On the higher-order edge-tenacity of a graph

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Abstract

The area of graph vulnerability concerns the question of how much communication in a network is disrupted by the deletion of edges from the graph. The most fundamental measure of graph vulnerability of a connected graph is the edge-connectivity of the graph. One of the motivations in studying the edge-tenacity of a graph is that it can be a more refined measure of vulnerability than that based on simple edge-connectivity.

The first-order edge-tenacity $T_1(G)$ of a graph $G$ is defined as

$$T_1(G) = \min \{ \frac{|X|+\tau(G-X)}{\omega(G-X)-1} | X \subseteq E(G) \text{ and } \omega(G-X) > 1 \}$$

The quantity $\omega(G-X) - 1$ can be interpreted as the number of additional components that are created by removing the set $X$ of edges from the connected graph $G$. Then the set $X$ that minimizes $\frac{|X|+\tau(G-X)}{\omega(G-X)-1}$ is the set whose removal minimizes the number of edges deleted and the size of the largest component, per additional component created. The smaller the edge-tenacity, the more vulnerable is the graph. The edge-tenacity becomes a more significant measurement in comparing the vulnerability of two graphs when they have the same edge-connectivity.

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1 INTRODUCTION

Throughout this paper, we allow multiple edges but no loops in all the graphs under consideration. Our terminology will be standard except as indicated; a good reference for any undefined terms is [1]. We use $V(G)$ and $\omega(G)$ to denote the vertex set and the number of components in a graph $G$, respectively. Vulnerability of graphs has attracted much attention among graph theorists and network designers [6, 7, 20]. The concept of tenacity of a graph $G$ was introduced in [2, 3] as a useful measure of the “vulnerability” of $G$. In [3] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs. In [11] we showed a complete proof for case three of the Harary Graphs. In [9], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs.

The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. The tenacity of a graph $G$, $T(G)$, is defined by $T(G) = \min \{ \frac{|V(G)-1|}{\omega(G-S)} | S \subseteq V(G) \}$, where the minimum is taken over all vertex cutsets $S$ of $G$. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph $G$ is called $T$-tenacious if $|S| + \tau(G-S) \geq 2\omega(G-S)$ holds for any subset $S$ of vertices of $G$ with $\omega(G-S) > 1$. If $G$ is not complete, then there is a largest $T$ such that $G$ is...
$T$-tenacious; this $T$ is the tenacity of $G$. On the other hand, a complete graph contains no vertex cutset and so it is $T$-tenacious for every $T$. Accordingly, we define $T'(K_p) = \infty$ for every $p$ ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a $T$-set of $G$ if $T(G) = \frac{|S| + \tau(G - S)}{\omega(G - S)}$

The Mix-tenacity $T_m(G)$ of a graph $G$ is defined as

$$T_m(G) = \min_{A \subseteq E(G)} \{ \frac{|A| + m(G - A)}{\omega(G - A)} \}$$

where $m(G - A)$ denotes the order (the number of vertices) of a largest component of $G - A$ and $\omega(G - A)$ is the number of components of $G - A$. Cozzens et al. in [2], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.

After the pioneering work of Cozzens, Moazzami, and Stueckle in [2, 3], several groups of researchers have investigated tenacity, and its related problems. In [15] and [16] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subseteq E(G)$ and the number of vertices of a largest component, $\tau(G - A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn’t seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [10]. The Edge-tenacity $T_e(G)$ of a graph $G$ is defined as

$$T_e(G) = \min_{A \subseteq E(G)} \{ \frac{|A| + \tau(G - A)}{\omega(G - A)} \}$$

where $\tau(G - A)$ denotes the order (the number of edges) of a largest component of $G - A$ and $\omega(G - A)$ is the number of components of $G - A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question “How difficult is it to recognize $T$-tenacious graphs?” has remained an interesting open problem for some time. The question was first raised by Moazzami in [12]. Our purpose in [4] was to show that for any fixed positive rational number $T$, it is $NP$-hard to recognize $T$-tenacious graphs. To prove this we showed that it is $NP$-hard to recognize $T$-tenacious graphs by reducing a well-known $NP$-complete variant of INDEPENDENT SET.

