Joint pricing and inventory control modelling for obsolescent products: a case study of the telecom industry

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Abstract: The current pricing context calls for better, faster and more precise pricing decision makings than ever before. These in turn, oblige companies to take a new stance with respect to the role of pricing in today’s overwhelming and complex marketing context. Similarly, decisions related to inventory control and types of goods are strongly linked to pricing as well. The aim of this paper is to present a mathematical programming model for joint inventory control and pricing for the bundling of obsolescent items. To this end, a mathematical programming model was developed following a comprehensive review of the literature in this area. The validity of the proposed model was then tested in the telecommunications industry as a case study. According to the obtained results, the proposed model had a fairly good validity.

Keywords: inventory control; pricing; obsolescent product; mathematical programming; telecom; bundling.

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1 Introduction

As is assumed in most inventory control models, the stocking of items can meet the future demands without any constraint. Nevertheless, such assumption does not normally come true in many real-world cases. Deterioration occurs when a group of commodities cannot be used anymore or they may have lost their utility all at once after a certain period of time. Food, medicines or films, for example, expire and become useless over time. Most clothes become old-fashioned or go out of style and some items such as petrol and petrochemical products must be stocked at the lowest amount. All three sources of loss identified above result in inventory deterioration. Deteriorating items are defined in various ways in the literature. In their paper, Goyal and Giri (2001) provide a comprehensive introduction on the deteriorating items inventory management and categorise the proposed models based on the literature of each group. They also introduce the key factors by which inventory goods are divided into three categories including (Vikas et al., 2016):

a. obsolescence
b. deterioration
c. no obsolescence/deterioration.

Obsolescence refers to the case where items lose part or all of their value over time because of the introduction of new technologies, new competing products, new trends or
changes in the customers’ preferences. The second category refers to the case in which items lose their utility or become damaged, spoiled, decayed, evaporated or expired over time. Items such as food, vegetables, grain, fruits, human blood or films fall in the category of deteriorating items, because they have expiration date and must be used before that specific date. Alcohol, gasoline and radioactive materials are other items in this category with the natural life cycle which gradually degrade. Finally the third category refers to goods with an indefinite life cycle. For instance, according to Chen and Chen (2007), commodity expiration and deterioration account for about $31.3 million dollars loss in the retail industry. Product deterioration also affects optimal inventory levels. In other words, retailers should order more than market demand because some items lose their value over time. To increase sales revenue for such products, pricing policies are of significant importance. Properly designed pricing strategy triggers success in the enterprise. Alternatively, if prices are significantly higher or lower than the actual value, income and demand will be directly affected and in both cases, income reduction will result in failure of the enterprise. Therefore, a well-designed pricing strategy plays a crucial role in the supply chain management (Chopra and Meindl, 2007). Furthermore, prices are not only important from a financial standpoint, but also from an operational point of view. Pricing is a tool which plays an important role in the regulation of inventory and production pressures (Bulut et al., 2009).

Firms can increase their profits by pursuing suitable pricing policies and tactics such as price discrimination and bundling. Bundling is one of the most common pricing strategies in which several products or services are offered for sale as a single product or service package. In other words, products or services are joined together in bundling strategy in order to be sold as a single combined item. As the bundled products have a higher level of appeal in terms of customers’ needs, the purchase intention level is increased in most customers (Kim et al., 2009).

This paper investigates the joint or concurrent determination of pricing and inventory modelling for a bundling product, prone to gradual obsolescence. A new mathematical nonlinear programming has been proposed as a multi-product constrained nonlinear one. One of the innovative aspects of this research is that a mathematical programming model for inventory control is developed and the price is associated with a decision variable and multi-product variable in the inventory model. Decision constraints are also added to the model. Another contribution of the paper is that it takes the combined demand function of a complementary obsolescent product into account, as well as the demand function dependent on advertising cost. Furthermore, this research adopts the price and advertisement-dependent probabilistic demand function and obsolescent function. There are two types of obsolescence (Barron, 2018) including sudden and gradual and this paper deals with the latter type. The study has been conducted in the telecommunication industry which is fairly underinvestigated in terms of concurrent or joint determination of pricing and inventory.

This paper is organised as follows: Section 2 addresses the theoretical foundation of the study. Section 3 presents the review of the related literature. Section 4 provides the research methodology. Section 5 defines assumptions, notations and the proposed mathematical model. Section 6 addresses the proposed model in a real context. Finally, discussion and conclusions is given in Section 7.
2 Theoretical background

Perishable items are defined as products which lose their quality and value over a specified time. Goods such as vegetables, meat, and other foodstuffs may be corrupted due to physical deterioration or devaluation (such as flight tickets or movie tickets). Sometimes technological changes resulting from changes in consumer preferences can lead to the perishing of a good (such as a fashion style) (Jia and Hu, 2011).

In general, obsolete, perishable or deteriorating items can be classified into two categories (Vikas et al., 2016): items which become decayed, damaged, evaporated or expired over time such as meat, vegetables, fruit, medicine or flowers, etc. fall into the first category. The second category refers to the items which lose some or all of their value over time because of new technologies or the introduction of alternatives like computer chips, mobile phones, fashion and seasonal goods, etc. Both categories are characterised by short lifecycle or natural life cycle and as a result, appropriate inventory strategies must be developed to avoid loss resulting from damage and expiration. In this research, the second category is investigated.

In addition to demand, deterioration rate is another key factor which can influence the inventory system in the supply chain. This factor is neglected in conventional and traditional inventory models on deteriorating products. Deterioration may be studied in different ways and modes. Nevertheless, deterioration rate is assumed to be similar in most of the studied models such as Bhunia and Maiti (1999) and Aggarwal (1978). Recently, researchers have investigated the relationship between time and deterioration rate in which time is considered as a variable. Several scenarios can be taken into account, like when deterioration rate is the linear increasing function of time (Mukhopadhyay et al., 2004), deterioration rate is a two-parameter Weibull distribution (Mahapatra and Maiti, 2005), deterioration rate is a three-parameter Weibull distribution (Chakrabarty et al., 1998) and deterioration rate is a function of time (Abad, 2001).

