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α-Derivable digraphs and its application in wireless sensor networking

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This paper considers derivable directed hypergraphs by undirected hypergraphs and generates undirected hypergraphs from directed hypergraphs. It tries to enumerate derivable directed hypergraphs and to find an upper bound for it. We introduce a positive relation on directed hypergraphs and derives digraphs via positive equivalence relation. We consider wireless sensor hypernetworks as directed hypergraphs and by clustering directed hypergraphs and positive equivalence relation obtain wireless sensor networks and show by cluster digraphs.

Keywords: α-Derivable digraphs; positive equivalence relation; WSN.

Mathematics Subject Classification 2020: 68R10, 05C15, 81Q30, 05C65

1. Introduction

A wireless sensor network (WSN) is a set of sensors deployed in a sensor field to monitor specific characteristics of the environment, to measure those characteristics, and to collect the data related to those phenomena. WSNs have gained worldwide attention in recent years, particularly with the proliferation of micro-electro-mechanical systems (MEMS) technology, which has facilitated the development of smart sensors. WSNs are used in numerous applications, such as environmental monitoring, habitat monitoring, prediction and detection of natural calamities, medical monitoring, and structural health monitoring [12, 17].

Graph Theory concepts can be used to describe, analyze and represent a WSN in a very clear way. Several of these concepts refer to structures and algorithms that had been previously used to address other aspects like topology management,

localization techniques and routing, not only in WSN but also in other types of wireless and wired networks. As to directed graphs, a well-known instance is the world wide web. A hyperlink can be thought of as a directed edge because given an arbitrary hyperlink we cannot expect that there certainly exists an inverse one, that is, the hyperlink based relationships are asymmetric. However, in many real-world problems, representing a set of complex relational objects as undirected or directed graphs is not complete.

Clustering is a fundamental mechanism to design scalable sensor network protocols. In general terms, clustering is the classification of similar objects into different groups or subsets. The formed subsets in some sense belong together, because they share one or more similar characteristics or behaviors. Every cluster would have a leader, commonly referred to as cluster-head (CH). A CH may be elected by the sensor nodes in the cluster or preassigned by the network designer [2, 9, 15].

Graph clustering is the task of grouping the vertices of the graph into clusters taking into consideration the edge structure of the graph in such a way that there should be many edges within each cluster and relatively few between the clusters [11].

The notion of hypergraph has been introduced by Berge as a generalization of graph around 1960 and one of the initial concerns was to extend some classical results of graph theory and the notion of hypergraph has been considered as a useful tool to analyze the structure of a system. Further materials regarding graph and hypergraph are available in the literature too [1, 4–8, 10, 19–22].

In many real-world problems, however, relationships among the objects of our interest are more complex than pairwise. Naively squeezing the complex relationships into pairwise ones will inevitably lead to loss of information which can be expected valuable for our learning tasks however. Therefore, we consider using hypergraphs instead to completely represent complex relationships among the objects of our interest, and thus the problem of learning with hypergraphs arises. A powerful technique for partitioning simple graphs is spectral clustering. Therefore, we generalize spectral clustering techniques to hypergraphs. A directed hypergraph is powerful tool to solve the problems that arise in different fields, including computer networks, social networks and collaboration networks [13, 14, 16, 18].

Regarding these points, this paper aims to endow the investigated objects with pairwise relationships, which can be illustrated as digraphs. Our main contribution in this paper is to generalize the powerful methodology of spectral clustering which originally operates on directed graphs to dihypergraphs and further develop properties for dihypergraph clustering approach. We need to organized our data in every wireless sensor hypernetwork according to our goals and so must obtained a special dihypergraph, hence, convert of undirected hypergraphs to dihypergraphs is an important research. Indeed, we consider a wireless sensor hypernetwork as a undirected hypergraph, convert to a regular dihypergraph and via positive relation construct a digraph as a WSN and so organize it.

2. Preliminaries

In this section, we recall some definitions and results are indispensable to our research paper.

Definition 2.1 ([1]). Let $H = \{x_1, x_2, \ldots, x_n\}$ be a finite set and for any $1 \leq i \leq n$, $E_i \subseteq H$. $\mathcal{H}' = (H, E_1, E_2, \ldots, E_m) = (H, \{E_i\}_{i=1}^{m})$ is called a hypergraph on H, if

(i) for all $1 \leq i \leq m$, $E_i \neq \emptyset$;
(ii) $\bigcup_{i=1}^{m} E_i = H$.
A *simple* hypergraph (*Sperner family*) is a hypergraph $\mathcal{H}' = (H, \{E_i\}_{i=1}^{m})$ such that
(iii) $E_i \subset E_j \Longrightarrow i = j$.

The elements $x_1, x_2, \ldots, x_n$ of H are called *vertices*, and the sets $E_1, E_2, \ldots, E_m$ are the *edges* (hyperedges) of the hypergraph. For any $1 \leq k \leq m$ if $|E_k| \geq 2$, then E_k is represented by a solid line surrounding its vertices, if $|E_k| = 1$ by a cycle on the element (loop). If for all $1 \leq k \leq m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph.

Definition 2.2 ([6]). Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^{n})$ be a hypergraph. Define a binary relation η on H as follows: $\eta_1 = \{(x, x) \mid x \in H\}$ and for every integer $k > 1$,

$$x\, \eta_k\, y \Leftrightarrow \exists\ E_i^s, \text{ such that } \{x, y\} \subseteq E_i^s, \text{ where } k = |E_i^s| = \min\{|E_t|;\, x, y \in E_t\}$$

and for all $1 \leq i,\, j \leq n$, there is no $E_i \neq E_i^s$, or $E_j \neq E_i^s$, such that $x \in E_i$, $y \in E_j$ and $|E_i| < k$, $|E_j| < k$. Obviously $\eta = \bigcup_{k \geq 1} \eta_k$ is a reflexive and symmetric relation on G. Let η^* be the *transitive closure* of η (the smallest transitive relation such that contains η).

Theorem 2.3 ([6]). *Let $\mathcal{H}' = (H, \{E_x\}_{x \in G})$ be a hypergraph, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\eta = \eta^*$. Then for any $i \in \mathbb{N}^*$, there exists an operation $*_i$ on H/η such that $H'/\eta = (H/\eta, *_i)$ is a graph.*

Definition 2.4 ([3]). Let G be a set and $F \subseteq P^*(G) \times P(G)$, where $P^*(G) = P(G) \smallsetminus \emptyset$ and $P(G)$ is the power set of G. Then $F = (T(F), H(F))$ is called a directed hyperedge or hyperarc, if $T(G) \cap H(G) = \emptyset$, where $T(G)$ is called the tail of G and $H(G)$ is called its head. A hypergraph $\mathcal{G}' = (G, \{F_i\}_{i=1}^{n} = \{(T(F_i), H(F_i))\}_{i=1}^{n})$ is called a directed hypergraph (dihypergraph) if for any $1 \leq i \leq n$, F_i is a directed hyperedge.

3. Relations between Dihypergraphs and Undirected Hypergraphs

In this section, we define the concept of derivable directed hypergraphs and obtain directed hypergraphs from undirected hypergraphs. Moreover, we find the upper bound for derivable directed hypergraphs from every undirected hypergraphs.

Definition 3.1. Let $\mathcal{G}' = (G, \{F_i\}_{i=1}^n)$ be an dihypergraph. Then we say $\mathcal{G}'$ is a simple dihypergraph, if for any $x \in G$, there exists a unique $1 \le i \le n$ such that $x \in T(F_i) \cup H(F_i)$.

Theorem 3.2. *From every undirected hypergraph $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$, can construct at least a dihypergraph $\mathcal{G}' = (G, \{F_i\}_{i=1}^m)$ such that*

(i) $G = H$,
(ii) $m = n$,
(iii) *for any* $1 \le i \le n$, $T(F_i) \cup H(F_i) = E_i$.