In [14] and [19], respectively, Nash-Williams and Tutte proved the following theorem.

**Theorem A.** A connected graph $G$ has a edge-disjoint spanning trees if and only if

$$|X| \geq s(\omega(G - X) - 1) \text{ for each } X \subseteq E(G)$$

An immediate consequence of Theorem A is Theorem 1.

**Theorem 1.** If graph $G$ has a edge-disjoint spanning trees then

$$|X| + \tau(G - X) \geq s(\omega(G - X) - 1) \text{ for each } X \subseteq E(G)$$

The following corollary can easily be concluded from Theorem 1.

**Corollary 1.** If graph $G$ has a edge-disjoint spanning trees then $T_1(G) \geq s$.

**Conjecture:** The first-order edge-tenacity of a graph is $NP$-complete.

The definition of the first-order edge-tenacity of a graph also admits a natural generalization as follows.

Let $k$ be a natural number. A subset $X$ of $E(G)$ is called a $k$-cut of $G$ if $\omega(G - X) > k$. Thus, a $1$-cut of $G$ is just an edge cut of $G$. We denote by $|\lambda^k(G)|$ the size of a smallest $k$-cut of $G$, called the $k$th-order edge-connectivity of $G$. Note that $|\lambda^k(G)|$ exists for each $k = 1, 2, \ldots, |V(G)| - 1$. Also, if $\omega(G) \leq k \leq |V(G)| - 1$ and $X \subseteq E(G)$ is a $k$th-order of size $|\lambda^k(G)|$ then $\omega(G - X) = k + 1$, whereas if $k < \omega(G)$ then $|\lambda^k(G)| = 0$.

For an integer $k$, $1 \leq k \leq |V(G)| - 1$, we define the $k$th-order edge-tenacity of a graph $G$ as

$$T_k(G) = \min \{ \frac{|X| + \tau(G - X)}{\omega(G - X) - k} \mid X \subseteq E(G) \text{ and } \omega(G - X) > k \}$$

As an immediate consequence of Theorem 1 we have

**Theorem 2.** If graph $G$ has a edge-disjoint spanning trees then
\(|X| + \tau(G - X) \geq s(\omega(G - X) - k)\) for each \(X \subseteq E(G)\).

**Corollary 2.** If graph \(G\) has \(s\) edge-disjoint spanning trees then \(T_k(G) \geq s\).

**Conjecture:** The \(k\)-th order edge-tenacity of a graph is NP-complete.

It is not clear whether the \(k\)-th order edge-tenacity of a graph can be computed in polynomial time. However, the maximum number of edge-disjoint spanning trees in a graph can be computed in polynomial time by matroid partitioning algorithms[5] see also [17]), and so by Corollary 2 the \(k\)-th order edge-tenacity of a graph can be very closely approximated.

In the remainder of this section, we shall recall some definitions and known results that will be used to prove our theorems in the subsequent sections.

**Theorem B.** (Marder [8]). If \(G\) is a connected vertex-transitive graph then 
\(\lambda(G) = \delta(G)\).

**Theorem C.** (Tindell [18]). If \(G\) is a connected regular edge-transitive graph then 
\(\lambda(G) = \delta(G)\).

Recall the arboricity of a graph \(G\) is the smallest number of forest whose union is \(G\). We shall also need the following result due to Nash-Williams [13].

**Theorem D.** The arboricity of a nonempty graph \(G\) is

\[
\max \left\lfloor \frac{|E(H)|}{|V(H)|-1} \right\rfloor,
\]

where the maximum is taken over all nontrivial subgraphs \(H\) of \(G\).

