An Analytic Solution of Water Transport in Unsaturated Porous Media

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ABSTRACT

One of the most well-known equations to describe the behavior of the infiltration of unsaturated zones in soil as a porous medium is known as Richards' equation. Although analytical approaches in simulating infiltration are few, there are many numerical researches to model this physical phenomenon. The Adomian decomposition method (ADM) is one of the most recent approaches used in solving nonlinear partial differential or algebraic equations, and is an easy way to achieve the analytic solution. In this article, two refined approaches in improving ADM are used to simulate volumetric water content via Richards' equation. The first modification was recently presented by Wazwaz using a new regrouping approach in Adomian series terms and the last is the Pade approximation. A comparison of the exact solution and (modified) ADM illustrate very good coverage and results.
### NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>unsaturated soil moisture content</td>
</tr>
<tr>
<td>$\theta_r$</td>
<td>residual soil moisture content</td>
</tr>
<tr>
<td>$\theta_s$</td>
<td>saturated soil moisture content</td>
</tr>
<tr>
<td>$K$</td>
<td>unsaturated hydraulic conductivity</td>
</tr>
<tr>
<td>$D$</td>
<td>unsaturated hydraulic diffusivity ($M^2/T$)</td>
</tr>
<tr>
<td>$K_s$</td>
<td>saturated hydraulic conductivity ($M/T$)</td>
</tr>
<tr>
<td>$\alpha, \lambda$</td>
<td>experimental parameters</td>
</tr>
<tr>
<td>$K_0, D_0, k$</td>
<td>soil parameters</td>
</tr>
<tr>
<td>$O_n$</td>
<td>nonlinear ordinary operator</td>
</tr>
<tr>
<td>$L$</td>
<td>linear time-differential term of nonlinear ODE or PDE equation ($\partial/\partial t$)</td>
</tr>
<tr>
<td>$L^{-1}$</td>
<td>integral of linear time-differential term of nonlinear ODE or PDE equation ($\int_0^1 \ldots dt$)</td>
</tr>
<tr>
<td>$R$</td>
<td>remainder term of the linear part ($L$) of the differential time</td>
</tr>
<tr>
<td>$N$</td>
<td>nonlinear term of nonlinear ODE or PDE equation</td>
</tr>
<tr>
<td>$u_i$</td>
<td>$i$th terms of estimation series</td>
</tr>
<tr>
<td>$\phi_i$</td>
<td>$i$th iteration of function estimation</td>
</tr>
<tr>
<td>$u_{ix}$</td>
<td>partial differential of $u_i$ over $x$</td>
</tr>
<tr>
<td>$z$</td>
<td>depth</td>
</tr>
<tr>
<td>$a, \gamma, b, c$</td>
<td>arbitrary coefficients</td>
</tr>
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### 1. INTRODUCTION

Several attempts have been made to model water movement in an unsaturated porous material. One of the first and most fundamental was that done by Richards (1931), who derived a governing equation for water flow in soil based on continuum mechanics. In the model, the continuity equation was coupled with Darcy’s law as a momentum equation, and the following equation, known as the one-dimensional form of Richards’ equation, was observed:

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial \theta}{\partial z} - K\right)
\]

(1)

where $\theta$ is unsaturated soil moisture content, $K$ is conductivity, and $D$ is soil water diffusivity.

The sophistication of the analytical and numerical methods that are available to solve the governing equation of unsaturated flow in soils (Richards’ equation) makes it necessary to have suitable models for parameters in the equation. Several methods are available for the estimation of such parameters, i.e., conductivity and water diffusivity (Russo, 1991). Basically, there are three commonly used models: (i) Brook-Corey’s model (Brook and Corey, 1964; Corey, 1994), (ii) the van Genuchten model (1980), and (iii) the exponential model (Gardner, 1958). Although the van Genuchten model matches the experimental data more satisfactorily than the other two, its functional form is complicated and restricts its usefulness for a large number of analytical solutions.