Proof. Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$ be an undirected hypergraph. Then for every $1 \le i \le n$, we consider $E_i = \{x^i \mid 1 \le i \le |E_i|\}$. Now, for every $1 \le i \le n$ define an equivalence relation R_i on E_i in such a way that $|E_i/R_i| = 2$. Thus, there exist x^i, $y^i \in E_i$ such that $E_i = R_i(x^i) \cup R_i(y^i)$ and for every $1 \le i \le n$, we set $F_i = (T(F_i) = R_i(x^i), H(F_i) = R_i(y^i))$. Clearly for any $1 \le i \le n, T(F_i) \cup H(F_i) = E_i$ and $\mathcal{G}' = (H, \{F_i\}_{i=1}^n)$ is a directed hypergraph. □

Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$ be an undirected hypergraph. We will call the dihypergraph $\mathcal{G}' = (G, \{F_i\}_{i=1}^m)$ which satisfied in Theorem 3.2, by derivable dihypergraph.

Example 3.3. Let $H = \{a, b, c, f, g, h, k\}$. Consider the undirected hypergraph $\mathcal{H}' = (H, E_1, E_2, E_3)$ in Fig. 1.

Now, we consider $F_1 = (\{a\}, \{b, c\}), F_2 = (\{c\}, \{f\})$ and $F_3 = (\{f\}, \{g, h, k\})$. Thus, we obtain the dihypergraph $\mathcal{G}'$ in Fig. 2.

Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be an undirected hypergraph. We will denote the set of all derivable simple dihypergraphs which obtained from undirected hypergraph $\mathcal{H}'$ by $\mathcal{D}_{\text{sh}} = \{\mathcal{G}' = (H, \{F_i\}_{i=1}^m) \mid \mathcal{G}'$ is a derivable simple dihypergraph$\}$ and $\mathcal{D}_{\text{sh}}^{(q)} = \{\mathcal{G}' = (G, \{(T(F_i), H(F_i))\}_{i=1}^m) \in \mathcal{D}_{\text{sh}} \mid |T(F_i)| = q\}$.

Theorem 3.4. *Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be an undirected hypergraph and $|H| = n$.*

(i) $|\mathcal{D}_{\text{sh}}^{(1)}| = \prod_{j=1}^{m} t^{(1,j)} = \prod_{i=1}^{m} |E_i|$.

(ii) *If* $1 \le j \le m$, *have* $t^{(2,j)} = \begin{cases} \binom{|E_j|}{2} \sum_{k=\lceil \frac{|E_j|-2}{2} \rceil}^{|E_j|-2} \binom{|E_j|-2}{k}, & \text{if } |E_j| > 2, \\ 0, & \text{if } |E_j| \le 2. \end{cases}$.

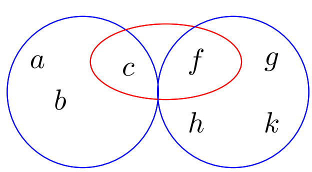


Fig. 1. Undirected hypergraph $\mathcal{H}' = (H, E_1, E_2, E_3)$.

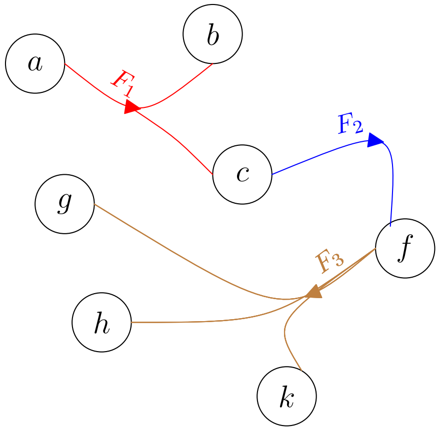


Fig. 2. Dihypergraph $\mathcal{G}' = (G, F_1, F_2, F_3)$.

(iii) $1 \leq |\mathcal{D}^{(2)}_{\text{sh}}| \leq \prod_{j=1}^{m} t^{(2,j)}$.

(iv) $|\mathcal{D}^{(|E_j|-1)}_{\text{sh}}| = \prod_{j=1}^{m} t^{(|E_i|-1,j)} = \prod_{i=1}^{m} |E_i|$.

Proof. Let $1 \leq i_0 \leq m$ and E_{i_0} be an arbitrary hyperedge of undirected hypergraph $\mathcal{H}'$. (i) By Theorem 3.2, we construct $F^{(1)}_{i_0}$ in such a way that $|T(F^{(1)}_{i_0})| = 1, |H(F^{(1)}_{i_0})| = |E_{i_0}| - 1$ and $T(F^{(1)}_{i_0}) \cap H(F^{(1)}_{i_0}) = \emptyset$. If $A^{(1)} = \{F^{(1)}_{i_0}\}$, a simple computation shows that $|A^{(1)}| = \binom{|E_{i_0}|}{|T(F^{(1)}_{i_0})|}\binom{|E_{i_0}|-1}{|H(F^{1}_{i_0})|} = \binom{|E_{i_0}|}{|T(F^{(1)}_{i_0})|} = |E_{i_0}|$. Since for $1 \leq i_0 \leq m$, E_{i_0} is an arbitrary hyperedge, we get $|\mathcal{D}^{(1)}_{\text{sh}}| = \prod_{i=1}^{m} t^{(1,j)} = \prod_{i=1}^{m} |E_i|$.

(ii), (iii) If $|E_{i_0}| \leq 2$, then $T(F^{(2)}_{i_0}) = \emptyset$ or $H(F^{(2)}_{i_0}) = \emptyset$, which is a contradiction. But for $|E_{i_0}| \geq 3$, in a similar way, by Theorem 3.2, we construct $F^{(2)}_{i_0}$ in such a way that $|T(F^{(2)}_{i_0})| = 2$, $|H(F^{(2)}_{i_0})| = |E_{i_0}| - 2$ and $T(F^{(2)}_{i_0}) \cap H(F^{(2)}_{i_0}) = \emptyset$. If $A^{(2)} = \{F^{(2)}_{i_0}\}$, a simple computation shows that:

$$|A^{(2)}| = \binom{|E_{i_0}|}{|T(F^{(2)}_{i_0})|}\left[\binom{|E_{i_0}|-2}{|H(F^2_{i_0})|} + \binom{|E_{i_0}|-2}{|H(F^2_{i_0})|-1} + \binom{|E_{i_0}|-2}{|H(F^2_{i_0})|-2} + \cdots + \binom{|E_{i_0}|-2}{|H(F^2_{i_0})|-(|E_{i_0}|-3)}\right].$$

Since $|T(F^{(2)}_{i_0})| = 2$, we can consider $H(F^{(2)}_{i_0}) = B_1 \cup B_2$ in such a way that $B_1 \cap B_2 = \emptyset$ and $1 \leq |B_1| \leq \lceil \frac{|E_{i_0}|-2}{2} \rceil$. For any $1 \leq k \leq \lceil \frac{|E_{i_0}|-2}{2} \rceil$, $\binom{|E_{i_0}|-2}{k} = \binom{|E_{i_0}|-2}{|E_{i_0}|-(2+k)}$

and $|T(F_{i_0}^{(2)})| = 2$ imply that $B_1 \cong B_2$, so

$$|A^{(2)}| = \binom{|E_{i_0}|}{|T(F_{i_0}^{(2)})|}\left[\binom{|E_{i_0}|-2}{|H(F_{i_0}^2)|} + \binom{|E_{i_0}|-2}{|H(F_{i_0}^2)|-1} + \binom{|E_{i_0}|-2}{|H(F_{i_0}^2)|-2} + \cdots \right.$$
$$\left. + \binom{|E_{i_0}|-2}{|H(F_{i_0}^2)| - \lceil\frac{|E_{i_0}|-2}{2}\rceil}\right].$$

Since for $1 \leq i_0 \leq m$, E_{i_0} is an arbitrary hyperedge, we get $1 \leq |\mathcal{D}_{\mathrm{sh}}^{(2)}| \leq \prod_{j=1}^m t^{(2,j)}$.