Product bundling provides customers with more than a few products or services for purchase as a single combined product or service package. Following the movement towards creating integrated solutions as a strategy and a means for competition in recent years, firms have paid much more attention to their products (Sandholm et al., 2006). Many studies have suggested that bundling can result in saving in production, distribution and transaction costs (Stremersch and Tellis, 2002). There are several types of bundling strategies including:

1. pure bundling which occurs when a consumer can only purchase the entire bundle and not the products separately
2. mixed bundling which applies to situations where consumers are offered a choice between purchasing the entire bundle or one of the separate parts of the bundle
3. unbundling in which products are only sold in separate parts
4. customised bundling which occurs when customers are allowed to choose pieces from a larger pool of products (Yang and Ng, 2010).
### Table 1: Summary of literature review on related studies

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3 Literature review

3.1 Inventory control models

Yang et al. (2009) formulated an inventory system for non-instantaneous deteriorating items with price-dependent demand. They developed a model in which shortage is allowed and partially backlogged is possible, where the backlogging rate is variable and depends on the waiting time for the next replenishment. The main objective of the research was to determine the optimal price, the length of time in which there was no inventory shortage and the replenishment cycle time simultaneously, such that the total profit per unit time had a maximum value. Wang and Tung (2011) proposed a pricing model for products which were gradually deteriorating. Dye (2012) considered a deterministic economic order quantity (EOQ) model with generalised demand, deterioration and unit purchase cost functions under two levels of trade credit policy. Soni and Patel (2013) developed an inventory model for non-instantaneous deteriorating items with imprecise deterioration free time and credibility constraint. Farughi et al. (2014) proposed a joint pricing and inventory control model for deteriorating items with price and time-dependent demand and time-dependent deteriorating rate with partial backlogging. Ghoreishi et al. (2015) proposed an economic order model for non-instantaneous deteriorating items with sales prices, demand influenced by inflation and customer returns. They assumed customer returns as a function of demand and price. The shortage and partial backlog were allowed in their study. Fu and Chen (2016) examined the issue of optimising the production inventory model with a producer and a retailer. In their paper, the products offered by the manufacturer were assumed as deteriorating items. Tiwari et al. (2018a) developed an inventory model for deteriorating items under a two-level partial trade credit with allowable shortages. The objective of their study was to determine the optimal selling price, the optimal replenishment cycle time and the time taken for the inventory to reach zero while maximising the total profit per unit. Banerjee and Agrawal (2017) proposed an inventory model for a deteriorating item whose demand depended initially only upon its selling price and within its life cycle on the freshness condition. Li et al. (2018) studied the inventory system for deteriorating drugs with fixed demand, certain lead time and shortage. Mishra et al. (2017) developed an EOQ model, taking into account the price and stock-dependent demand with permitted shortages. Finally Tiwari et al. (2018b) developed an inventory system to minimise both the total inventory and carbon emission costs simultaneously.

3.2 Joint inventory control and pricing models

Chatwin (2000) investigated the continuous-time inventory problem in retail industry. Bitran and Caldentey (2003) investigated the dynamic pricing policies and their relationship with revenue management. Bitran et al. (2006) examined pricing policies within the range of alternative perishable products with correlated demand. They studied the vendor’s problem to choose the optimal pricing strategy which maximised revenue within a limited sales range. In another study by Chew et al. (2014), the order and price were determined for non-deteriorating products with several life cycles. Demand for products was assumed with different life cycles depending on the price of the alternative products. Chew et al. (2014) studied the problem of two-period lifecycle products and
developed a stochastic dynamic programming model. Rabbani et al. (2014) proposed an integrated dynamic pricing and inventory control model for deteriorating products to reflect the dynamic specifications of the sales price problem as a function of time from the initial sale price and pre-specified discount rate. They included the unrestrained planning horizon and time-dependent deteriorating rate in their study and the demand rate was also assumed to be dependent on advertising. Maihami et al. (2017) proposed a model to control inventory and pricing in which the demand for a product was assumed as a probabilistic distribution function with a certain time-free distribution function and deterioration rate. Jadidi et al. (2017) proposed a joint pricing and inventory decision problem with transport capacity and cost in a single period of time. They considered product demand as stochastic and price-dependent. Yao (2017) considered an infinite horizon continuous-review stochastic inventory system in which cumulative customers’ demand is price-dependent and is modelled as a Brownian motion. Yao attempted to determine a pricing strategy and an inventory strategy, simultaneously, to maximise the expected long-run average profit. The inventory model and pricing with the shortage of deteriorating products were examined by Soni and Chauhan (2018). Rabbani et al. (2018) proposed an EOQ model with shortage in the form of partial backorder under VMI policy. They analysed a multi-item, multi-constraint problem by including storage space, time period and budget constraints. Tiwari et al. (2018c) modelled the interdependencies among price of a product, demand for the product and the integration among the retailer and the supplier under various policies. Tai et al. (2019) developed six practical models for an inventory system with deterioration rate, depending on the maximum lifetime of items. Duan et al. (2018) investigated the joint dynamic pricing and production decisions of deteriorating items with uncertain demand over a finite selling season, where demand was price-dependent and followed a stochastic process. Table 1 summarises the major related research in this field.

4 Research methodology

Research falls into the three major categories of applied, fundamental or basic and developmental in terms of objectives (Saunders et al., 2009). The present study is an applied research aimed to address and solve a problem in the real world. A deductive-mathematical approach is adopted in this study, since mathematical modelling is performed on the basis of a deductive approach (Wacker, 1998). According to deductive research principles, previous models must be investigated and reviewed to identify the existing mathematical equations and variables in the concurrent models of inventory control and pricing. Having examined the existing theoretical gap in the proposed models in the next step, the researchers must apply their own assumptions, while taking into account the resources, facilities and existing limitations. Finally a new model is developed.

This paper presents the rationale for selecting the case study research strategy in the empirical study of a particular phenomenon in a real context. Case study research strategy is in contrast to experimental strategy in which research is conducted in a controlled environment. Furthermore, case study strategy is used in exploratory and explanatory researches (Saunders et al., 2009). The complementary products in this research are the SIM card and the mobile phone of Irancell Company. After developing the original
model, data required for modelling was collected from the marketing department of Irancell Company.

As for timeframe, this study is cross-sectional, since it is conducted over a distinct period of time. Having conducted an extensive overview of the research literature, experts (university professors and managers of Irancell Corporation) were interviewed and an initial mathematical model was created and later solved using the collected data.

5 Proposed model

As mentioned above, the objective of this paper is to make decisions on inventory control and pricing strategies for a bundling of deteriorating goods in the telecommunication industry (case study: Irancell Company), in order to maximise total profit. In other words, the product (cellphone) and service (SIM card) bundling are investigated in this paper. The studied case attempts to offer MTN mobile phone, a 4G SIM card along with 6 GB internet as a bonus in the form of a single product which can be considered as a bundling policy. Since these SIM cards will be outdated when a better technology is introduced, the product under investigation could be considered as a gradually obsolescent product. In other words, with the advent of new technologies by the manufacturer or its competitors, the demand of this product will decrease gradually. This section is designed to introduce the modelling process which follows in the next part.