## 2 Bounds for \(T_k(G)\)

**Theorem 3.** Let \(G\) be a graph of order \(p\) and let \(k\) be an integer with \(1 \leq k \leq p-1\). Then

\[
\frac{\lambda(G)}{2(p-k)} \leq T_k(G) \leq |\lambda^k(G)| \leq \lambda^k(G).
\]

\[
\frac{|X| + \tau(G - X)}{s(G - X) - k} \leq T_k(G) \leq |\lambda^k(G)| + \tau(G - \lambda^k(G)).
\]

**Proof.** The result is clear if \(\omega(G) > k\). We may assume that \(\omega(G) \leq k\). Then there exists a \(k\)-cut \(X_0\) of \(G\) with \(|\lambda^k(G)| \leq |X_0|\) and, hence, \(\omega(G - X_0) = k + 1\). Thus,

\[
T_k(G) \leq \frac{|X_0| + \tau(G - X_0)}{s(G - X_0) - k} = |\lambda^k(G)| + \tau(G - \lambda^k(G)).
\]

On the other hand, let \(X\) be a \(k\)-cut of \(G\) minimizing \(\frac{|X| + \tau(G - X)}{s(G - X) - k}\). Let \(H_1, H_2, \ldots, H_n\) be the components of \(G - X\). Then \(\omega > k\). For each \(i\), let \(X_i\) be the set of all edges in \(X\) with one end in \(H_i\). By the minimality of \(X\), no edge of \(X\) has both of its ends in the same component of \(G - X\). Thus,

\[
2|X| = \sum_{i=1}^n |X_i| \geq \omega|\lambda(G)|.
\]

since \(|X_i| \geq |\lambda(G)|\) for each \(i\). From this we have

\[
T_k(G) = \frac{|X| + \tau(G - X)}{s(G - X) - k} \geq \frac{\omega|\lambda(G)|}{2(p-k)} \geq \frac{|\lambda(G)|}{2(p-k)}.
\]

**Corollary 3.** Let \(G\) be a graph of order \(p\), size \(q\) and let \(k\) be an integer, with \(1 \leq k \leq p-1\). Then \(T_k(G) \leq \frac{q}{p-k}\).

**Proof.** Suppose \(|X| = q\). Then the number of components is \(p\), and the size of largest component is \(0\) therefore \(T_k(G)\) will be \(\frac{q}{p-k}\). In any other case (i.e \(|X| < q\) \(T_k(G)\) should be less than or equal to \(\frac{q}{p-k}\).

**Theorem 4.** Let \(G\) be a graph with \(p\) vertices and \(q\) edges, and let \(s = \frac{q}{p-k}\), where \(k\) is an integer satisfying \(1 \leq k \leq p-1\). Then \(T_k(G) = s\) if and only if

\[
|E(H)| \leq s(|V(H)| - 1)
\]

for every subgraph \(H\) of \(G\).

**Proof.** Suppose that \(T_k(G) = s\). Let \(H\) be any subgraph of \(G\) with \(p'\) vertices and \(q'\) edges. Without loss of generality, we may assume that \(H\) is an induced subgraph of \(G\). If \(p' > p - k\) then it is obvious that \(q' \leq s(p'-1)\). Otherwise, let \(x\) be the cardinality of the \(k\)-cut \(X = E(G) - E(H)\) of \(G\). Then \(q = q' + x\) and \(\omega(G - X) \geq p - p' + 1\). Therefore, we have

\[
s \leq \frac{|X| + \tau(G - X)}{s(G - X) - k} \leq \frac{x + \tau(G - X)}{p - p' + 1}.
\]
or

\[ s(p - k) - s(p' - 1) \leq x + \tau(G - X) = q' - q + \tau(G - X). \]

Hence,

\[ q' \leq s(p' - 1) + \tau(G - X) \]

On the other hand, when \( T_k(G) = \frac{q}{p-k} \), we should select all the edges to be in our edge-cutset and so the size of largest component is zero so

\[ q' \leq s(p' - 1), \]

as required.