(iv) It is similar to (i). $\square$

Theorem 3.5. *Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be an undirected hypergraph. Then*

$$1 \leq |\mathcal{D}_{\mathrm{sh}}^{(3)}| \leq \begin{cases} \prod_{j=1}^m t^{(3,j)} & \text{if } |E_j| > 3, \\ 0 & \text{if } |E_j| \leq 3, \end{cases} \qquad \text{where, } t^{(3,j)} = \binom{|E_j|}{3}(2^{(|E_j|-3)} - 1 + u)$$

and

$$u = \sum_{\substack{a+b+c=|E_j|-3 \\ 1\leq c\leq b\leq a\leq |E_j|-5}} \binom{|E_j|-3}{a}\binom{|E_j|-3-a}{b}\binom{c}{c}\Bigg).$$

Proof. Let $1 \leq i_0 \leq m$ and E_{i_0} be an arbitrary hyperedge of undirected hypergraph $\mathcal{H}'$. By Theorem 3.2, we construct $F_{i_0}^3$ in such a way that $|T(F_{i_0}^3)| = 3$, $|H(F_{i_0}^3)| = |E_{i_0}| - 3$, $T(F_{i_0}^3) \cap H(F_{i_0}^3) = \emptyset$ and $A^3 = \{F_{i_0}^3\}$. If $|E_{i_0}| \leq 3$, then $T(F_{i_0}^{(3)}) = \emptyset$ or $H(F_{i_0}^{(3)}) = \emptyset$, so $|\mathcal{D}_{\mathrm{sh}}^{(3)}| = 0$.

Since $|T(F_{i_0}^{(2)})| = 3$, we can consider two partitions for $H(F_{i_0}^{(3)})$ as $P_1 = \{B_1, B_2\}$ or $P_2 = \{C_1, C_2, C_3\}$. Let P_1 be a partition of $H(F_{i_0}^{(3)})$. Then $|T(F_{i_0}^{(3)})| = 3$ implies that $B_1 \ncong B_2$ and so

$$|P_1| = \binom{|E_{i_0}|}{|T(F_{i_0}^{(3)})|}\left[\binom{|E_{i_0}|-3}{|H(F_{i_0}^3)|} + \binom{|E_{i_0}|-3}{|H(F_{i_0}^3)|-1} + \binom{|E_{i_0}|-3}{|H(F_{i_0}^3)|-3} + \cdots \right.$$
$$\left. + \binom{|E_{i_0}|-3}{1}\right].$$

If P_2 is a partition of $H(F_{i_0}^{(3)})$. Then

$$|P_2| = |\{(c_1, c_2, c_3) \mid c_1 + c_2 + c_3 = |E_{i_0}| - 3 \text{ and } 1 \leq c_1 \leq c_2 \leq c_3 \leq |E_j| - 5|\}.$$

Computations show that

$$P_2 = \Big\{(|E_{i_0}|-5,1,1),(|E_{i_0}|-6,2,1),(|E_{i_0}|-7,3,1),\ldots,$$
$$\left(\left\lfloor\frac{|E_{i_0}|}{3}\right\rfloor,\left\lfloor\frac{|E_{i_0}|-2}{3}\right\rfloor,\left\lfloor\frac{|E_{i_0}|-4}{3}\right\rfloor\right)\Big\}$$

and so

$$|P_2| = \binom{|E_{i_0}|}{|T(F_{i_0}^{(3)})|}\left[\binom{|E_{i_0}|-3}{|E_{i_0}|-5}\binom{2}{1}+\binom{|E_{i_0}|-3}{|E_{i_0}|-6}\binom{3}{2}+\binom{|E_{i_0}|-3}{|E_{i_0}|-7}\binom{4}{3}\right.$$
$$\left. +\cdots+\binom{|E_{i_0}|-3}{\left\lfloor\frac{|E_{i_0}|}{3}\right\rfloor}\binom{\left\lfloor\frac{|E_{i_0}|-2}{3}\right\rfloor+\left\lfloor\frac{|E_{i_0}|-4}{3}\right\rfloor)}{\left\lfloor\frac{|E_{i_0}|-2}{3}\right\rfloor}\right]$$
$$= \binom{|E_{i_0}|}{|T(F_{i_0}^{(3)})|}\left[\sum_{(c_1,c_2,c_3)\in P_2}\binom{|E_{i_0}|-3}{c_1}\binom{c_2+c_3}{c_2}\right].$$

Since $P_1 \cap P_2 = \emptyset$, we get that $|t^{(3,j)}| = |P_1| + |P_2|$. Since for $1 \le i_0 \le m$, E_{i_0} is an arbitrary hyperedge, we get $1 \le |\mathcal{D}_{\mathrm{sh}}^{(3)}| \le \prod_{j=1}^{m} t^{(3,j)}$. ◻

Example 3.6. Let $H = \{a,b,c,d,e,f\}$. Consider the undirected hypergraph $\mathcal{H}' = (H, E_1, E_2)$ in Fig. 3.

By Theorem 3.5, for hyperedge $E_1 = \{a,b,c\}$, we have

$$|t^{(1,1)}| = 3, |t^{(2,1)}| = \binom{3}{2}\binom{1}{1} = 3 \text{ and for every } \ k \ge 3, t^{(k,1)} = 0.$$

For hyperedge $E_2 = \{c,d,e,f\}$, we have

$$|t^{(1,2)}| = 4, |t^{(2,2)}| = \binom{4}{2}\left(\binom{2}{1}+\binom{2}{2}\right) = 18, |t^{(3,2)}| = \binom{4}{3} = 4.$$

and for every $k \ge 4, t^{(k,3)} = 0$. Hence,

$$|\mathcal{D}_{\mathrm{sh}}^{(1)}| = \prod_{j=1}^{2} t^{(1,j)} = 3 \times 4 = 12, |\mathcal{D}_{\mathrm{sh}}^{(2)}| = \prod_{j=1}^{2} t^{(1,j)} = 3 \times 18 = 54$$

and for every $k \ge 3$, $|\mathcal{D}_{\mathrm{sh}}^{(k)}| = 0$. So $|\mathcal{D}_{\mathrm{sh}}| \ge 12 + 54 = 66$.

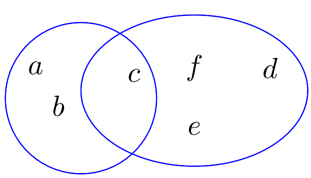


Fig. 3. Undirected hypergraph $\mathcal{H}' = (H, E_1, E_2)$.

Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be an undirected hypergraph, $E_j \in \{E_i\}_{i=1}^m$ be an arbitrary hyperedge of $\mathcal{H}'$ and $3 \leq q$, $r \in \mathbb{N}$. Then we will denote $|E_j| = e_j$, $e_j - q = f_j$, $\mathcal{A}^{(r)} = \{(x_1, x_2, \ldots, x_r) \mid x_1 + x_2 + \cdots + x_r = f_j \text{ and } 1 \leq x_1 \leq x_2 \leq \cdots \leq x_r \leq f_j - 2\}$ and

$$\varphi_j(f_j, x_1, \ldots, x_r) = \binom{f_j}{x_1}\binom{f_j - x_1}{x_2} \cdots \binom{f_j - \sum_{i=1}^{r-1} x_i}{x_{r-1}}\binom{x_r}{x_r}.$$

Corollary 3.7. *Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be an undirected hypergraph.*

(i) *If $q \geq 3$, then $1 \leq |\mathcal{D}_{\text{sh}}^{(q)}| \leq \prod_{j=1}^m t^{(q,j)}$, where for any $1 \leq j \leq m$,*

$$t^{(q,j)} = \begin{cases} \binom{e_j}{q}\left[(2^{e_j - q} - 1) + \sum_{r=3}^{f_j} \sum_{\mathcal{A}^{(r)}} \varphi_j(f_j, x_1, \ldots, x_{n-j})\right], & \text{if} \quad |E_j| > q, \\ 0, & \text{if} \quad |E_j| \leq q. \end{cases}.$$

(ii) $|\mathcal{D}_{\text{sh}}| \geq \sum_{q=1}^{n-1} \prod_{j=1}^m t^{(q,j)}$.

Theorem 3.8. *From every dihypergraph $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$, can construct at least an undirected hypergraph $\mathcal{G}' = (G, \{F_i\}_{i=1}^m)$ such that*

(i) $G = H$,
(ii) $m = n$,
(iii) *for any $1 \leq i \leq n$, $T(F_i) \cup H(F_i) = E_i$.*

Proof. Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$ be a dihypergraph. Then for every $1 \leq i \leq n$, we consider $F_i = T(E_i) \cup H(E_i)$. It is easy to see that $\bigcup_{i=1}^n F_i = H$ and so $\mathcal{G}' = (G, \{F_i\}_{i=1}^m)$ is a undirected hypergraph. □

Example 3.9. Let $H = \{a, b, c, d, e\}$. Consider the dihypergraph $\mathcal{H}' = (H, E_1, E_2)$ in Fig. 4.