Model assumptions

The proposed model assumptions are as follows:

1. There are two obsolescent items complement each other.
2. The demand function of each item depends on the time, price and advertisement. Furthermore, the demand function depends on the time-independent random variable.
3. Lead time is assumed to be negligible.
4. The time horizon of the inventory system is equivalent to the life cycle of the products (365 days).
5. The order quantity of both items is assumed to be unlimited.
6. Shortages are allowed in the system; however, only a part of the demand with short supply is considered as partially backlogged. The backlog rate is calculated according to Abad (1996) as follows:

   \[ \beta(t) = k_0 e^{-\delta t} \quad 0 < \delta, 0 < k_0 \leq 1 \]  

where \( t \) is the waiting time for replenishment, \( k_0 \) is called backordering intensity and \( \delta \) is called backordering resistance. It is worth noting that if \( \beta(t) = 0 \) for all \( t \), then all shortage of backlog is lost. Conversely, shortage is completely backlogged when \( \beta(t) = 1 \) for all \( t \).
The greater value of $\delta$ indicates that a smaller fraction of deferred orders can be replaced. In other words, $\delta$ represents customer impatience and the more the customers are impatient to receive the goods, the smaller the $\delta$ will be. The other portion of shortage is considered as a lost sale represented by the function $1 - \beta(t)$. Deterioration rate follows a three-parameter Weibull distribution. It has been empirically observed that the expected failure and expected life cycle of many items can be shown by the Weibull distribution. These empirical observations have encouraged researchers to employ deterioration rate by the Weibull distribution (Begum et al., 2012). As a result, according to Covert and Philip (1973), when the deterioration rate of goods (in this paper obsolescence) increases with age, Weibull distribution will explain the probability density function which is compatible with this feature. The following notations are used throughout the paper.

**Decision variables**

- $p_1$: the optimal selling price of the first product
- $p_2$: the optimal selling price of the second product
- $Q_1$: the optimal order of the first product
- $Q_2$: the optimal order of the second product
- $t_1$: the duration of stock availability of the first product (no shortage)
- $t_2$: the duration of stock availability of the second product
- $A$: advertisement cost
- $\epsilon_1$: time-independence random variable according to Maihami et al. (2017)
- $\epsilon_2$: time-independence random variable according to Maihami et al. (2017)
- $T$: replenishment duration
- $I_0^1$: the first product inventory level
- $I_0^2$: the second product inventory level.

**Parameters**

- $c_1$: the prime cost of the first product
- $c_2$: the prime cost of the second product
- $h_1(t)$: the holding cost of product 1 at time $t$
- $h_2(t)$: the holding cost of product 2 at time $t$
- $t_{d1}$: the length of time in which product 1 exhibits no obsolescence.
- $t_{d2}$: the length of time in which product 2 exhibits no obsolescence.
- $S_1$: the maximum allowed backlogged of product 1
- $S_2$: the maximum allowed backlogged of product 2
Joint pricing and inventory control modelling for obsolescent products

\[ \sigma_1 \] the backorder cost of product 1 per time unit
\[ \sigma_2 \] the backorder cost of product 2 per time unit
\[ \omega_1 \] the lost sale cost of product 1 per unit of time
\[ \omega_2 \] the lost sale cost of product 2 per unit of time
\[ \delta_1 \] the positive backlog parameter in \( \beta_1 \) function
\[ \delta_2 \] the positive backlog parameter in \( \beta_2 \) function
\[ k_1^0 \] parameter in \( \beta_1 \)
\[ k_2^0 \] parameter in \( \beta_2 \)
\[ \theta_1(t) \] the first product obsolescence rate at time \( t \)
\[ \theta_2(t) \] the second product obsolescence rate at time \( t \).

A mathematical model must be developed to determine the optimum order quantity and the optimal price. Furthermore, the operation of system needs to be explained in order to develop the mathematical model. \( I_1^0 \) and \( I_2^0 \) represent the inventory or order quantity of products 1 and 2 at the beginning of each cycle, respectively. As shortage is allowed, inventory may drop to zero during the cycle and the shortage which occurs is regarded as partial backlog in the paper. Partial backlog indicates that a part of lack or shortage is replenished at the end of the given period. As a result, the part which is regarded as a lost sale indicates that the customer refers to other companies to meet the demand. The following graphical representation demonstrates the inventory level and shortage model in this paper.

**Figure 1** Graphical representation of the inventory level for product 1 on a life cycle
The following functions demonstrate the inventory level:

\[
I_1^1(t) \quad \text{the inventory level of the first product at time } t \in [0, t_d_1]
\]

\[
I_2^2(t) \quad \text{the inventory level of the second product at time } t \in [0, t_d_2]
\]

\[
I_1^2(t) \quad \text{the inventory level of the first product at time } t \in [t_d_1, t_1]
\]

\[
I_2^2(t) \quad \text{the inventory level of the second product at time } t \in [t_d_2, t_2]
\]

\[
I_1^3(t) \quad \text{the inventory level of the first product at time } t \in [t_1, T]
\]

\[
I_2^3(t) \quad \text{the inventory level of the second product at time } t \in [t_2, T]
\]

The \(TP(p_1, p_2, A, t_1, t_2, T)\) function represents the amount of total profit per unit of time. The demand rate function per unit of time is calculated by the following differential equation:

\[
D_1(p_1, p_2, t, A) = \left( a_1 p_1^{-\kappa_1} p_2^{-\beta_1} \right) (a_1 A^\beta) e^{\lambda_1 t} + \epsilon_1
\]

where \(a_1 > 0, l_1 \geq 0 \) and \(\kappa_1 \geq 0\) are constant and are estimated by calculating the impact of prices of the first and second products on the first product demand. Obviously, the price of the first product is related to its demand. In other words, increased price of the first product will reduce its demand and vice versa. Since two products are analysed in the model, an increase in the price of a product decreases the demand for its complimentary product. Therefore, if two products are complementary, price drop of the second one will not only increase its demand, it will also increase the demand of the first product. Furthermore, it is supposed that \(l_1\) is non-negative. If two products are substitutive, \(l_1\) is considered to be a non-negative value which is known as iso-price-elastic model and cross price elasticity (Chen and Simchi-Levi, 2012).

The function of \(e^{\lambda_1 t}\) presents the impact of time on the demand rate of the first product, where \(t\) is time variable and \(\lambda_1\) is a fixed value derived from the logical estimation by decision makers. If \(\lambda_1\) is positive, the demand for goods increases over time. However, if it is negative, the demand would decrease over time. In addition, \(\lambda_1\) may be zero. In this case, the demand rate will be time-independent (Tsao and Sheen, 2008).

A polynomial function \(a_1 A^\beta\) is used as well, to indicate the influence of advertisement on the demand rate. Hence \(a_1 > 0\) and \(0 \leq \beta_1 < 1\) are parameters which are identified by decision makers and the available historical data. Furthermore, it is assumed in this paper that the demand rate of the first product depends on the random variable \(\epsilon_1\). Since the random variable is added to the demand rate, it is called additive demand in the literature (Chen and Simchi-Levi, 2012).

