetilde{B}_C)$.

**Lemma 3.6.** *Let* $(R, \oplus, \otimes)$ *be a fuzzy hyperring. Then*

(1) If $x_{rs} \in a \otimes b$ then

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \leq \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{r1}^{r(s-1)}, a, b, x_{r(s+1)}^{rk_r}], \ldots, [x_{n1}^{nk_n}]).$$

(2) If $x_{rs} \in a \oplus b$ then

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \leq \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{(r-1)1}^{(r-1)k_{r-1}}], [x_{r1}^{r(s-1)}, a, x_{r(s+1)}^{rk_r}], [x_{r1}^{r(s-1)}, b, x_{r(s+1)}^{rk_r}], [x_{(r+1)1}^{(r+1)k_{r+1}}], \ldots, [x_{n1}^{nk_n}]).$$

(3) If $x_{rs} \in B = \widetilde{B}_C([y_{11}^{1l_1}], \ldots, [y_{m1}^{ml_m}])$, then

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \leq \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{r1}^{r(s-1)}, B, x_{r(s+1)}^{rk_r}], \ldots, [x_{n1}^{nk_n}]) = \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{(r-1)1}^{(r-1)k_{r-1}}], [.], [x_{(r+1)1}^{(r+1)k_{r+1}}], \ldots, [x_{n1}^{nk_n}]),$$

where $[.] = [x_{r1}^{r(s-1)}, y_{11}^{1l_1}, x_{r(s+1)}^{rk_r}], \ldots, [x_{r1}^{r(s-1)}, y_{m1}^{ml_m}, x_{r(s+1)}^{rk_r}]$.

**Proof.** (1) Let $x_{rs} \in a \otimes b$ and $\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])(y) > 0$. Then there exists $\sigma \in S_n, \sigma_r \in S_{k_r}$, such that $(\widetilde{\sum}_{i=1}^{n}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)})(y) > 0$, if $\sigma(u) = r$ and $\sigma_r(v) = s$, then we have $x_{\sigma(u)\sigma_{\sigma(u)}(v)} = x_{rs}$, such that $(a \otimes b)(x_{\sigma(u)\sigma_{\sigma(u)}(v)}) = 1$, we have

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])(y) = (\widetilde{\sum}_{i=1}^{n}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)})(y)$$
$$= ((\widetilde{\sum}_{i=1}^{u-1}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}) \oplus (\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes x_{rs} \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}) \oplus (\widetilde{\sum}_{i=u+1}^{n}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}))(y),$$

we set:

$$\widetilde{\sum}_{i=1}^{u-1}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} = \mu,$$
$$\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes x_{rs} \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)} = \nu,$$

and

$$\widetilde{\sum}_{i=u+1}^{n}\widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)} = \eta.$$

Then

$$\widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])(y) = \bigvee(\mu \oplus \nu \oplus \eta)(y)$$
$$= \bigvee\{\bigvee_{p\in R}[(\mu \oplus \nu)(p) \wedge (p \oplus \eta)(y)]\}$$
$$= \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}\nu(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$$
$$= \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes x_{rs} \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)})(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$$
$$= \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\bigvee_{z\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes x_{rs})(z) \wedge (z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$$
$$= \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\bigvee_{z\in R}(\bigvee_{t\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)})(t) \wedge (t \otimes x_{rs})(z) \wedge (z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$$
$$\leq \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\bigvee_{z\in R}(\bigvee_{t\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)})(t) \wedge (t \otimes x_{rs})(z) \wedge (a \otimes b)(x_{rs}) \wedge (z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y))]\}$$
$$\leq \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\bigvee_{z\in R}(\bigvee_{t\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)})(t) \wedge \bigvee_{w\in R}[(t \otimes w)(z) \wedge (a \otimes b)(w)] \wedge (z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y))]\}$$
$$\leq \bigvee\{\bigvee_{p\in R}[(\bigvee_{q\in R}(\bigvee_{z\in R}(\bigvee_{t\in R}(\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)})(t) \wedge (t \otimes (a \otimes b))(z) \wedge (z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y))]\}$$