Since $F_1 = (\{a, b\}, \{c, d\})$ and $F_2 = (\{b\}, \{e\})$, we get $E_1 = \{a, b, c, d\}$, $E_2 = \{b, e\}$ and so the undirected hypergraph $\mathcal{G}' = (H, \{E_i\}_{i=1}^2)$ in Fig. 5.

Positive relation on dihypergraphs

In this section, we will define an equivalence relation on dihypergraph $\mathcal{H}'$ as α, in such a way that, we can construct a quotient $\mathcal{H}'/\alpha$ and show that it is a digraph. Moreover, the concept of fundamental digraph is defined and it is shown that some digraphs are fundamental graphs.

Definition 3.10. Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ be a dihypergraph. Then define, $\alpha_1 = \{(x, x) \mid x \in H\}$ and for every integer $n \geq 2$, α_n is defined as follows:

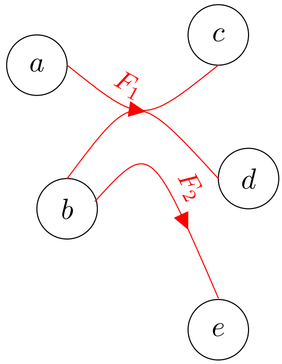


Fig. 4. Dihypergraph $\mathcal{G}' = (G, F_1, F_2)$.

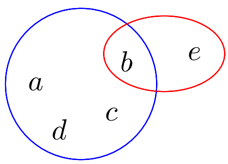


Fig. 5. Undirected hypergraph $\mathcal{H}' = (H, E_1, E_2)$.

$x\alpha_n y \Longleftrightarrow \exists 1 \leq k \leq m$ such that $\{x, y\} \subseteq T(E_k) \cup H(E_k)$, where for any $1 \leq i \neq k \leq m, x, y \notin T(E_i) \cup H(E_i)$ and $n = |T(E_k) \cup H(E_k)|$.

Obviously the relation $\alpha = \bigcup_{n \geq 1} \alpha_n$ is an equivalence relation on $\mathcal{H}'$. We denote the set of all equivalence classes of α by $\mathcal{H}'/\alpha$. Hence $\mathcal{H}'/\alpha = \{\alpha(x) \mid x \in H\}$.

Example 3.11. Let $H = \{a, b, c, d, e, f, g, h\}$. Consider the dihypergraph $\mathcal{H}' = (H, E_1, E_2, E_3, E_4)$ in Fig. 6.

Since

$$E_1 = (\{a, b\}, \{c, d, e\}), E_2 = (\{c, d\}, \{f, g\}),$$

$$E_3 = (\{d\}, \{h\}) \text{ and } E_4 = (\{h\}, \{e\}),$$

by definition, we get that $\alpha_1 = \{(x, x) \mid x \in H\}$. Also,

$$\{a, b\} \subseteq T(E_1) \cup H(E_1), \text{such that for all }, 1 \leq i \neq k \leq 4, x, y \notin T(E_i) \cup H(E_i),$$

$$\{c\} \subseteq T(E_1) \cup H(E_1), \text{and } \{c\} \subseteq T(E_2) \cup H(E_2),$$

$$\{d\} \subseteq T(E_1) \cup H(E_1), \{d\} \subseteq T(E_2) \cup H(E_2) \text{and } \{d\} \subseteq T(E_3) \cup H(E_3),$$

$$\{e\} \subseteq T(E_1) \cup H(E_1), \text{and } \{e\} \subseteq T(E_4) \cup H(E_4),$$

$$\{h\} \subseteq T(E_3) \cup H(E_3), \text{and } \{h\} \subseteq T(E_4) \cup H(E_4),$$

$$\{a, b\} \subseteq T(E_2) \cup H(E_2), \text{such that for all }, 1 \leq i \neq k \leq 4, x, y \notin T(E_i) \cup H(E_i),$$

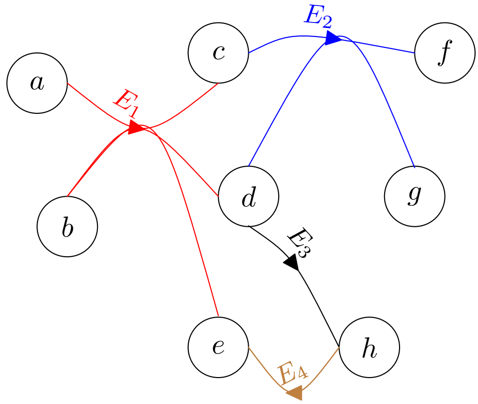


Fig. 6. Dihypergraph.

imply that $a\alpha_5 b$, $c\alpha_1 c$, $d\alpha_1 d$, $e\alpha_1 e$, $h\alpha_1 h$ and $f\alpha_4 g$ and so

$$\alpha(a) = \{a, b\}, \alpha(c) = \{c\}, \alpha(d) = \{d\},$$

$$\alpha(e) = \{e\}, \alpha(f) = \{f, g\} \text{ and } \alpha(h) = \{h\}.$$

Hence, we obtain $\mathcal{H}'/\alpha = \{\alpha(a), \alpha(c), \alpha(d), \alpha(e), \alpha(f), \alpha(h)\}$.

Theorem 3.12. *Let $\mathcal{H}' = (H, \{E_i\}_{i=1}^n)$ be a dihypergraph. Then there exists a relation "$*$" on $\mathcal{H}/\alpha$ such that $(\mathcal{H}'/\alpha, *)$ is a digraph.*

Proof. Let $\alpha(x)$, $\alpha(y) \in \mathcal{H}'/\alpha$. Then define an operation "$*$" on $\mathcal{H}'/\alpha$ by

$$\alpha(x) * \alpha(y) = \begin{cases} (\overrightarrow{\alpha(x), \alpha(y)}) & \text{if } \exists 1 \leq k \leq n, \alpha(x) \cap T(E_k) \neq \emptyset \quad \text{and} \\ & \alpha(y) \cap H(E_k) \neq \emptyset, \\ \overrightarrow{\emptyset} & \text{otherwise,} \end{cases}$$

where for any x, $y \in H$, $(\overrightarrow{\alpha(x), \alpha(y)})$ is represented as an ordinary directed edge from vertex $\alpha(x)$ to vertex $\alpha(y)$ and $\overrightarrow{\emptyset} = \alpha(x) * \alpha(y)$ means that there is no edge between $\alpha(x)$ and vertex $\alpha(y)$ (so $\overrightarrow{\alpha(x)} = \overrightarrow{\emptyset}$ and has not a loop). We show that a relation "$*$" is a well-defined relation. Let $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$. Then there exists uniquely $1 \leq k$, $s \leq n$ such that $\{x, x'\} \subseteq T(E_k) \cup H(E_k)$, and $\{y, y'\} \subseteq T(E_s) \cup H(E_s)$. If $\alpha(x) * \alpha(y) = (\overrightarrow{\alpha(x), \alpha(y)})$, then by definition, there exists $1 \leq m \leq n$ such that $\alpha(x) \cap T(E_m) \neq \emptyset$ and $\alpha(y) \cap H(E_m) \neq \emptyset$. It follows that $x \in T(E_k) \cap T(E_m) \neq \emptyset$ and so $\alpha(x') \cap T(E_m) \neq \emptyset$. In a similar way $\alpha(y') \cap H(E_m) \neq \emptyset$ and so $\alpha(x') * \alpha(y') = (\overrightarrow{\alpha(x'), \alpha(y')}) = (\overrightarrow{\alpha(x), \alpha(y)})$. If $\alpha(x) * \alpha(y) = \overrightarrow{\emptyset}$, then for any $1 \leq m \leq n$, $\alpha(x) \cap T(E_m) = \emptyset$ or $\alpha(y) \cap H(E_m) = \emptyset$. It follows that $T(E_k) \cap T(E_m) = \emptyset$ and so $\alpha(x') \cap T(E_m) = \emptyset$. In a similar way