Similarly, the following function is regarded as a demand function of the second product:

\[
D_2(p_1, p_2, t, A) = \left( a_2 p_2^{-\kappa_2} p_1^{-\beta_2} \right) (a_2 A^\beta) e^{\lambda_2 t} + \epsilon_2
\]

where \(\kappa_2 \geq 1, a_2 > 0, l_2, a_2 > 0, 0 \leq \beta_2 < 1\) and \(\epsilon_2\) is a random variable.
The inventory level of the first product on the interval \([0, t_0]\) depends solely on the demand. Therefore, the inventory level of the first product follows the equation below:

\[
\frac{dI_1(t)}{dt} = -D_1 \left( p_1, p_2, t, A \right), \quad \forall t \in \left[ 0, t_0 \right] \tag{4}
\]

In the same line and for the second product, we have:

\[
\frac{dI_2(t)}{dt} = -D_2 \left( p_1, p_2, t, A \right), \quad \forall t \in \left[ 0, t_2 \right] \tag{5}
\]

The inventory level of the products at time \(t\) is obtained through the following equation where \(I_1^0\) and \(I_2^0\) are inventory levels of the first and second products, respectively:

\[
I_1(t) = \left( a_1 p_1^{-\alpha_1} p_2^{-\alpha_2} \right) \left( \alpha_1 A^\beta \right) \left( \frac{1-e^{\lambda t}}{\lambda_1} \right) - \rho \theta t + I_1^0, \quad \forall t \in \left[ 0, t_0 \right] \tag{6}
\]

\[
I_2(t) = \left( a_2 p_2^{-\alpha_1} p_2^{-\alpha_2} \right) \left( \alpha_2 A^\beta \right) \left( \frac{1-e^{\lambda t}}{\lambda_2} \right) - \rho \theta t + I_2^0, \quad \forall t \in \left[ 0, t_2 \right] \tag{7}
\]

The inventory level of the first product at time \([t_0, t_1]\) also depends on the obsolescence rate. The inventory level of the first product at time \([t_0, t_1]\) follows the equation below:

\[
\frac{dI_1(t)}{dt} = -D_1 \left( p_1, p_2, t, A \right) - \theta(t) I_1^0(t), \quad \forall t \in \left[ t_0, t_1 \right], I_1^0(t_0) = 0 \tag{8}
\]

Since the inventory level of first product is zero at \(t_1\), we have the boundary condition of \(l_1^2(t_1) = 0\). Likewise, the inventory level for the second product on the interval \([t_0, t_1]\) follows the equation below:

\[
\frac{dI_2(t)}{dt} = -D_2 \left( p_1, p_2, t, A \right) - \theta(t) I_2^0(t), \quad \forall t \in \left[ t_0, t_1 \right], I_2^0(t_0) = 0 \tag{9}
\]

Inventory function of the first product at time \([t_0, t_1]\) is obtained by the following equation:

\[
I_2(t) = -\frac{1}{F(t)} \int_{t_0}^{t} F(t) \left( \left( a_1 p_1^{-\alpha_1} p_2^{-\alpha_2} \right) \left( \alpha_1 A^\beta \right) e^{\lambda t} + \epsilon \right) dt + \frac{1}{F(t)} \int_{t_0}^{t} F(t) \left( \left( a_1 p_1^{-\alpha_1} p_2^{-\alpha_2} \right) \left( \alpha_1 A^\beta \right) e^{\lambda t} + \epsilon \right) dt \quad \forall t \in \left[ t_0, t_1 \right] \tag{10}
\]

where

\[
F(t) = e^{\theta(t)} \tag{11}
\]
Given the continuity of the inventory level which is \( I_2(t_{d_2}) = I_2(t_{d_1}) \), the initial inventory of the first product is calculated by the following equation:

\[
I_0 = \frac{1}{F_1(t_0)} \int_{t_0}^{t_{d_1}} F_1(\tau) \left( (a_1 p_1^{\alpha_1} p_2^{\beta_1}) \left( \alpha_1 A_1^\beta \right) e^{\lambda_1 \tau} + \varepsilon_1 \right) d\tau
- (a_1 p_1^{\alpha_1} p_2^{\beta_1}) \left( \alpha_1 A_1^\beta \right) \left( \frac{1 - e^{\lambda_1 t_{d_1}}}{\lambda_1} \right) + \varepsilon_1 t_{d_1}
\]

(12)

In addition, by putting the above equation into equation (6), we have:

\[
I_1(t) = (a_1 p_1^{\alpha_1} p_2^{\beta_1}) \left( \alpha_1 A_1^\beta \right) \left( \frac{1 - e^{\lambda_1 t}}{\lambda_1} \right) + \varepsilon_1 (t_{d_1} - t)
+ \frac{1}{F_1(t_{d_1})} \int_{t_{d_1}}^{t} F_1(\tau) \left( (a_1 p_1^{\alpha_1} p_2^{\beta_1}) \left( \alpha_1 A_1^\beta \right) e^{\lambda_1 \tau} + \varepsilon_1 \right) d\tau
- (a_1 p_1^{\alpha_1} p_2^{\beta_1}) \left( \alpha_1 A_1^\beta \right) \left( \frac{1 - e^{\lambda_1 t_{d_1}}}{\lambda_1} \right) + \varepsilon_1 t_{d_1} \quad \forall t \in [0, t_{d_1}]
\]

(13)

Similarly, the inventory level of the second product on \([t_{d_2}, t_2]\) is calculated by the following equations:

\[
I_2(t) = \frac{1}{F_2(t)} \int_{t_{d_2}}^{t} F_2(\tau) \left( (a_2 p_2^{\alpha_2} p_1^{\beta_2}) \left( \alpha_2 A_2^\beta \right) e^{\lambda_2 \tau} + \varepsilon_2 \right) d\tau
+ \frac{1}{F_2(t)} \int_{t_{d_2}}^{t} F_2(\tau) \left( (a_2 p_2^{\alpha_2} p_1^{\beta_2}) \left( \alpha_2 A_2^\beta \right) e^{\lambda_2 \tau} + \varepsilon_2 \right) d\tau \quad \forall t \in [t_{d_2}, t_2]
\]

(14)

where

\[
F_2(t) = e^{\lambda_2 t}
\]

(15)

Moreover, we have

\[
I_2 = \frac{1}{F_2(t_{d_2})} \int_{t_{d_2}}^{t} F_2(\tau) \left( (a_2 p_2^{\alpha_2} p_1^{\beta_2}) \left( \alpha_2 A_2^\beta \right) e^{\lambda_2 \tau} + \varepsilon_2 \right) d\tau
- (a_2 p_2^{\alpha_2} p_1^{\beta_2}) \left( \alpha_2 A_2^\beta \right) \left( \frac{1 - e^{\lambda_2 t_{d_2}}}{\lambda_2} \right) + \varepsilon_2 t_{d_2}
\]

(16)

and
Joint pricing and inventory control modelling for obsolescent products

\[ I_1^1(t) = \left( a_2 p_2^{-12} p_2^{-\lambda_2} \right) \left( \alpha_2 A^2 \right) \left( \frac{1 - e^{\lambda_1 t}}{\lambda_2} \right) + \epsilon_2 \left( t_{d_2} - t \right) \]

\[ + \frac{1}{F_2^2(t_{d_2})} \int_{t_{d_2}}^{t} F_2^2(\tau) \left[ \left( a_2 p_2^{-12} p_2^{-\lambda_2} \right) \left( \alpha_2 A^2 \right) e^{\lambda_1 \tau} + \epsilon_2 \right] d\tau \]

\[ - \left( a_2 p_2^{-12} p_2^{-\lambda_2} \right) \left( \alpha_2 A^2 \right) \left( \frac{1 - e^{\lambda_1 t}}{\lambda_2} \right), \quad \forall t \in [0, t_{d_2}] \]