Conversely, let \( H \) be any subgraph of \( G \) and \( X = E(G) - E(H) \) be a \( k \)-cut of \( G \) that separating \( G \) into \( \omega \) components \( H_1, H_2, \ldots, H_\omega \). Then, for each component \( H_i \),

\[ |E(H_i)| \leq s(|V(H_i)| - 1) \]

Thus,

\[ s(p - k) = q = \sum_{i=1}^{\omega} |E(H_i)| + |X| \leq \sum_{i=1}^{\omega} s(|V(H_i)| - 1) + |X| \]

\[ = s(p - \omega) + |X| = sp - s\omega + |X|. \]

Therefore,

\[ s \leq \frac{|X + \omega(G - X)|}{p-k} \]

and, by definition, \( T_k(G) \geq s \). On the other hand, by the Corollary 3, \( T_k(G) \leq s \). So, we have the desired result.

Corollary 4. Let \( G \) be a graph with \( p \) vertices and \( q \) edges, and let \( s = \frac{2q}{p^2} \). Then \( T_1(G) = s \) if and only if

\[ |E(H)| \leq s(|V(H)| - 1) \]

for every subgraph \( H \) of \( G \).

\[ 3 \] \hspace{1cm} REGULAR GRAPHS

In this section, we shall find the edge tenacity of special class of regular graphs whose edge-connectivity and regularity are equal. We shall first recall the following theorem.

The following result is an immediate consequence of Corollary 4.

Theorem 5. Let \( G \) be a graph of order \( p \) with edge-connectivity \( \lambda \). Then \( T_1(G) = \frac{pe}{2(p-1)} \) if and only if \( G \) is \( \lambda \)-regular.

The following consequences are straightforward. Corollary 5 can be concluded by using the fact that a \( r \)-regular graph of order \( p \) and \( r \geq \lceil \frac{p}{2} \rceil \) has edge-connectivity \( r \).

Corollary 5.

(a) If \( G \) is \( r \)-regular of order \( p \) and \( r \geq \lceil \frac{p}{2} \rceil \), then \( T_1(G) = \frac{pe}{2(p-1)} \). If

\[ r = \frac{k-1}{2}, \]

then \( T_1(G) = \frac{pe}{2(p-1)} \).

(b) If \( G \) is a vertex-transitive graph or a regular edge-transitive graph of order \( p \) and size \( q \), then \( T_1(G) = \frac{1}{q-1} \).

The cartesian product \( G = G_1 \times G_2 \) has \( V(G) = V(G_1) \times V(G_2) \), and two vertices \((u_1, u_2) \) and \((v_1, v_2) \) of \( G \) are adjacent if and only if either \( u_1 = v_1 \) and \( u_2v_2 \in E(G_2) \) or \( u_2 = v_2 \) and \( u_1v_1 \in E(G_1) \). The \( n \)-cube \( Q_n \) is the graph \( K_2 \) if \( n = 1 \), while for \( n \geq 2 \), \( Q_n \) is defined inductively as \( Q_{n-1} \times K_2 \).

Corollary 6. Let \( G_i (i = 1, 2) \) be a connected graph of order \( p_i \), size \( q_i \), and edge-connectivity \( \lambda_i \). If \( G_i \) is \( \lambda_i \)-regular, then \( T_1(G_1 \times G_2) = \frac{(q_1p_2 + q_2p_1)}{(p_1p_2 - 1)} \).

Corollary 7.

(a) \( T_1(K_n \times C_m) = \frac{(nm\lceil n+1 \rceil)}{(2nm-1)} \).