Brooks and Corey modified Kozney’s (1927) and Carman’s (1937) equations, and combined them with some experimental observations. They suggested a model to predict conductivity and soil water diffusivity of an isotropic porous media as follows:

\[
K(\theta) = K_s \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{3+2/(\lambda)}
\]

(2)

\[
D(\theta) = \frac{K_s}{\alpha \lambda (\theta_s - \theta_r) \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{2+1/(\lambda)}}\]

(3)

where $K_s$ is the saturated hydraulic conductivity, $\theta_r$ is residual water content, $\theta_s$ is saturated water content, and $\alpha$ and $\lambda$ are experimentally determined parameters. Brooks and Corey (1964) used the parameter $\lambda$ as an index of pore-size distribution. They
reasoned that for media having a uniform pore size, the index would be a large number, which theoretically could approach infinity. On the other hand, media with a very wide range of pore sizes have small values of $\lambda$, which theoretically could also approach zero.

The Brooks-Corey model introduces a well-defined air-entry value that is associated with the largest pore size, assuming complete wettability. Clearly, such a maximum pore radius can exist in any soil. Above this air-entry value, the soil is considered to be saturated. In contrast, the Van Genuchten (1980) model describes a continuous function without such an air-entry value. The resulting water retention curve is continuously differentiable, which simplifies the analytical solution of Richards’ equation [Eq. (1)]. Brooks-Corey model soils can be simplified to the following equations by some further considerations (Witelski, 1997, 1998 and 2005):

$$K(\theta) = K_0 \theta^k k \geq 1$$  \hspace{1cm} (4)

$$D(\theta) = D_0 (n + 1) \theta^n n \geq 0$$  \hspace{1cm} (5)

where $K_0$, $D_0$, $n$, and $k$ are constants representing soil properties such as pore-size distribution, particle shape, etc. In these relations, $\theta$ is scaled between 0 and 1, and the form of diffusivity is normalized so that $\int_0^1 D(\theta) d\theta = 1$ for all $n$. Most of the results in this paper can be applied to the general values of $k$ and $n$.

It is believed that the Brook-Corey model is widely used because of its well-defined configuration. There are several analytical and numerical solutions to Richards’ equation considering the Brook-Cory model. The choice $n = 0$ and $k = 2$ in Eqs. (4) and (5) yields the classic Burger’s equations studied by many authors, such as Whitam (1974), Broadbridge and Rogers (1990), and Basha (2002). For general values of $k$ and $n$, the generalized Burgers’ equation, which was the focus of several researchers including Grundy (1983), is obtained. In the special case $n = 0$, Richards’ equation reduces to a linear equation.

Richards’ equation with other models was also solved numerically, and by various innovative and common analytical methods. Some of these solutions are limited to very simple geometrical and initial conditions. Therefore, in the last 30 years many finite difference and finite element numerical solutions were developed, even in 2D and 3D, such as Haverkamp et al. (1977), Paniconi et al. (1994), and Simunek et al. (1999). Another numerical method, the finite volume method, looks quite promising for solving Richards’ equation, especially when sharp infiltration fronts develop and must be approximated on unstructured multidimensional grids. [For a general introduction to this topic, see Farthing and Miller (2003) for porous media applications]. Parlange et al. have used some techniques to encounter Richard equations (1975, 1987, 1992, 1997, 1999).

Some currently defined techniques are employed to solve Richards’ equation. For instance, a few researchers have tried to solve the equation by the Adomian decomposition method (ADM) as an analytical series solution (Serrano and Adomian, 1996; Serrano, 1998 and 2004, Pamuk, 2005).

In this study, a simplified Brooks-Corey model [Eqs. (4) and (5)] was applied in Richards’ equation [Eq. (1)]. However, three cases for conductivity exponents (with equal soil water diffusivity) and the solutions were tried. A current widely used technique, ADM, which is used to solve ordinary or partial differential equations analytically, was employed to solve the equation. This method is expressed in the next section. In order to have more reliable results, some modifications were applied to the model.