(17)

It is assumed that after \( t_1 \) to the end of the cycle, the shortage of first product will continue and only part of it will be provided as a backlogged order for customers. Deferred demand of the first product is governed by the following equation:

\[ \frac{d I_1^1(t)}{dt} = -D_1 \left( p_1, t, A \right) \beta_1(T - t), \quad \forall t \in [t_1, T], \quad I_1^1(t_1) = 0 \]  

(18)

\( \beta_1(T - t) = k_1^1 e^{\delta_1(T - t)} \) represents part of the demand of the first product which will be satisfied and the demand will be met where \( \delta_1 \geq 0 \) and \( 0 < k_1^1 \leq 1 \) are constant, determined by decision makers. If \( \beta_1 \) equals zero, the whole demand is lost. However, if \( \beta_1 \) is equal to one, all demands will be fulfilled. The inventory level on this interval for the first product is calculated by the following differential equation:

\[ I_1^1(t) = -k_1^1 e^{-\delta_1 T} \left( a_1 p_1^{-12} p_1^{-\lambda_1} \right) \left( \alpha_1 A^1 \right) \left( e^{\delta_1 T} - e^{\delta_1 T} \right) \]

\[ - \frac{\delta_1 k_1^1}{\delta_1} \left( e^{\delta_1 (T - t)} - e^{\delta_1 (T - t)} \right), \quad \forall t \in [t_1, T] \]  

(19)

In a similar manner for the second product, we have:

\[ \frac{d I_2^1(t)}{dt} = -D_2 \left( p_2, t, A \right) \beta_2(T - t), \quad \forall t \in [t_2, T], \quad I_2^1(t_1) = 0 \]  

(20)

and

\[ I_2^1(t) = -k_2^1 e^{-\delta_2 T} \left( a_2 p_2^{-12} p_2^{-\lambda_2} \right) \left( \alpha_2 A^2 \right) \left( e^{\delta_2 T} - e^{\delta_2 T} \right) \]

\[ - \frac{\delta_2 k_2^1}{\delta_2} \left( e^{\delta_2 (T - t)} - e^{\delta_2 (T - t)} \right), \quad \forall t \in [t_2, T] \]  

(21)

In addition, the shortage of the first and second products which are replenished as deferred demand is determined by the following equation:

\[ S_1 = -I_1^1(T), \quad S_2 = -I_2^1(T) \]  

(22)

The demand order of the first product \( Q_1 \), over a cycle, is obtained by the following equation:
The demand order of the first product $Q_1$, over a cycle, is calculated by the following equation:

\[
Q_1 = I_0^1 + s_1 = \frac{1}{F_1(t_d)} \int_{t_d}^{t_1} F_1(t) \left( \left( a_1 p_1^{-\kappa_1} p_2^{-\kappa_2} \right) (\alpha_1 A^\beta) e^{\delta t} + \varepsilon_1 \right) dt \\
- \left( a_1 p_1^{-\kappa_1} p_2^{-\kappa_2} \right) (\alpha_1 A^\beta) \left( \frac{1 - e^{\delta t_d}}{\delta t_d} \right) + \varepsilon_1 t_d \\
+ \frac{k_1^2 e^{-\delta T}}{\delta t_d} \left( a_1 p_1^{-\kappa_1} p_2^{-\kappa_2} \right) (\alpha_1 A^\beta) (e^{\delta \lambda t_d} - e^{\delta \lambda t_d} h) + \frac{k_1^2 \delta t_d}{\delta t_d} \left( 1 - e^{\delta (t_1 - T)} \right)
\]

(23)

The demand order of the first product $Q_2$, over a cycle, is calculated by the following equation:

\[
Q_2 = I_0^2 + s_2 = \frac{1}{F_2(t_d)} \int_{t_d}^{t_2} F_2(t) \left( \left( a_2 p_2^{-\kappa_2} p_1^{-\kappa_1} \right) (\alpha_2 A^\beta) e^{\delta t} + \varepsilon_2 \right) dt \\
- \left( a_2 p_2^{-\kappa_2} p_1^{-\kappa_1} \right) (\alpha_2 A^\beta) \left( \frac{1 - e^{\delta t_d}}{\delta t_d} \right) + \varepsilon_2 t_d \\
+ \frac{k_2^2 e^{-\delta T}}{\delta t_d} \left( a_2 p_2^{-\kappa_2} p_1^{-\kappa_1} \right) (\alpha_2 A^\beta) (e^{\delta \lambda t_d} - e^{\delta \lambda t_d} h) + \frac{k_2^2 \delta t_d}{\delta t_d} \left( 1 - e^{\delta (t_2 - T)} \right)
\]

(24)

Now the costs and revenues can be calculated through the following equations:

- The ordering cost:
  \[
  OC = K_1 + K_2
  \]
  (25)

- The storage cost:
  \[
  HC = \int_0^{t_1} h(t) I_1^1(t) dt + \int_{t_1}^{t_2} h(t) I_1^2(t) dt + \int_{t_2}^{t} h(t) I_2^2(t) dt + \int_0^{t_1} h(t) I_2^1(t) dt + \int_{t_2}^{t} h(t) I_2^2(t) dt
  \]
  (26)

- The backlog cost:
  \[
  SC = -\sigma_1 \int_{t_1}^{T} I_1^1(t) dt - \sigma_2 \int_{t_2}^{T} I_2^2(t) dt \\
  = -\sigma_1 \left( \frac{k_1^2 e^{-\delta T}}{\delta t_d} \left( a_1 p_1^{-\kappa_1} p_2^{-\kappa_2} \right) (\alpha_1 A^\beta) \left( \frac{e^{\delta \lambda t_d} - e^{\delta \lambda t_d} h}{\lambda_1 + \delta_1} \right) - \left( T - t_1 \right) e^{\delta \lambda t_d} h \right) \\
  - \frac{k_1^2 \delta t_d}{\delta} \left( 1 - e^{\delta (t_1 - T)} \right) \\
  - \sigma_2 \left( \frac{k_2^2 e^{-\delta T}}{\delta t_d} \left( a_2 p_2^{-\kappa_2} p_1^{-\kappa_1} \right) (\alpha_1 A^\beta) \left( \frac{e^{\delta \lambda t_d} - e^{\delta \lambda t_d} h}{\lambda_2 + \delta_2} \right) - \left( T - t_2 \right) e^{\delta \lambda t_d} h \right) \\
  - \frac{k_2^2 \delta t_d}{\delta} \left( 1 - e^{\delta (t_2 - T)} \right)
  \]
  (27)
Joint pricing and inventory control modelling for obsolescent products

The lost sale cost:

\[
PC = a_1 \left( \int_{t_1}^{t_2} D_1(p_1, p_2, t, A) \, dt - s_1 \right) + a_2 \left( \int_{t_1}^{t_2} D_2(p_1, p_2, t, A) \, dt - s_2 \right)
\]

\[
= a_1 \left( \frac{1}{\lambda_1} \left( e^{\lambda_1 T} - e^{\lambda_1 t_1} \right) - s_1 \right)
+ a_2 \left( \frac{1}{\lambda_2} \left( e^{\lambda_2 T} - e^{\lambda_2 t_2} \right) - s_2 \right)
\]  \hspace{1cm} (28)