$\alpha(y') \cap H(E_m) = \emptyset$ and so $\alpha(x') * \alpha(y') = (\overrightarrow{\alpha(x'), \alpha(y')}) = (\overrightarrow{\alpha(x), \alpha(y)})$. It is easy to see that $(\mathcal{H}/\alpha, *)$ is a digraph. $\square$

Example 3.13. Consider the dihypergraph $\mathcal{H}' = (H, E_1, E_2, E_3, E_4)$ in Fig. 6, in Example 3.11. Since

$$\alpha(a) \cap T(E_1) \neq \emptyset \ \text{ and } \alpha(c) \cap H(E_1) \neq \emptyset,$$
$$\alpha(a) \cap T(E_1) \neq \emptyset \ \text{ and } \alpha(d) \cap H(E_1) \neq \emptyset,$$
$$\alpha(a) \cap T(E_1) \neq \emptyset \ \text{ and } \alpha(e) \cap H(E_1) \neq \emptyset,$$
$$\alpha(c) \cap T(E_2) \neq \emptyset \ \text{ and } \alpha(f) \cap H(E_2) \neq \emptyset,$$
$$\alpha(d) \cap T(E_2) \neq \emptyset \ \text{ and } \alpha(f) \cap H(E_2) \neq \emptyset,$$
$$\alpha(d) \cap T(E_3) \neq \emptyset \ \text{ and } \alpha(h) \cap H(E_3) \neq \emptyset,$$
$$\alpha(h) \cap T(E_4) \neq \emptyset \ \text{ and } \alpha(e) \cap H(E_4) \neq \emptyset,$$

we get that

$$\alpha(a) * \alpha(c) = (\overrightarrow{\alpha(a), \alpha(c)}), \alpha(a) * \alpha(d) = (\overrightarrow{\alpha(a), \alpha(d)}),$$
$$\alpha(a) * \alpha(e) = (\overrightarrow{\alpha(a), \alpha(e)}), \alpha(c) * \alpha(f) = (\overrightarrow{\alpha(c), \alpha(f)}),$$
$$\alpha(d) * \alpha(f) = (\overrightarrow{\alpha(d), \alpha(f)}), \alpha(d) * \alpha(h) = (\overrightarrow{\alpha(d), \alpha(h)}),$$
$$\text{and } \alpha(h) * \alpha(e) = (\overrightarrow{\alpha(h), \alpha(e)}).$$

So, by Theorem 3.12, we obtain the directed graph $(\mathcal{H}'/\alpha, *)$ in Fig. 7.

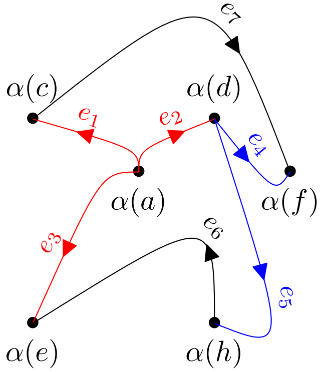


Fig. 7. Digraph $(\mathcal{H}/\alpha, *)$.

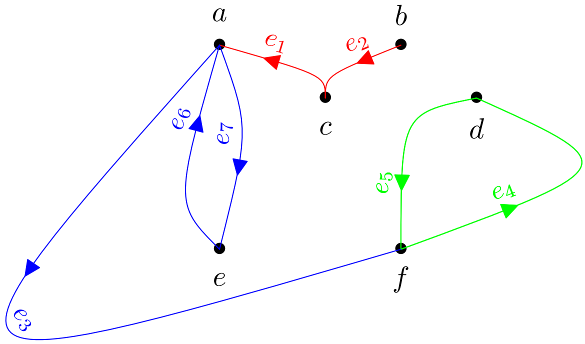


Fig. 8. Digraph G.

Definition 3.14. A digraph $G = (V, E)$ is said to be:

(i) an *α-derivable digraph* if there exists a nontrivial dihypergraph $\mathcal{H}' = (H, \{E_k\}_{k=1}^{n})$ such that $\mathcal{H}'/\alpha = (H, \{E_k\}_{k=1}^{n})/\alpha \cong G = (V, E)$ and $\mathcal{H}'$ is called an associated dihypergraph with digraph G. In other words, it is equal to the quotient of nontrivial dihypergraph on α up to isomorphic;

(ii) an α-self derivable digraph, if it is an α-derivable digraph by itself.

Lemma 3.15. *Any digraph is a trivial dihypergraph.*

Proof. Let $G = (V, E)$ be a digraph, where $G = \{a_1, a_2, \ldots, a_n\}$ and $|E| = m$. We rearrange $E = \{e_{ij} = (a_i, a_j) \mid 1 \le i, j \le n\}$. Now consider $\mathcal{H}' = (V, \{E_{ij}\}_{1\le i,j\le n})$ in such a way that for any $1 \le i, j \le m$, $E_{ij} = (\{a_i\}, \{a_j\})$. It is easy to see that $\mathcal{H}'$ is a dihypergraph and is trivial. $\square$

Example 3.16. Consider the digraph $G = (V, E)$ in Fig. 8, where $V = \{a, b, c, d, e, f\}$.

Now we construct a dihypergraph $\mathcal{H}' = (H, \{E_i\}_{i=1}^{5})$ in Fig. 9.

Clearly $\mathcal{H}' = (H, \{E_i\}_{i=1}^{5})$ is a nontrivial dihypergraph, where $H = \{a, b, c, d, e, f\}$ and

$$E_1 = (\{a\}, \{f, e\}), E_2 = (\{c\}, \{b\}), E_3 = (\{e, c\}, \{a\}),$$

$$E_4 = (\{f\}, \{d\}) \text{ and } E_5 = (\{d\}, \{f\}).$$

Computations show that for any $x \in H$, $\alpha(x) = \{x\}$. It is easy to see that digraph $\mathcal{H}/\alpha$ is obtained in Fig. 10.
Clearly $\mathcal{H}'/\alpha \cong G$. Since $|H| = |V|$, we have digraph $G = (V, E)$ is an α-self derivable digraph.

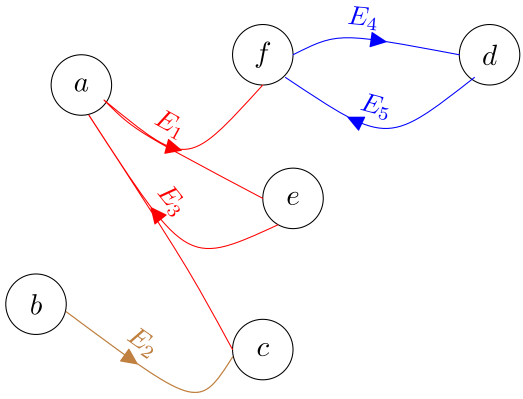


Fig. 9. Dihypergraph.

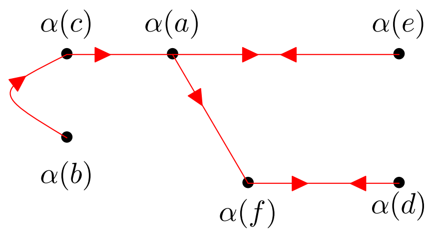


Fig. 10. Digraph G.

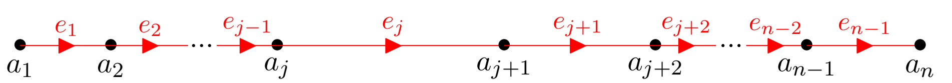


Fig. 11. Linear ditree Dt_n^l.

Let $V = \{a_1, a_2, \dots, a_n\}$. Then we denote the linear directed tree (ditree) on V in Fig. 11 and denote it by Dt_n^l.