(b) \( T_1(K_n \times K_m) = \frac{(nm\lceil n+m \rceil - 2)}{(2nm-1)} \).
(c) \( T_1(C_n \times C_m) = \frac{\binom{2mn}{nm-1}}{nm-1} \);

(d) \( T_1(K_n(m) \times C_r) = \frac{\binom{nm(n+m)}{2(nm-1)}}{2nm-1} \);

(e) \( T_1(K_n(m) \times K_r) = \frac{\binom{nm(n+m-r)}{2(nm-1)}}{2nm-1} \);

(f) \( T_1(K_n(m) \times K_{r+1}) = \frac{\binom{nm(n+n+1)}{2(nm-1)}}{2nm-1} \);

(g) \( T_1(Q_n) = \frac{(2n-1)}{2} \).

\[ (a) \quad K_{1} \times C_{3} \quad (b) \quad K_{4} \times K_{2} \]

\[ (c) \quad C_{3} \times C_{3} \quad (d) \quad K_{2} \times C_{3} \]

\[ (e) \quad K_{2} \times K_{3} \quad (f) \quad K_{2} \times K_{3} \]

\[ (g) \quad Q_{3} \]

In closing this section we note that \( T_1(G) = \frac{|E(G)|}{|V(G)| - 1} \) if and only if the arboricity of \( G \) is \( \left\lceil \frac{|E(G)|}{|V(G)| - 1} \right\rceil \) by theorem 6 and the definition of first-order edge-tenacity. Thus we have

Theorem 6.

The arboricity of a vertex-transitive graph or a regular edge-transitive graph of order \( p \) and regularity \( r \) is \( \left\lceil \frac{p}{2(r-1)} \right\rceil \).

4 CONSTRUCTIONS

Theorem 7. Given any two integers \( r, s \) satisfying \( \frac{r}{2} < s \leq r \) there exists a graph \( G \) such that \( \lambda(G) = r \) and \( T_1(G) = s \).

The aim of this section is to prove the theorem above by constructing three infinite families of graphs with the required properties.
Construction I

Let \( r \) and \( s \) be positive integers satisfying \( \frac{r}{2} \leq s \leq r \). Consider the complete graph \( K_{2s} \). Let \( G \) be the graph obtained from \( K_{2s} \), deleting any set of \( r - s \) independent edges \( A \) and by adding a new vertex \( w \) that is joined to the ends of the edges in \( A \) and to any other \( 2s - r \) vertices. (By definition, a set of edges is independent if they are mutually nonadjacent.) Clearly \( \lambda(G) = r \), \( |V(G)| = 2s + 1 \) and \( |E(G)| = 2s^2 \). We need to show that \( T_1(G) = s \).

Let \( H \) be any subgraph of \( G \) with \( p' \) vertices and \( q' \) edges. If \( w \not\in V(H) \), then

\[
q' \leq \frac{p'(p' - 1)}{2} \leq s(p' - 1)
\]

because \( H \) is a subgraph of \( K_{p'} \) and \( p' \leq 2s \). If \( w \in V(H) \), then \( H \) is obtained from \( K_{p' - 1} \) by deleting \( t(\leq r - s) \) edges and by adding the vertex \( w \) and at most \( 2t + 2s - r + (r - s - t) \) edges. Thus

\[
q' \leq |E(K_{p' - 1})| + s = \frac{(p' - 1)(p' - 2)}{2} + s \leq s(p' - 2) + s = s(p' - 1)
\]

since \( p' \leq 2s + 1 \). Therefore, by Corollary 4, \( T_1(G) = s \), as desired.

Clearly, the construction can be applied repeatedly to obtain infinitely many graphs of different orders and sizes with the required properties.

Construction II

Let \( n, d \) and \( m \) be positive integers with \( d \leq m \). We denote by \( G(n, d, m) \) the set of graphs, each consisting of two disjoint complete graphs \( K_n \) and \( K_m \) such that each vertex of \( K_n \) is adjacent to exactly \( d \) distinct vertices of \( K_m \).

Theorem 8.