The purchasing cost:

\[
PC = c_1 Q_h + c_2 Q_s = c_1 \left( \frac{1}{F_1(t_0)} \int_{t_0}^{t_1} F_1(t) \left( \frac{1}{\lambda_1} \left( e^{\lambda_1 \tau} - e^{\lambda_1 t} \right) + \epsilon_1 \right) \, dt \right.
\]

\[
+ \frac{k_1}{\lambda_1} \left( e^{-\lambda_1 \tau} (a_1 p_1^{-\lambda_1} p_2^{-\lambda_2}) (a_1 A^{\lambda_1}) (e^{\lambda_1 t} - e^{\lambda_1 t_0}) \right)
\]

\[
+ c_2 \left. \left( \frac{1}{F_2(t_2)} \int_{t_2}^{t_3} F_2(t) \left( \frac{1}{\lambda_2} \left( e^{\lambda_2 \tau} - e^{\lambda_2 t} \right) + \epsilon_2 \right) \, dt \right) \right)
\]

\[
= \frac{k_1}{\lambda_1} \left( e^{-\lambda_1 \tau} (a_1 p_1^{-\lambda_1} p_2^{-\lambda_2}) (a_1 A^{\lambda_1}) (e^{\lambda_1 t} - e^{\lambda_1 t_0}) \right)
\]

\[
+ \frac{k_2}{\lambda_2} \left( e^{-\lambda_2 \tau} (a_2 p_2^{-\lambda_2} p_1^{-\lambda_1}) (a_2 A^{\lambda_2}) (e^{\lambda_2 t} - e^{\lambda_2 t_0}) \right)
\]  \hspace{1cm} (29)

Advertisement cost:

\[
CA = c_3 A
\]  \hspace{1cm} (30)

Sale revenue:

\[
p_1 \left( \int_{0}^{t_1} D_1(p_1, p_2, t, A) \, dt + s_1 \right) + p_2 \left( \int_{0}^{t_2} D_2(p_1, p_2, t, A) \, dt + s_2 \right)
\]

\[
= p_1 \left( \frac{1}{\lambda_1} \left( e^{\lambda_1 T} - e^{\lambda_1 t} \right) + \epsilon_1 + s_1 \right)
+ p_2 \left( \frac{1}{\lambda_2} \left( e^{\lambda_2 T} - e^{\lambda_2 t} \right) + \epsilon_2 + s_2 \right)
\]  \hspace{1cm} (31)

Then, the profit in the unit of time is:
\[ TP(p_1, p_2, A, h, t_2, T) = \frac{SR - AC - PC - SC - HC - OC - OPC}{T} \]

\[= \frac{1}{T} \left[ p_1 \left( \frac{e^{\lambda T}}{\lambda_i} \right) + \epsilon_T \right] + p_2 \left( \frac{e^{\epsilon_T}}{\lambda_T} \right) + \epsilon_T \]

\[+ \frac{k_1^T e^{-\lambda T}}{\lambda_1 + \delta_1} (a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) (e^{(\lambda_1 h)T} - e^{(\lambda_1 h)h}) + \frac{k_1^2 \epsilon_T}{\lambda_1} (1 - e^{h(\lambda_1 - T)}) \]

\[+ \frac{k_2^T e^{-\lambda T}}{\lambda_2 + \delta_2} (a_2 p_2^{x_2} p_2^{x_2}) (\alpha_2 A^0) (e^{(\lambda_2 h)T} - e^{(\lambda_2 h)h}) + \frac{k_2^2 \epsilon_T}{\lambda_2} (1 - e^{h(\lambda_2 - T)}) \]

\[-c_A A - K_4 - K_2 - c_1 \int \frac{1}{F_1 (t_0)} \int F_1 (t) \left( (a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) \right) e^{\epsilon_T} + \epsilon_T \right] \]

\[-(a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) \left( \frac{1 - e^{\lambda_0 h}}{\lambda_i} \right) + \epsilon_T \]

\[+ \frac{k_1^T e^{-\lambda T}}{\lambda_1 + \delta_1} (a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) (e^{(\lambda_1 h)T} - e^{(\lambda_1 h)h}) + \frac{k_1^2 \epsilon_T}{\lambda_1} (1 - e^{h(\lambda_1 - T)}) \]

\[-c_2 \int \frac{1}{F_2 (t_0)} \int F_2 (t) \left( (a_2 p_2^{x_2} p_2^{x_2}) (\alpha_2 A^0) \right) e^{\epsilon_T} \]

\[+ \epsilon_T \int \frac{1}{F_2 (t_0)} \int F_2 (t) \left( (a_2 p_2^{x_2} p_2^{x_2}) (\alpha_2 A^0) \right) e^{\epsilon_T} + \epsilon_T \right] \]

\[-(a_2 p_2^{x_2} p_2^{x_2}) (\alpha_2 A^0) \left( \frac{1 - e^{\lambda_0 h}}{\lambda_2} \right) + \epsilon_T \]

\[+ \frac{k_1^T e^{-\lambda T}}{\lambda_1 + \delta_1} (a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) (e^{(\lambda_1 h)T} - e^{(\lambda_1 h)h}) + \frac{k_1^2 \epsilon_T}{\lambda_1} (1 - e^{h(\lambda_1 - T)}) \]

\[+ c_3 \int \frac{k_1^T e^{-\lambda T}}{\lambda_1 + \delta_1} (a_1 p_1^{x_1} p_2^{x_2}) (\alpha_1 A^0) \left( \frac{(e^{(\lambda_1 h)T} - e^{(\lambda_1 h)h})}{(\lambda_i + \delta_i)} \right) (T - h) e^{(\lambda_1 h)h} \]

\[- k_1^2 \epsilon_T \left( \frac{1 - e^{(\lambda_1 h)T}}{\lambda_i} \right) (T - h) e^{(\lambda_1 h)h} \]

\[+ c_3 \int \frac{k_2^T e^{-\lambda T}}{\lambda_2 + \delta_2} (a_2 p_2^{x_2} p_2^{x_2}) (\alpha_2 A^0) \left( \frac{(e^{(\lambda_2 h)T} - e^{(\lambda_2 h)h})}{(\lambda_i + \delta_i)} \right) (T - h) e^{(\lambda_2 h)h} \]

\[- k_2^2 \epsilon_T \left( \frac{1 - e^{(\lambda_2 h)T}}{\lambda_i} \right) (T - h) e^{(\lambda_2 h)h} \]

\[- \int_{t_0}^{t_0} h_2(t) F_2(t) dt - \int_{t_0}^{t_0} h_2(t) F_2(t) dt - \alpha_1 \left( a_1 p_1^{x_1} p_2^{x_2} (\alpha_1 A^0) \left( \frac{e^{(\lambda_1 h)T} - e^{(\lambda_1 h)h}}{\lambda_i} \right) \right) \left. - k_1 \right|_{t_0}^{t_0} \]

\[-\alpha_2 \left( a_2 p_2^{x_2} p_2^{x_2} (\alpha_2 A^0) \left( \frac{e^{(\lambda_2 h)T} - e^{(\lambda_2 h)h}}{\lambda_i} \right) \right) \left. - s_2 \right|_{t_0}^{t_0} \]
Since \( TP \) depends on random variables, the expected value of the objective function is maximised and it is supposed that:

\[
E(v_1) = \mu_1, \ E(v_2) = \mu_2
\]  

(33)

Examining the properties of \( E(TP) \), an algorithm is developed to find the maximum value for the function. It is assumed in this paper that the minimum and maximum cost of advertisement are \( A_{\text{min}} \) and \( A_{\text{max}} \), respectively.