Theorem 3.17. *Let* $2 \leq n \in \mathbb{N}$. *Then*

(i) Dt_n^l *is an* α*-derivable digraph.*
(ii) Dt_2^l *is not an* α*-self derivable digraph.*

Proof. (i) Let $V = \{a_1, a_2, \dots, a_n\}$. Then for any $a, b \notin V$ consider $E_1 = (\{a_1, a\}, \{a_2\})$, for any $2 \leq i \leq n-2, E_i = (\{a_i\}, \{a_{i+1}\})$ and $E_{n-1} = (\{a_{n-1}\}, \{a_n, b\})$. If $H = V \cup \{a, b\}$, then $\mathcal{H} = (H, \{E_i\}_{i=1}^{n-1})$ is a nontrivial dihypergraph. It can see that for $\alpha(a_1) = \alpha(a) = \{a_1, a\}, \alpha(a_n) = \alpha(b) = \{a_n, b\}$ and for any $2 \leq i \leq n-1, \alpha(a_i) = \{a_i\}$. Since for any $1 \leq i \leq n$ which is an odd,

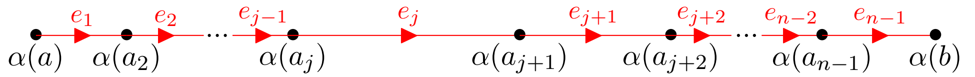


Fig. 12. Linear ditree Dt_n^l.

we have $\alpha(a_i) \cap T(E_i) \neq \emptyset$ and for any $1 \leq i \leq n$ which is an even, we have $\alpha(a_i) \cap H(E_i) \neq \emptyset$, we get that $\alpha(a_i) * \alpha(a_{i+1}) = \overrightarrow{(\alpha(a_i), \alpha(a_{i+1}))} = e_{ii+1}$. Hence, we obtain a digraph in Fig. 12. Clearly, $\mathcal{H}'/\alpha \cong Dt_n^l$ and so for any $n \geq 2, Dt_n^l$ is an α-derivable digraph.

(ii) Let Dt_2^l be an α-self derivable digraph. Then there exists an associated dihypergraph $\mathcal{H}' = (H, \{E_i\}_{i=1}^m)$ with digraph Dt_2^l such that $\mathcal{H}'/\alpha \cong Dt_2^l$ and $|H| = 2$. Suppose that $H = \{x, y\}$, since $\mathcal{H}'$ is a nontrivial dihypergraph, there exists $1 \leq i \leq m$ such that $E_i = (T(E_i), H(E_i))$, $|T(E_i)| \geq 2$ or $|H(E_i)| \geq 2$. $|H| = 2$ implies $|T(E_i)| = 2$ or $|H(E_i)| = 2$ and follows that $T(E_i) \cap H(E_i) \neq \emptyset$ which is a contradiction. $\square$

Corollary 3.18. *Let $2 \leq n \in \mathbb{N}$. Then Dt_n^l is an α-derivable digraph but is not an α-self derivable digraph.*

We introduce digraphs G_1 and G_2 in Fig. 13.

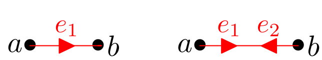
From now on, we apply these digraphs in Fig. 13, in the following theorem.

Theorem 3.19. *Let $G = (V, E)$ be a digraph. Then the following properties hold.*

(i) *$G = G_2$ is not an α-derivable digraph.*
(ii) *If $|V| \geq 3$ and there exists at least a vertex $x \in G$ such that $\deg^+(x) \neq \deg^-(x)$, then G is an α-derivable digraph.*
(iii) *If $G \cong C_n^*$, then G is not an α-derivable digraph.*

Proof. (i) Let $G_2 = (V_2, E_2)$ be an α-derivable digraph. Then there exist a nontrivial hypergraph $\mathcal{H}' = (H = \{x_1, x_2, \ldots, x_n\}, \{E_i\}_{i=1}^m)$ and $1 \leq k,\ l \leq n$ such that $\{\alpha(x_k), \alpha(x_l)\} = \mathcal{H}'/\alpha \cong G_2$. Since $|V_2| = 2$, we get $m = 2$. In addition, $\alpha(x_k) * \alpha(x_l) = \overrightarrow{(\alpha(x_k), \alpha(x_l))}, \alpha(x_l) * \alpha(x_k) = \overrightarrow{(\alpha(x_l), \alpha(x_k))}$ implies that there exists $1 \leq i \leq m$ such that $\alpha(x_k) \cap T(E_i) \neq \emptyset$ and $\alpha(x_l) \cap H(E_i) \neq \emptyset$ or $\alpha(x_k) \cap H(E_i) \neq \emptyset$ and $\alpha(x_l) \cap T(E_i) \neq \emptyset$. It follows that $H(E_1) \cap T(E_2) \neq \emptyset$ and $H(E_2) \cap T(E_1) \neq \emptyset$ and so $|\mathcal{H}'/\alpha| \geq 3$ which is a contradiction.

(ii) Let $G = (V, E), V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{e_{ij} \mid 1 \leq i \neq j \leq n\}$, where $e_{ij} = (x_i, x_j)$. If there exists $1 \leq k \leq n$ such that $\deg^+(x_k) \neq \deg^-(x_k)$, then



(a) Digraph Dt_2^l (b) Digraph G_2

Fig. 13. Digraphs G_1 and G_2.

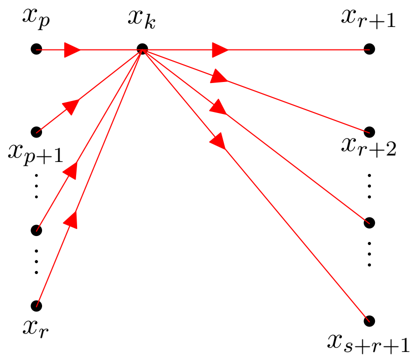


Fig. 14. Disubgraph G'.

without loss of generality, we suppose that $\deg^+(x_k) = r > s = \deg^-(x_k)$. Now, consider the subgraph $G' = (V' = \{x_p, x_{p+1}, \ldots, x_r, x_{r+1}, \ldots, x_{s+r+1}\} \cup \{x_k\}, E')$ in Fig. 14, where $1 \leq p < r$ and the dihypergraph $\mathcal{H}_1 = (H_1, \{E_i\}_{i=p}^{r+s+2})$, where $H_1 = V'$. Now, for any $p \leq i \leq r$, and for any $r+1 \leq j \leq r+s+1$, we construct $E_i = (\{x_i\}, \{x_k\})$, $E_j = (\{x_k\}, \{x_j\})$ and $E_{s+r+2} = (\{x_k\}, \{x_{r+1}, x_{r+2}, \ldots, x_{s+r+1}\})$. Clearly, for any $p \leq i \leq s+r+1, \alpha(x_i) = \{x_i\}$ and

$$\mathcal{H}_1/\alpha = \{\alpha(x_p), \alpha(x_{p+2}), \ldots, \alpha(x_{r+s+1})\}.$$

Computations show that for any $p \leq i \leq r$, $\alpha(x_i) * \alpha(x_k) = (\overrightarrow{\alpha(x_i), \alpha(x_k)})$ and for any $r+1 \leq j \leq r+s+1$, $\alpha(x_k) * \alpha(x_j) = (\overrightarrow{\alpha(x_k), \alpha(x_j)})$. Hence $(\mathcal{H}_1, *) \cong (V', E') = G'$. Now, we consider the dihypergraph $\mathcal{H}' = (H, \{E_i\}_{i=p}^{r+s+1} \cup \{E_{tu}\}_{1 \leq t \neq u}^{n})$ in such a way that $i \notin \{t, u\}$ and for any $1 \leq i \leq n$, $E_{tu} = (\{x_t\}, \{x_u\})$. Clearly, $\mathcal{H}'$ is a nontrivial dihypergraph and for any $1 \leq i \leq n$, $\alpha(x_i) = \{x_i\}$ and for any $1 \leq i \neq j \leq n$, $\alpha(x_i) * \alpha(x_j) = (\overrightarrow{\alpha(x_i), \alpha(x_j)}) = e_{ij}$. It is easy to see that $\mathcal{H}'/\alpha \cong G$.

(iii) Let $G = (V, E)$, $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{e_{ij} \mid 1 \leq i \neq j \leq n\}$, where $e_{ij} = (x_i, x_j)$. Since for any $1 \leq i \neq j \leq n$, $\deg^+(x) = \deg^-(x) = 1$, we rearrange the digraph $G = (V, E)$ in Fig. 15.