2. ANALYSIS OF THE METHOD

Most of the frontier problems in applied physics and mathematics are in the nonlinear partial differential equation (PDE) category. There are many techniques such as the perturbation method, Cole-Hopf transformation (Cole, 1951), variational iteration (He, 1997), etc., to solve partial differential equations. ADM is one of the most recent widely used approaches ap-
plied to solve even nonlinear ordinary differential equations (ODEs) or PDEs. The method was first presented by Adomian in the early 80s. This is one of the most powerful series approximations to solve integral and differential equations, especially nonlinear PDEs (Adomian, 1994). These methods depend greatly on the initial condition (function) of the partial differential equation, and by using iterative integration, the result is gradually improved. Consider the nonlinear operator

$$O_u = g$$  \hspace{1cm} (6)

where $O_u$ represents a general form of the nonlinear ordinary operator and includes both linear and nonlinear parts, and $g$ is a given function. The terms of operator $O$ may be decomposed to three parts: $L$ as the linear term of time differential, $R$ as the remainder term of the linear differential term, and $N$ as the nonlinear differential term, and $u(\beta) = \alpha$ may also be regarded as an initial condition, and it is assumed that $u$ is the exact solution. Therefore, applying these assumptions in Eq. (6), we have

$$L_u + R_u + N_u = g$$

$$L^{-1}(L_u) = -L^{-1}(R_u) - L^{-1}(N_u) + L^{-1}(g)$$

$$u(\beta) = \alpha$$  \hspace{1cm} (7)

In the Adomian method, $N_u$ can be simplified into terms of a series, which are defined as

$$\phi_n = \sum_{i=0}^{n} u_i, \quad u = \sum_{i=0}^{\infty} u_i$$  \hspace{1cm} (8)

and

$$u_0 = u(\beta) + L^{-1}g$$

$$u_i = -L^{-1}(R_{u_{i-1}}) - L^{-1}(A_{n-1}) \quad 1 \leq i \leq n$$  \hspace{1cm} (9)

where $A_n$ is a part of the approximation series that is replaced by the nonlinear terms, and may be obtained as follows (Adomian, 1994):

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N_u \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \bigg|_{\lambda=0}$$  \hspace{1cm} (10)

There are some modifications that improve both the accuracy of the approximation and the convergence (Adomian and Rach, 1996; Wazwaz, 1997, 1999a, 2000; Jiao et al., 2002). For example, a modification was based on categorizing terms of the series solution suggested by Wazwaz (2000). In this modification, when the nonlinear term is in the form of $u_x u$, the approximation series (modified Adomian polynomials) may be computed faster, simpler, and more accurately by rewriting the nonlinear terms in the form of approximated series components as $(u_{1x} + u_{2x} + u_{3x} + u_{4x} + u_{5x} + ...) (u_1 + u_2 + u_3 + u_4 + u_5 + ...)$. Basically, the first five terms of $u_x u$ are

$$A_0 = u_0 u_0$$

$$A_1 = u_0 u_1 + u_0 u_{1x}$$

$$A_2 = u_0 u_2 + u_1 u_{1x} + u_{2x} u_0$$

$$A_3 = u_0 u_3 + u_1 u_2 + u_{2x} u_{1x} + u_{3x} u_0$$

$$A_4 = u_0 u_4 + u_1 u_3 + u_{2x} u_{2x} + u_{3x} u_{1x} + u_{4x} u_0$$

(11)

Similarly, when the nonlinear term is $u^a u_x$, the first four terms of the multistage Adomian decomposition method (MADM) are

$$A_0 = u_0^a u_0$$

$$A_1 = 2 u_0^a u_1 u_0 + u_0^2 u_{1x}$$

$$A_2 = 2 u_0^a u_0 u_2 + u_0^2 u_{1x} + 2 u_{2x} u_0 + u_1^2 u_0$$

$$A_3 = 2 u_0^a u_0 u_3 + u_0^2 u_{3x} + 2 u_{2x} u_{1x} + u_1^2 u_1 + u_{1x}$$

(12)