(a) Let \( t \) and \( s \) be positive integers with \( t + 1 \leq s \leq 2t \). Then any graph \( G \in G(n, d, m) \) with \( n = 4t - 2s + 1 \), \( d = 2s - 2t \), and \( m = 2s \) satisfies \( \lambda(G) = 2t \) and \( T_1(G) = s \).

(b) Let \( t \) and \( s \) be positive integers with \( t + 1 \leq s \leq 2t + 1 \). Then any graph \( G \in G(n, d, m) \) with \( n = 4t - 2s + 3 \), \( d = 2s - 2t - 1 \), and \( m = 2s \) satisfies \( \lambda(G) = 2t + 1 \) and \( T_1(G) = s \).

Proof: (a) Clearly \( |V(G)| = 4t + 1 \), \( |E(G)| = 4st \) and \( \lambda(G) = 2t \). Thus \( s = \frac{|E(G)|}{|V(G)|} \). We denote by \( H(u, w) \) any subgraph of \( G \) that contains \( u \) vertices of \( K_n \) and \( w \) vertices of \( K_m \). Note that \( d \leq s \). Thus if \( 0 \leq u \leq d \), then we have

\[
|E(H(u, w))| \leq \left( \begin{array}{c} u \\ 2 \end{array} \right) + \left( \begin{array}{c} w \\ 2 \end{array} \right) + uw
\]

\[
= \frac{1}{2} (u^2 - u + w^2 - w + 2uw)
\]

\[
= \frac{1}{2} (u + w)(u + w - 1)
\]

\[
\leq \frac{1}{2} (n + d)(u + w - 1)
\]
= \frac{1}{2}(2t + 1)(u + w - 1) < s(u + w - 1)

If \( d < w \leq s + 1 \), then the number of edges in \( S_{H(u,w)} \), where \( S = V(K_m) \cap V(H(u,w)) \), is not larger than \( \left( \begin{array}{c} d \\ 2 \end{array} \right) + (w - d)s \), as the degree of each vertex in \( S_{H(u,w)} \) not in a given set of \( d \) vertices is at most \( s \). Thus

\[
|E(H(u,w))| \leq \left( \begin{array}{c} n \\ 2 \end{array} \right) + \left( \begin{array}{c} d \\ 2 \end{array} \right) + (w - d)s + ud
\]

\[
= \frac{1}{2}(u + d)(u + d - 1) + (w - d)s
\]

\[
< s(u + d - 1) + (w - d)s
\]

\[
= s(u + w - 1).
\]

For the case \( m \geq w > s + 1 \), we first show that \( |E(H(u,w))| \leq s(u + m - 1) \). If \( u \geq 2t - s \), then since the degree of any vertex in \( K_n \) not in a given set of \( u \) vertices is at least \( d + u \), we have

\[
|E(H(u,m))| \leq \left( \begin{array}{c} n \\ 2 \end{array} \right) - (n - u)(d + u) + \left( \begin{array}{c} m \\ 2 \end{array} \right) + nd
\]

\[
= n(2t - s) + s(m - 1) + (n - u)(d + u) - (n - u)(2s - 2t) - (n - u)(d + u)
\]

\[
\leq s(n + m - 1) - (n - u)(2s - 2t + 2t - s)
\]

\[
= s(u + m - 1).
\]

If \( 0 \leq u < 2t - s \), then in \( H(u,m) \), each vertex of \( K_n \) is adjacent to less than \( d + u \) vertices. Thus

\[
|E(H(u,m))| \leq \left( \begin{array}{c} m \\ 2 \end{array} \right) + u(d + u)
\]

\[
< s(m - 1) + u(2s - 2t + 2t - s)
\]

\[
= s(u + m - 1).
\]

Now we are ready to consider our last case. If \( m \geq w > s + 1 \), then

\[
|E(H(u,w))| \leq |E(H(u,m))| - w(m - w)
\]

\[
< s(u + m - 1) - s(m - w)
\]

\[
= s(u + w - 1).
\]

Therefore by Corollary 4, \( T_1(G) = s \), as required.