**Lemma 1:** For the fixed value of \( p_1, p_2, t_1, t_2, T \), the function \( E[TP(p_1, p_2, ..., t_1, t_2, T)] \) attains its maximum level at \( A_{\text{min}} \) or \( A_{\text{max}} \).

**Proof:** \( E(TP) \) can be represented by the following formula:

\[
E[TP(p_1, p_2, A, t_1, t_2, T)] = a_1 A^{\delta_1} G_1(p_1, p_2, t_1, t_2, T, \mu_1, \mu_2) + a_2 A^{\delta_2} G_2(p_1, p_2, t_1, t_2, T, \mu_1, \mu_2) + K
\]

(34)

where \( K \) is a fixed value. Considering \( p_1, p_2, t_1, t_2, T \), as constant and the specific distribution for random variables, \( E[TP(p_1, p_2, ..., t_1, t_2, T)] \) will be either ascending or descending with respect to \( A \). As a result, the maximum value is obtained at the beginning or at the end of the domain.

Lemma 1 indicates that the values at the beginning or at the end of the domain can be compared, so as to find the maximum value of \( E(TP) \) function. Therefore, to optimise \( E[TP(p_1, p_2, A_2, t_1, t_2, T)] \), one can maximise and separately and choose the best attained solution. While the advantage of this model is the reduction of problem size, the optimisation problems must be solved, too.

### 6 The proposed model in a real context

As mentioned above, this paper investigated Irancell Company as an Iranian telecommunication company, operating in the field of mobile network services, mobile internet services and WiMAX. Irancell Company cooperates with 25 distributors, 11,797 dealers and 302 service centres. This investigation was carried out at two levels of manufacturer and distributor. Having introduced the proposed model in the previous section, this part deals with the procedure for reaching the optimal solution of the proposed mathematical programming model. Cellphone and SIM card are two complementary commodities investigated in this study. It was assumed that a mobile phone and SIM card would cost 2,500,000 and 300,000 IRR, respectively. Moreover, the cellphone and SIM card life cycles were considered to be 180 and 270 days, respectively. Finally, cellphone and SIM card setting up (ordering registration) costs were assumed to be 50 and 10 million IRR, respectively. The maintenance cost function per unit for each day was calculated according to the following equation:

\[
h_i(t) = \frac{1}{365} \max (10000, 10000 + 50(t - 180))
\]

(35)

In other words, the above function which was proposed by Shah et al. (2013) reveals that maintenance costs will increase by 500 tomans per day after 180 days. The maintenance cost for SIM card is assumed to be 2,000 IRR per year:
Gradual obsolescence rate follows a three-parameter Weibull distribution. It has been empirically observed that the expected failure and life cycle of many of items can be shown by the Weibull distribution. These empirical observations have encouraged researchers to show deterioration rate by the Weibull distribution (Begum et al., 2012). The obsolescence rate function of the phone is calculated by the following equation:

$$h_1(t) = \frac{2000}{365}$$

(36)

where $\rho_1 > 0$, $\tau_1 > 0$, and $\gamma_1 > 0$ are the scales, shape parameter and situation parameter, respectively. Obsolescence rate of the SIM card is calculated by the following equation:

$$\theta_2(t) = \frac{2000}{365}$$

(37)

where $\rho_2 > 0$ is the scales parameter, $\tau_2 > 0$ is the shape parameter and $\gamma_2 > 0$ is the situation parameter.

According to coefficients in Table 2, the following optimal value $p_1$, $p_2$, $A_2$, $t_1$, $t_2$, $T$, $Q_1$, $Q_2$, $TP$ is calculated for the following coefficients $a_1$, $a_2$, $p_1$, $p_2$, where $p_1$, $p_2$ is assumed to be the multiple of 100,000, time variables $t_1$, $t_2$, $T$ are divided by 365. $Q_1$ and $Q_2$ are the multiple of 10,000 as well.

Table 2  Parameters values used in the proposed model

<table>
<thead>
<tr>
<th>Fixed values included in the model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1 = 40,000$</td>
</tr>
<tr>
<td>$\sigma_2 = 3,000$</td>
</tr>
<tr>
<td>$\sigma_1 = 120,000$</td>
</tr>
<tr>
<td>$\sigma_2 = 2,400$</td>
</tr>
<tr>
<td>$k_1 = 1$</td>
</tr>
<tr>
<td>$k_2 = 1$</td>
</tr>
<tr>
<td>$S_2 = 1,000$</td>
</tr>
<tr>
<td>$S_2 = 5,000$</td>
</tr>
<tr>
<td>$\delta_1 = 0.01$</td>
</tr>
<tr>
<td>$\delta_2 = 0.01$</td>
</tr>
<tr>
<td>$\lambda_1 = -1/365$</td>
</tr>
<tr>
<td>$\lambda_2 = -4/3,650$</td>
</tr>
<tr>
<td>$\kappa_1 = 4$</td>
</tr>
<tr>
<td>$\kappa_2 = 2$</td>
</tr>
<tr>
<td>$l_1 = 0.1$</td>
</tr>
<tr>
<td>$l_2 = 0.2$</td>
</tr>
<tr>
<td>$\gamma_1 = 180$</td>
</tr>
<tr>
<td>$\gamma_2 = 270$</td>
</tr>
<tr>
<td>$\alpha_1 = 60$</td>
</tr>
<tr>
<td>$\alpha_2 = 15$</td>
</tr>
<tr>
<td>$\beta_1 = 0.3$</td>
</tr>
<tr>
<td>$\beta_2 = 0.5$</td>
</tr>
<tr>
<td>$\mu_1 = 0.5$</td>
</tr>
<tr>
<td>$\mu_2 = 0.5$</td>
</tr>
<tr>
<td>$r_1 = 2$</td>
</tr>
<tr>
<td>$r_2 = 2$</td>
</tr>
<tr>
<td>$c_4 = 2 \times 10^6$</td>
</tr>
</tbody>
</table>