Applying the traveling wave technique (Elwakil et al., 2004; Wazwaz, 2005; Witelski, 2005) instead of time and depth, a new variable is found as a linear combination of the two. The hyperbolic tangent function is commonly applied to solve these transform
equations (Elwakil et al., 2002; Soliman, 2006; Abdou and Soliman, 2006). Therefore, the general form of Burgers’ equation in the order of $(n, 1)$ is (Wazwaz, 2005)

$$\theta_t + \alpha \theta^n \theta_x - \theta_{xx} = 0$$  \hspace{1cm} (13)

and its exact solution is

$$\theta(z, t) = \left[ \frac{\gamma}{2} + \frac{\gamma}{2} \tanh(A_1[x - A_2t]) \right]^{1/n}$$

$$A_1 = \frac{-\alpha n + n |\alpha|}{4(1 + n)} $$ \hspace{1cm} (n \neq 0)

$$A_2 = \frac{\gamma |\alpha|}{1 + n} $$  \hspace{1cm} (14)

By assigning $t = 0$, the initial condition for the Adomian approximation can be found. Table 1 shows two cases of initial function and conductivity. In these cases, Richards’ equation coincides with Burgers’ equation $(n, 1)$, where $\theta$ is water content, $z$ is depth (cm), and $\gamma$ is an arbitrary coefficient (which may be selected to be 1). The Pade approximation is generally employed to improve the results (Wazwaz, 1999b; Wazwaz, 1999c; Chrysos et al., 2002). In the next section, three cases are studied; the units of depth and time in these cases are centimeters and days respectively.

### Table 1

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K$</th>
<th>$\theta(z, 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{\theta^2}{2}$</td>
<td>$\frac{1}{2} \left( 1 + \tanh \frac{-z}{4} \right)$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\theta^3}{3}$</td>
<td>$\frac{1}{2} \left( 1 + \tanh \frac{-z}{3} \right)^{1/2}$</td>
</tr>
</tbody>
</table>

$$\theta_z = \left( \frac{\theta_z}{c} \right) \frac{K_0}{c} \theta_c$$  \hspace{1cm} (15)

The above equation is known as the “one-dimensional heat conduction-diffusion equation” with a simple exact solution as follows:

$$\theta(z, t) = \frac{a}{cK_0 + c^2} e^{(cK_0 + c^2)(t+cz)} + b$$  \hspace{1cm} (16)

where $a$, $b$, and $c$ are arbitrary coefficients that are related to soil properties. These parameters can be determined with field soil tests. This case was solved using the ADM method. Here $a$, $b$ (maximum water content), and $c$ are, respectively, $1.25 \times 10^{-3}$, 0.8, and $-0.5$. Assuming $t = 0$ in Eq. (13), the initial condition can be derived. The ADM recursive procedure, explained in Eq. (10), was repeated for the initial condition 20 times in order to approach $\phi_{20}$. Figure 1 shows the convergence of the results in this case. Figure 2 is an exact solution to the equation. In

![Figure 1. Approximated for linear infiltration equation by applying 20 terms of ADM series](image-url)
order to avoid any ambiguity for the amounts of water content smaller than the minimum allowable values, in both ADM and the exact solution all values less than 0.02 are replaced with 0.02. The relative error of the exact and approximated solutions is shown in Fig. 3. It may be observed that relative error increases sharply in the beginning of the transition zone, but after a few seconds the error approaches zero. The first three terms of the approximated results in the ADM are as follows:

\[
\begin{align*}
\theta_0 &= 0.8 - 0.005e^{0.125x} \\
\theta_1 &= 0.0005x e^{0.125x} t \\
\theta_2 &= -2.9 \times 10^{-5}e^{0.125x} t^2 \\
\end{align*}
\]

(17)