$= \bigvee\{ \bigvee_{p\in R} [( \bigvee_{q\in R} ( \bigvee_{z\in R} (\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes (a \otimes b))(z) \wedge$
$(z \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(q)$
$\wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$
$= \bigvee\{ \bigvee_{p\in R} [( \bigvee_{q\in R} (\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes a \otimes b \otimes$
$\widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)})(q) \wedge (\mu \oplus q)(p)) \wedge (p \oplus \eta)(y)]\}$
$= \bigvee\{ \bigvee_{p\in R} [(\mu \oplus (\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes a \otimes b \otimes$
$\widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}))(p)] \wedge (p \oplus \eta)(y)]\}$
$= \bigvee((\widetilde{\sum}_{i=1}^{u-1} \widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}) \oplus (\widetilde{\prod}_{j=1}^{v-1} x_{r\sigma_{\sigma(r)}(j)} \otimes$
$a \otimes b \otimes \widetilde{\prod}_{j=v+1}^{k_u} x_{r\sigma_{\sigma(r)}(j)}) \oplus$
$(\widetilde{\sum}_{i=u+1}^{n} \widetilde{\prod}_{j=1}^{k_{\sigma(i)}} x_{\sigma(i)\sigma_{\sigma(i)}(j)}))(y),$
$= \widetilde{B}_C([x_{11}^{1k_1}], \ldots, [x_{r1}^{r(s-1)}, a, b, x_{r(s+1)}^{rk_r}], \ldots, [x_{n1}^{nk_n}])$

and the proof is complete.
The proof of (2) is similar to (1) and (3) obtains from (1) and (2).

**Lemma 3.7.** *Lemma 3.6, is true for the fuzzy geometric space* $(\Delta, \widetilde{B}_R)$.

**Theorem 3.8.** *If* $(R, \oplus, \otimes)$ *be a fuzzy hyperfield then*

*(1) The fuzzy geometric space* $(\Delta, \widetilde{B}_R)$ *is a strongly transitive fuzzy geometric space.*
*(2) The fuzzy geometric space* $(\Delta, \widetilde{B}_C)$ *is a strongly transitive fuzzy geometric space.*

**Proof.** Let $B_1 = \widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])$, $B_2 = \widetilde{B}_R([y_{11}^{1l_1}], \ldots, [y_{m1}^{ml_m}])$ be two fuzzy block of $\widetilde{B}_R$ such that

$$B_1 \wedge B_2 > 0, \ B_2(x) > 0.$$

Let for $b \in R$, we have $B_1(b) > 0$ and $B_2(b) > 0$. Since $R$ is a fuzzy hyperfield, thus there exist $u_1^n \in R$ and $v \in R$ such that $(u_i \otimes x)(x_{ik_i}) = 1$ and $(b \otimes v)(x) = 1$. Then we have $x_{ik_i} \in u_i \otimes x$, $x \in b \otimes v$. By Lemma 3.6 we have

$B_1 = \widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])$
$\leq \widetilde{B}_R([x_{11}^{1(k_1-1)}, u_1, x], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x])$
$\leq \widetilde{B}_R([x_{11}^{1(k_1-1)}, u_1, b, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, b, v])$
$\leq \widetilde{B}_R([x_{11}^{1(k_1-1)}, u_1, B_2, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B_2, v]).$

Moreover, since $(b \otimes v)(x) = 1$, $B_1(b) > 0$, then $\bigvee_b [\widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}])(b) \wedge (b \otimes v)(x)] > 0$, that is equal to $(\widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \otimes v)(x) > 0$, then by Lemma 3.6,

$(\widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, x], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x]) \otimes$
$v)(x) > 0.$

So by Lemma 3.3,

$0 < \widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, x], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x]) \otimes v$
$= \widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, x, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x, v]),$

since $B_2(x) = \widetilde{B}_R([y_{11}^{1l_1}], \ldots, [y_{m1}^{ml_m}])(x) > 0$, and by Lemma 3.6, we have