If $G = (V, E)$ is an α-derivable digraph, we consider the smallest associated dihypergraph $\mathcal{H} = (H, \{E_i\}_{i=1}^m)$, where there exists $1 \leq t \leq m$ in such a way that $2 \in \{|T(E_t)|, |H(E_t)|\}$ and for any $1 \leq i \neq t \leq m$, $|T(E_t)| = |H(E_t)|$. Since for any $1 \leq i \leq n$, $\deg^+(x_i) = \deg^+(x_i) = 1$, for all $1 \leq i \neq j \leq n$, we get that $\{x_i, x_j\} \nsubseteq T(E_i), \{x_i, x_j\} \nsubseteq H(E_i), \{x_i, x_j\} \nsubseteq T(E_j)$ and $\{x_i, x_j\} \nsubseteq H(E_j)$. Hence, there exists $x' \notin H$ such that $x' \in T(E_t) \cup H(E_t)$ and so $m = n$. In addition, for some $1 \leq k \leq n, x_t \in (T(E_t) \cup H(E_t)) \cap (T(E_k) \cup H(E_k))$ implies that $\alpha(x_t) \neq \alpha(x')$ and for $1 \leq i \leq n, \alpha(x_i) = \{x_i\}$. It follows that $\mathcal{H}'/\alpha = \{\alpha(x'), \alpha(x_1), \ldots, \alpha(x_n)\}$ and so $\mathcal{H}'/\alpha \ncong G$, which is a contradiction. □

Corollary 3.20. *Let $G = (V, E)$ be a digraph. Then every digraph $G \ncong G_2$ and $G \ncong C_n^*$ is an α-derivable digraph.*

Let $V = \{a_1, a_2, \ldots, a_n\}$. Then linear directed tree $Dt_n^l = (V, \{e_i\}_{i=1}^{n-1} \cup \{e\})$ is called a nonsimple linear directed tree (ditree) on V if, there exists at least $1 \leq i \leq n$ such that Dt_n^l has a two-sided edge e_i, is denoted it by St_n^l and is shown in Fig. 16.

Theorem 3.21. *Let $G = (V, E)$ be a digraph. Then St_n^l is an α-self derivable digraph.*

Proof. By definition of digraph St_n^l, there exists $1 \leq j \leq n-1$ such that $e_j = (\overrightarrow{a_j, a_{j+1}})$ and $e = (\overrightarrow{a_{j+1}, a_j})$. We consider $G = \{a_1, a_2, \ldots, a_n\}$, $E_{j-1} = (\{a_{j-1}, a_{j+1}\}, \{a_j\})$, $E_j = (\{a_j\}, \{a_{j+1}\})$ and for any $1 \leq k \leq n-1$, where $t \notin \{j-1, j\}$, $E_k = (\{a_k\}, \{a_{k+1}\})$. Clearly, $\mathcal{H}' = (V, \{E_i\}_{i=1}^{n-1})$ is a nontrivial directed hypergraph and a rutin computations show that for any $1 \leq i \leq n$, $\alpha(a_i) = \{a_i\}$. It is easy to see that $\mathcal{H}'/\alpha \cong G$, so St_n^l an α-derivable digraph and $G = V$ implies that St_n^l is an α-self derivable digraph. $\square$

Example 3.22. Let $V = \{a_1, a_2, a_3, a_4\}$. Consider the nonsimple linear ditree St_4^l in Fig. 17.

Now we construct the dihypergraph $\mathcal{H}' = (V, E_1, E_2, E_3)$ in Fig. 18. Clearly, $\mathcal{H}'/\alpha \cong \mathrm{St}_4^l$ and so St_4^l is an α-self derivable digraph.

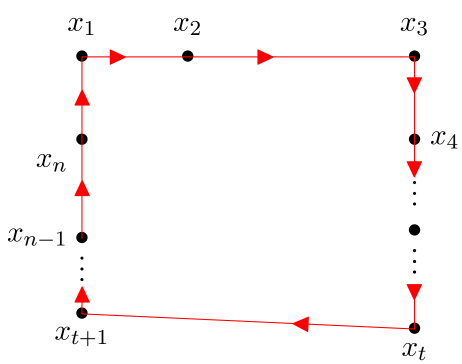


Fig. 15. Directed cycle graph C_n^*.

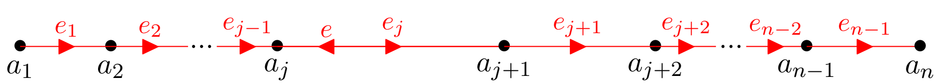


Fig. 16. Nonsimple linear directed tree St_n^l.

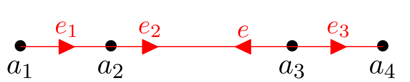


Fig. 17. Nonsimple linear directed tree St_4^l.

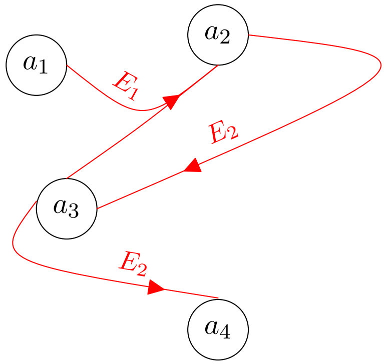


Fig. 18. Dihypergraph $\mathcal{H}' = (G, E_1, E_2, E_3)$.

Corollary 3.23. *Let $G = (V, E)$ be a digraph, $|V| \geq 3$ and $2 \leq k \in \mathbb{N}$. If there exists $x \in V$ such that $k \in \{\deg^+(x), \deg^-(x)\}$, then G is an α-self derivable digraph.*

3.1. *Applications of α-derivable digraphs*

In this subsection, we describe some applications of the concept of α-derivable digraphs and dihypergraphs.

Complex networks plays a main role in the important area of information sciences, social sciences, biology and engineering science in such a way that, include information networks. Data clustering, including problems such as finding network communities, can be put into a systematic framework by means of a systematic methodology for interpreting complex data sets and for evaluating model hypotheses. We represented these systems by simple or directed hypergraphs (graphs) that consist of sets of nodes representing the objects or group under investigation, joined together in pairs by links if the corresponding nodes or sets are related by some kind of relationship. Groups of more than two nodes in any hypergraph is related as a hypergraph. Consequently, we will formally apply the hypergraph concept as a generalization for representing complex networks and will call them complex hypernetworks. A cluster in WNS consists of three main different elements: sensor nodes (SNs), base station (BS), and CH. The SNs are the set of sensors present in the network, arranged to sense the environment and collect the data. The main task of an SN in a sensor field is to detect events, perform quick local data processing, and then transmit the data. The BS is the data processing point for the data received from the SNs, and where the data are accessed by the end-user. It is generally considered fixed and at a far distance from the SNs. The CH acts as a gateway between the SNs and the BS. The function of the cluster–head is to perform common functions for all the nodes in the cluster, like aggregating the data before sending it to the BS. In some way, the CH is the sink for the cluster nodes, and the BS is the sink for the CHs. This structure formed between the SNs, the sink, and the BS can be replicated as many times as it is needed, creating the different layers

of the hierarchical WSN. The SNs and the communication links between them can be represented by an undirected graph $G = (V, E)$, where each vertex $v \in V$ (the set of vertices in the graph) represents a sensor node with a unique ID. An edge $(u, v) \in E$ (the set of edges in the graph) represents a communication link if the corresponding nodes u and v are within the transmission range of each other. We apply the concept of directed hypergraphs for clustering WSNs via the notation of positive relation and obtain directed clustering graphs.

Example 3.24. The proposed protocol weight-based clustering routing (WCR) is a clustering-based, energy-efficient protocol for WSN. The objective of the protocol is to reduce the energy dissipation of nodes for routing data to the BS and prolong the network lifetime. In WCR, a CH selection algorithm is designed for periodically selecting CHs based on the node position information and residual energy of node. This CHs selection scheme is a central controlled algorithm performed by the BS which is assumed to have no energy constraint. Distributed weight-based energy-efficient hierarchical clustering (DWEHC) as an algorithm, aims at high energy efficiency by generating balanced cluster sizes and optimizing the intracluster topology. DWEHC algorithm has been shown to generate more well-balanced clusters as well as to achieve significantly lower energy consumption in intracluster and intercluster communication. Let $H = \{a, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ be a set of nodes in a WSN as a hyper network. Figure 19, shows a multi-level cluster generated by DWEHC, where a is the CH, first level children are a_0, a_1, a_2, second level children are a_3, a_4, a_5, a_6 and a_7, and the third level children are a_8 and a_9.