3.2. Case II

In the second case, \( n \) is assumed to be 1 in Eq. (13). Consequently, Richards' equation is converted to a classical Burgers' equation. Several authors have tried to solve Burgers' equation using the ADM including El-Sayed and Kaya (2004), Inc (2005), and Gorguis (2006). This equation is one of the most well-known equations in many branches of science, especially in applied mathematics. In order to derive an exact solution, the tanh function, instead of transforms such as the Cole-Hopf method, can be used as an alternative. Equation (14) and Table 1 show the initial function and the general exact solution to Burgers' equation. Applying the modification proposed by Wazwaz (2000) and improving the results by the Padé approximation method, the consequence of applying the ADM are shown. The first three terms of approximated ADM series solution are

\[
\begin{align*}
\theta_0 &= \frac{1}{2} \left( 1 + \tanh \left( \frac{-z}{4} \right) \right) \\
\theta_1 &= \frac{t(0.0625 + 0.1 \times \tanh[z/4])}{\sinh[z/4]^2} \\
\theta_2 &= \left\{ \cosh[z/4](t^2/10) + (t^2/10) \cosh[3z/4] + (2t^2/1000)(\sinh[z/4] + \sinh[3z/4])^3/\sinh[z/4]^5 \right\} \\
\end{align*}
\]

(18)

However, Richards' equation is approximated by repeating the ADM recursive relation Eq. (11) six times (\( \phi_T \)). Figures 4 and 5 show the approximated ADM and the exact solution to this case, respectively. In this stage, significant fluctuation occurred in the upper and lower computational time domain. As shown in Fig. 4, approaching the boundaries (±∞), the value of \( \theta \) attains its minimum or maximum values in its range \( (K = 0^2/2 \text{ cm/h and } D = 1 \text{ cm}^2/\text{h}) \).
3.3. Case III

In the last case, as shown in Table 1, a conductivity term is selected as a function of cubic water content, and constant value $K = 0.3$ cm/h and $D = 1$ cm$^2$/h. Using a modified recursive Eq. (12), the approximated terms of the series solution will be generated. For example, the first two terms of the approximation MADM series are

$$
\theta_0 = \frac{1}{2} \left[ 1 + \tanh \left( \frac{-z}{3} \right) \right]^{1/2},
$$

$$\theta_1 = e^{2z/3} (0.11 + 0.11e^{2z/3}) \left( \frac{1}{1 + e^{2z/3}} \right)^{2.5} t \quad (19)
$$

Using the first five terms of the MADM series ($\phi_k$), Fig. 8 will be generated. The three-dimensional plot of the exact solution is shown in Fig. 9. The discrepancy between the MADM approximation and the
Figure 7. Comparison of exact solution and improved modified ADM (order 7) series at a) $z = 0$, b) $z = 5$, c) $z = 10$ by Pade (3/3).

Figure 8. Approximated Burger (2/1) by first four terms of modified ADM series.

Figure 9. Exact solution of Burger (2/1) by tanh function.

Figure 10. Error of approximated (exact solution - ADM solution).

The exact solution is shown in Fig. 10. As in case II, the error will be increased in the transition zone of the computational domain, but the upper and lower bounds of depth satisfy the exact solution very well. The behavior of the MADM and the exact solution in three transition points ($z = 1, 4, 7$) are plotted versus time in Fig. 11. Considerable improvement can be observed in Fig. 12 using the Pade approximation (2/2) on $\phi_6$, in comparison with the previous three critical cross sections. Considering the figures, since the exact and approximated solutions are rather equivalent, no further modifications are required.
4. CONCLUDING REMARKS

This study attempts to suggest an analytical solution to Richards' equation considering the simplified Brooks-Corey model and applying a very widely used technique known as the Adomian decomposition method (ADM). To verify the approximations gain by ADM, the results (cases II and III) were fully compared with the exact solutions achieved by the tanh function method. In the case of the linear model for conductivity, in order to obtain more accurate results from the approximation, more terms in the series solution are required. However, in nonlinear forms, fewer terms may lead to more accurate results. All of the figures indicate that ADM and MADM result in more accurate outcomes when going far from the transition zone, i.e., boundaries. This method is very sensitive to the rapid variation of the initial function (as are many other methods). This shortcoming can be greatly improved by applying some additional modifications such as the Pade approximation (case II). However the findings presented in this paper show that the ADM and its modifications are suitable approaches for a semiexact solution of a number of nonlineear partial differential equations.
ACKNOWLEDGMENT

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An Analytic Solution of Water Transport