$0 < \widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, x, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, x, v])(x)$
$\leq \widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, B_2, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B_2, v])(x)$

therefore there exists $0 < \alpha \leq 1$ such that

$\widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, B_2, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B_2, v])(x)$
$\geq \alpha.$

By the concept of fuzzy point $x_\alpha$, we have

$\widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, B_2, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B_2, v]) \geq$
$x_\alpha.$

Therefore

$\widetilde{B}_R([x_{11}^{1k_1}], \ldots, [x_{n1}^{nk_n}]) \vee x_\alpha \leq$
$\widetilde{B}_R([x_{11}^{1(k_1-1}, u_1, B_2, v], \ldots, [x_{n1}^{n(k_n-1)}, u_n, B_2, v]).$

and the fuzzy geometric space $\widetilde{G} = (\Delta, \widetilde{B}_R)$ is strongly transitive. In a similar way we obtain (2).

**Definition 3.9.** Let $(R, \oplus, \otimes)$ be a fuzzy hyperring. We define the relation $\Gamma$ as follows:

$x\Gamma y \Longleftrightarrow \exists n \in N, k_i \in N, \exists(x_{i1}, \ldots, x_{ik_i}) \in$
$R^{k_i}, 1 \leq i \leq n;$
$\widetilde{\sum}_{i=1}^{n}(\widetilde{\prod}_{j=1}^{k_i} x_{ij})(x) > 0, \widetilde{\sum}_{i=1}^{n}(\widetilde{\prod}_{j=1}^{k_i} x_{ij})(y) > 0.$

**Definition 3.10.** Let $(R, \oplus, \otimes)$ be a fuzzy hyperring. We define the relation $\alpha$ as follows:
$x\alpha y \Longleftrightarrow \exists n \in N, k_i \in N, \exists\sigma \in$
$S_n, \exists(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, \exists\sigma_i \in S_{k_i} 1 \leq i \leq n;$

$$\widetilde{\sum}_{i=1}^{n}(\widetilde{\prod}_{j=1}^{k_i} x_{ij})(x) > 0, \ \widetilde{\sum}_{i=1}^{n}(A_i)(y) > 0,$$

where $A_i = \widetilde{\prod}_{j=1}^{k_i} x_{i\sigma_i(j)}$.

The relation $\alpha$ and $\Gamma$ are reflexive and symmetric. We take $\Gamma^*$, $\alpha^*$ be the transitive closure of $\Gamma$, $\alpha$. Then $\Gamma^*$, $\alpha^*$ is an equivalence relation on $R$.

**Theorem 3.11.** Let $(R, \oplus, \otimes)$ be a hyperfield. Then

(1) $\Gamma = \Gamma^*$.
(2) $\alpha = \alpha^*$.

**Proof.** (2) Let $(R, \oplus, \otimes)$ be a fuzzy hyperfield. Then the relation $\sim$ defined on fuzzy geometric space $(\Delta, \widetilde{B}_C)$ coincides with the relation $\alpha$ on the fuzzy hyperfield $(R, \oplus, \otimes)$. Also the relation $\approx$ defined on the fuzzy geometric space $(\Delta, \widetilde{B}_C)$ coincides the relation $\alpha^*$ on the fuzzy hyperfield $(R, \oplus, \otimes)$. Now, if $(R, \oplus, \otimes)$ is a hyperfield then the fuzzy geometric space $(\Delta, \widetilde{B}_C)$ is strongly transitive by Theorem 2.13, we have

$$\alpha = \sim = \approx = \alpha^*.$$

The proof of (1) is similar.

## Conclusion

The study of hyperoperations was initiated by Marty in [18] and continued by others (see [2 , 4–7, 9, 11, 13, 14, 20]). The above discussion shows that fuzzy geometric space can be done for fuzzy hyperrings, which have recently appeared in the previous paper (for more see [1]). This paper provides useful condition for doing new research in the field of fuzzy geometric space associated to fuzzy hypermodules and fundamental relation for fuzzy hyperstructures.