As shown in Table 3, an increase in the fixed coefficients of the demand function (the initial demand) improves the profit. Table 3 reflects the sensitivity analysis of the proposed model according to the fixed coefficients of the demand function and the deterioration function.
<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$A$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$T$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$TP$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 10^2 \times 365^2$</td>
<td>$0.05 \times 365^2$</td>
<td>0.01</td>
<td>0.01</td>
<td>33.28</td>
<td>0.44</td>
<td>100</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>3.14</td>
<td>17.2</td>
<td>$9.46 \times 10^3$</td>
</tr>
<tr>
<td>$1.8 \times 10^2 \times 365^2$</td>
<td>$0.05 \times 365^2$</td>
<td>0.01</td>
<td>0.01</td>
<td>33.27</td>
<td>0.46</td>
<td>100</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>2.82</td>
<td>16.4</td>
<td>$8.59 \times 10^3$</td>
</tr>
<tr>
<td>$2 \times 10^2 \times 365^2$</td>
<td>$0.06 \times 365^2$</td>
<td>0.01</td>
<td>0.01</td>
<td>33.26</td>
<td>0.46</td>
<td>100</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>3.14</td>
<td>19.2</td>
<td>$9.64 \times 10^3$</td>
</tr>
<tr>
<td>$1.8 \times 10^2 \times 365^2$</td>
<td>$0.06 \times 365^2$</td>
<td>0.01</td>
<td>0.01</td>
<td>33.26</td>
<td>0.47</td>
<td>100</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>2.82</td>
<td>18.5</td>
<td>$8.77 \times 10^3$</td>
</tr>
<tr>
<td>$2 \times 10^2 \times 365^2$</td>
<td>$0.05 \times 365^2$</td>
<td>0.03</td>
<td>0.01</td>
<td>33.28</td>
<td>0.44</td>
<td>100</td>
<td>0.80</td>
<td>0.82</td>
<td>0.82</td>
<td>3.14</td>
<td>17.2</td>
<td>$9.46 \times 10^3$</td>
</tr>
<tr>
<td>$1.8 \times 10^2 \times 365^2$</td>
<td>$0.05 \times 365^2$</td>
<td>0.03</td>
<td>0.01</td>
<td>33.28</td>
<td>0.46</td>
<td>100</td>
<td>0.80</td>
<td>0.82</td>
<td>0.82</td>
<td>2.82</td>
<td>16.4</td>
<td>$8.59 \times 10^3$</td>
</tr>
<tr>
<td>$2 \times 10^2 \times 365^2$</td>
<td>$0.06 \times 365^2$</td>
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<td>0.01</td>
<td>33.28</td>
<td>0.46</td>
<td>100</td>
<td>0.80</td>
<td>0.82</td>
<td>0.82</td>
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<td>19.2</td>
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</tr>
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<td>$0.06 \times 365^2$</td>
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<td>0.01</td>
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<td>0.82</td>
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</tr>
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<td>0.46</td>
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<td>0.80</td>
<td>0.82</td>
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<td>0.03</td>
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<td>0.47</td>
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<td>0.80</td>
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<td>0.03</td>
<td>0.03</td>
<td>33.27</td>
<td>0.46</td>
<td>100</td>
<td>0.80</td>
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<td>0.80</td>
<td>0.82</td>
<td>0.82</td>
<td>2.82</td>
<td>17.2</td>
<td>$9.46 \times 10^3$</td>
</tr>
</tbody>
</table>
Since the inflation rate is high in Iranian market, Irancell asked the researchers to investigate the behaviour of the provided solutions of the model under the perturbation of SIM card and cell phone cost. To this end, Irancell provided the researchers with nine prospective scenarios for the cost of products. To investigate the behaviour model for each scenario, the researchers set the parameters similar to the former case. In addition, the following was set:

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Quantity of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = 2 \times 100^2 \times 365^2$</td>
<td>$a_2 = 0.05 \times 365^2$</td>
</tr>
</tbody>
</table>

Table 5 summarises the results obtained using the model. It is readily seen that products cost rise results in their optimal price increase. Furthermore, the results indicate that demand lowers considerably after increase of products, leading to the total profit drop. However, the time variables have not changed significantly. In fact, the model proposed the same solutions for the time variables under each scenario.

<table>
<thead>
<tr>
<th>Table 5</th>
<th>Optimal values obtained in the proposed model in scenario 2 for $a_1$, $a_2$, $\rho_1$, $\rho_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$c_2$</td>
</tr>
<tr>
<td>25</td>
<td>0.3</td>
</tr>
<tr>
<td>25</td>
<td>0.4</td>
</tr>
<tr>
<td>25</td>
<td>0.5</td>
</tr>
<tr>
<td>26</td>
<td>0.3</td>
</tr>
<tr>
<td>26</td>
<td>0.4</td>
</tr>
<tr>
<td>26</td>
<td>0.5</td>
</tr>
<tr>
<td>27</td>
<td>0.3</td>
</tr>
<tr>
<td>27</td>
<td>0.4</td>
</tr>
<tr>
<td>27</td>
<td>0.5</td>
</tr>
</tbody>
</table>

7 Discussion and conclusions

Firms are increasingly looking for ways to fulfil goals such as achieving a specific market share and credit, preserving existing market share and customers, entering new markets, finding new customers, minimising costs and maximising profits. Bundle pricing strategy is one of the innovative techniques and methods in this area to increase firms’ sales revenue and profit margins.

The EOQ and the economic production quantity are among the most fundamental and basic inventory models. Modifying the underlying assumptions of these models is one of the significant changes which have been brought about over the years following the introduction of these models. One of the innovations of this research was the development of an inventory control model by setting price as a multi-product decision variable in the inventory and adding decision constraints to the model. Joint inventory control and pricing strategy of product bundling have not been previously studied thoroughly in the industry. Therefore, it is hoped that the results and findings of this research, as well as, the optimal inventory control model and the joint pricing strategy for bundling which were presented in this paper help telecommunication companies such as
Joint pricing and inventory control modelling for obsolescent products

Irancell to solve the problem of jointly finding the optimal bundling mechanism and provide opportunities for them to improve their products and services, attract new customers, increase market penetration and maximise their profit. It is also expected that solving this model will result in the optimal amount of inventory, replenishment cycles and pricing for obsolescent products complement each other in the supply chain of Irancell Cooperation.

All research studies have limitations and this study is no exception. One of the limitations of this research was the lack of historical data for the product under study, because the studied product was a new bundle. It is recommended that future studies take advantage of grey and fuzzy system theories as appropriate methods for taking data uncertainties into account. Another limitation of this study was that the errors in the demand functions were considered as time independent. Future studies could develop the proposed demand functions by considering errors as time-dependent variables. Additionally, an integrated pricing and inventory control model for obsolescence products can be investigated in future research within a bi-level supply chain for a multi-product. Extending the proposed model within a multi-level supply chain could be done by considering quantity discounts. Last but not least, the demand function can be extended in future studies. For instance, Jaggi and Kausar (2009, 2010) studied the impact of credit period and trade credit on demand function. Jaggi and Khanna (2008) have taken the inflation rate into account in their production-inventory model for deteriorating products. Future studies, too, can develop the proposed demand functions by analysing the impact of the credit and economic indicators like inflation rate on demand rate.

References


Joint pricing and inventory control modelling for obsolescent products