Since

$$\begin{aligned} &\alpha(a) = \{a\}, \alpha(a_1) = \{a_1\}, \alpha(a_2) = \{a_2\}, \alpha(a_3) = \{a_3\}, \\ &\alpha(a_4) = \{a_4, a_5\}, \alpha(a_6) = \{a_6\}, \alpha(a_7) = \{a_7\}, \text{ and} \\ &\alpha(a_8) = \{a_8, a_9\}, \end{aligned}$$

we get the α-derivable digraph $\mathcal{H}'/\alpha$ in Fig. 20.

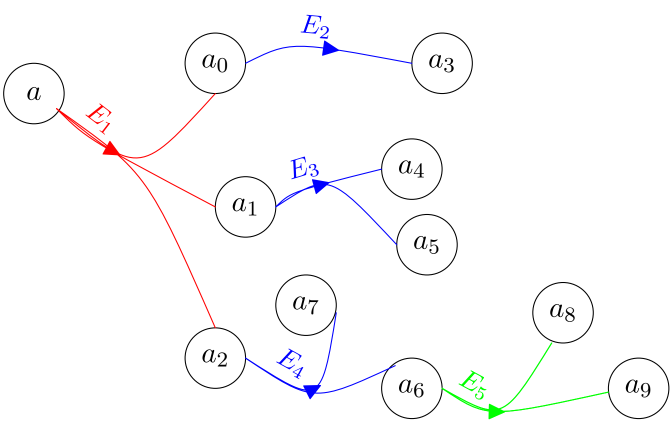


Fig. 19. DWEHC multi-hop intracluster topology.

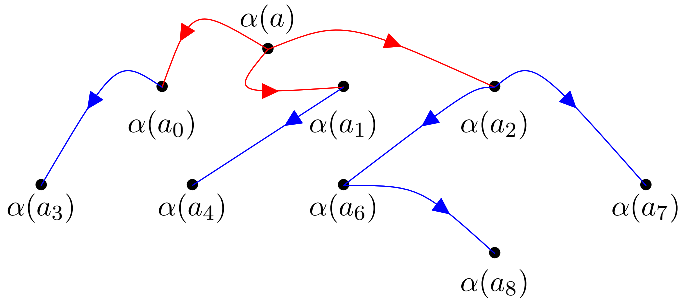


Fig. 20. Digraph G.

Example 3.25. Clustering scheme are applied to many networking scenarios, such as WSN and are constructed clustering mechanism efficient in these networks. These networks comprise of a set of SNs scattered arbitrarily over some region. The clustering scheme provides an useful service that can be leveraged by different applications to achieve scalability. The cluster-based network of graph G (CNet(G)) is a spanning tree of G. The nodes in the WSNs are grouped into disjoint clusters, where the backbone is a tree consisting of the CHs and gateway nodes. A cluster of G is a star subgraph of G, where the CH has an edge to each other cluster member and no edges exist between two cluster members in the cluster. To minimize the number of clusters, two CHs cannot be neighbors with each other. Thus, they form a maximal independent set in G. To guarantee the communication between CHs, any two CHs are joined through one special cluster member called a gateway node, which is an intersection of neighbors in G of two CHs. The cluster members that do not act as gateways are called pure cluster members. A transmission between a CH and its members is called local transmission, and a transmission between CHs is called backbone transmission. Figure 21, shows an example of a clustering scheme, where nodes $3, 7, 10$ and 16 are CH, nodes 6 and 13 are gateway nodes, and the remaining nodes are pure cluster members. (CH) and gateway nodes form the network backbone. Hyperedges and the vertices they connect form the (CNet(G)) in G.

First, we convert the hypernetwork in Fig. 21 to directed hypergraph in Fig. 22. Clearly,

$$\alpha(1) = \{1, 2\} = \alpha(2), \alpha(3) = \{3\}, \alpha(4) = \{4, 5\} = \alpha(5),$$
$$\alpha(6) = \{6\}, \alpha(7) = \{7\}, \alpha(8) = \{8, 9, 11, 12\}, \alpha(10) = \{10\},$$
$$\alpha(13) = \{13\}, \alpha(16) = \{16\} \text{ and } \alpha(14) = \{14, 15\}.$$

Thus, we obtain the α-derivable graph $\mathcal{H}'/\alpha$ in Fig. 23.

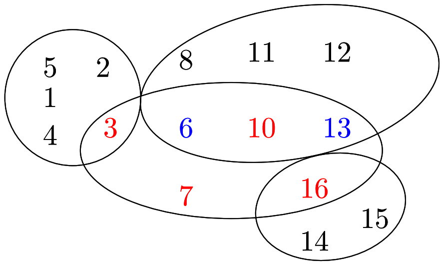


Fig. 21. CNet(G) clustering scheme.

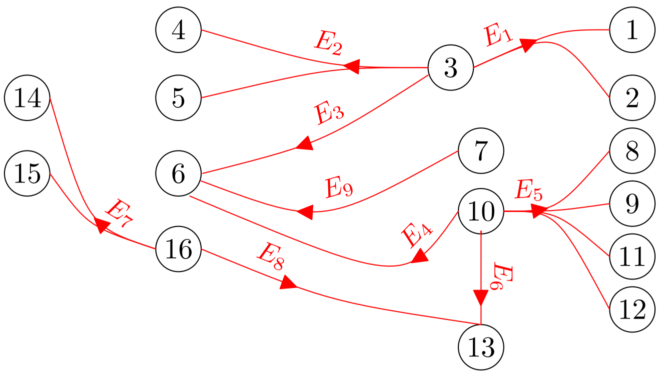


Fig. 22. Dihypergraph $\mathcal{H}'$.

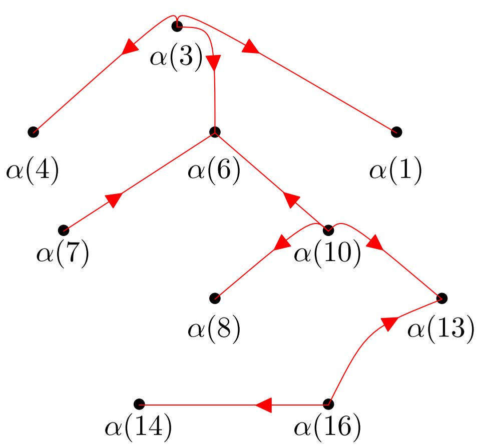


Fig. 23. Directed tree $\mathcal{H}'/\alpha$.

4. Conclusion

The current paper considered the concept of directed hypergraphs as a generalization of digraphs, constructed directed hypergraphs from undirected hypergraphs and investigated some relations between them. Also:

(i) The notation of derivable dihypergraphs is introduced based on undirected hypergraphs and is computed the cardinal of set of all derivable dihypergraphs from any given undirected hypergraph.
(ii) An equivalence relation is defined on dihypergraph $\mathcal{H}'$ as α, in such a way that, it can construct a quotient $\mathcal{H}'/\alpha$ which is a digraph and so the concept of α-(self) derivable digraph is introduced.
(iii) This study obtained some conditions that a digraph is an α-(self)derivable graph.
(iv) As an application, it is considered directed hypergraphs(as complex hypernetworks) corresponding to wireless sensor hypernetworks such that vertices (V) represent the nodes and the set of links (E) represents the connections between nodes.
(v) Using the relation α, it is constructed $\mathcal{H}'/\alpha$ as nodes cluster($\mathcal{H}'$ is a complex hypernetwork) and is investigated on associated complex network $\mathcal{H}'/\alpha$.
(vi) It is extracted digraphs corresponding to WSNs such that are obtained from quotient of wireless sensor hypernetworks on α.

We hope that these results are helpful and encourage for further studies in graph theory. In our future studies, we hope to obtain more results regarding digraphs, digraphs based on rough sets or soft sets, WSNs and their applications.

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