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## References

[1] R. Ameri, M. Asghari-Larimi and N. Firouzkouhi, Fuzzy geometric spaces associated to fuzzy hypersemigroups, *Journal of Intelligent and Fuzzy Systems* **34**(2018), 703–709.
[2] R. Ameri, On categories of hypergroups and hypermodules, *Journal of Discrete Mathematical Sciences and Cryptography* **6**(2-3) (2003), 121–132.
[3] R. Ameri, T. Nozari and M. Norouzi, On fuzzy geometric spaces, submitted for publication.
[4] S.M. Anvariyeh and B. Davvaz, Strongly transitive geometric spaces associated to hypermodules, *Journal of Algebra* **322**(4) (2009), 1340–1359.
[5] M. Al Tahan, H.-M. Šarkab and B. Davvaz, An overview of topological hypergroupoids, *Journal of Intelligent and Fuzzy Systems* **34**(3) (2018), 1907–1916.
[6] G. Chowdhury, Fuzzy transposition hypergroups, *Iranian Journal of Fuzzy Systems* **6**(3) (2009), 37–52.
[7] J. Chvalina and S. Hoskova-Mayerova, On certain proximities and preorderings on the transposition hypergroups of linear first-order partial differential operators, *An Stintt Univ Ovidius Constanta Ser Mat* **22**(1) (2014), 85–103.
[8] P. Corsini, *Prolegomena of Hypergroup Theory*, second edition Aviani editor, 1993.
[9] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publications, 2003.
[10] P. Corsini and I. Tofan, *On fuzzy hypergroups*, *PU M A* **8** (1997), 29–37.
[11] B. Davvaz and V. Leoreanu, Hyperring theory and applications, *Interna* (2007).
[12] D. Freni, Strongly transitive geometric spaces: Applications to hypergroups and semigroups theory, *Commu Algebra* **32**(3) (2004), 969–988.
[13] D. Freni, On a strongly regular relation in hypergroupoids, *Pure Mathematics and Applications* **3**(3-4) (1992), 191–198.
[14] D. Freni, A new characterization of the derived hypergroup via strongly regular equivalences, *Communications in Algebra* **30**(8) (2002), 3977–3989.
[15] M. Koskas, Gropoids, semi-hypergroups and hypergroups, *J Math Pures* **49**(2) (1970), 155.
[16] V. Leoreanu-Fotea and B. Davvaz, Fuzzy hyperrings, *Fuzzy Sets and Systems* **160**(16) (2009), 2366–2378.
[17] V. Leoreanu-Fotea, Fuzzy hypermodules, *Computers and Mathematics with Applications* **57**(3) (2009), 466–475.
[18] F. Marty, Sur une generalization de la notion de groupe, *In 8th Congress Math Scandinaves*, 1934, 45–49.
[19] S. Mirvakili and B. Davvaz, Strongly transitive geometric spaces: Applications to hyperrings, *Rev Un Mat Argentina* **53**(1) (2012), 43–53.
[20] S. Mirvakili, S.M. Anvariyeh and B. Davvaz, Transitivity of ⌈-relation on hyperfields, *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie* (2008), 233–243.
[21] M.K. Sen, R. Ameri and G. Chowdhury, Fuzzy hypersemigroups, *Soft Computing* **12**(9) (2008), 891–900.
[22] K. Serafimidis, A. Kehagias and M. Konstantinidou, The L-fuzzy Corsini join hyperoperation, *Italian Journal of Pure and Applied Mathematics* (2002), 83–90.
[23] S. Spartalis and T. Vougiouklis, The fundamental relations on $H_v$-rings, *Math Pura Appl* **13** (1994), 7–20.
[24] T. Vougiouklis, The Fundamental Relations in Hyperrings, *The General Hyperfield Proc 4th International Congress in Algebraic Hyperstructures and Its Applications(AHA 1990) World Scientific*, 1990, 203–211.