The proof of (b) is similar and is omitted.

**Construction III**

Let \( r \) and \( s \) be positive integers satisfying \( \frac{r}{2} < s \leq r \). Let \( G \) be the graph obtained from the union of two disjoint copies of \( K_{2s} \) by deleting any set of \( r - s \) independent edges and by adding a set \( A \) of \( r \) edges joining the two copies of \( K_{2s} \) such that (i) each end of a deleted edge is incident with exactly one edge in \( A \) and (ii) no edge in \( A \) joins two ends of the deleted edges. Thus \( 2s - r \) of the edges in \( A \) are not incident with any ends of the deleted edges. Clearly, \( G \) has \( 4s \) vertices, \( s(4s - 1) \) edges and \( \lambda(G) = r \). Now we show that \( T_1(G) = s \).

![Graph](image)

Take any subgraph \( H \) of \( G \). Suppose that the complement (in \( H \)) of \( H \) contains \( t \) of the deleted edges and that the number of vertices of \( H \) in the two copies of \( K_{2s} \) are \( m \) and \( n \), respectively. Then clearly, \( t \leq r - s \) and since \( H \) contains at most \( 2t + (r - s - t) \) of the vertices incident to the edges in \( A \), we have

\[
|E(H)| \leq \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} n \\ 2 \end{array} \right) + 2t + (r - s - t) + (2s - r) - t
\]

\[
\leq \frac{1}{2}(m(m - 1) + n(n - 1)) + s
\]

\[
\leq \frac{1}{2}(2s)(m + n - 2) + s
\]

\[
\leq s(m + n - 1).
\]

Thus by Corollary 4, \( T_1(G) = s \), as required.
5 Relations between $T_k(G)$'s

Let $G$ be a graph of order $p$. Recall that, for any integer $k = 1, 2, \ldots, p - 1$, we have

$$T_k(G) = \min \{ \frac{\lambda^k(G) + \tau(G - \lambda^k(G))}{i+1-k} \mid k \leq i \leq p - 1 \}.$$ 

Thus, it is easy to see that $T_i(G) < T_{i+1}(G)$ for any connected graph $G$ and any $i = 0, 1, 2, \ldots, p - 2$. In the following theorem, we show a relationship between $T_i(G)$ and $T_{i-n}(G)$ for any $i$ and $n$ such that $1 \leq n < i \leq b(G)$.

**Theorem 9.** Let $G$ be a connected graph of order $p$, and $i$ and $n$ be integers satisfying $1 \leq n < i \leq b(G)$. Then

$$\frac{T_i(G)}{T_{i-n}(G)} > \frac{p+n-i}{p-i}.$$

**Proof.** We shall prove the result for $n = 1$. The rest follows by induction on $n$. Let $i$ be an integer such that $2 \leq i \leq b(G)$. Then

$$T_i(G) = \frac{\lambda^i(G) + \tau(G - \lambda^i(G))}{i+1-i}$$ for some $i, t \leq p - 1.$

and

$$T_{i-1}(G) = \frac{\lambda^{i-1}(G) + \tau(G - \lambda^{i-1}(G))}{(i+1)-(i-1)}$$ for some $s, 1 \leq s \leq p - 1.$

By the definition of $T_{i-1}(G)$, we have

$$\frac{\lambda^{i-1}(G) + \tau(G - \lambda^{i-1}(G))}{(i+1)-(i-1)} \geq \frac{\lambda^i(G) + \tau(G - \lambda^i(G))}{(i+1)-i} = T_i(G).$$

Thus,

$$\frac{T_i(G)}{T_{i-1}(G)} > \frac{p+n-i}{p-i}.$$

Induction on $n$:

Let $i$ and $n$ be integers such that $1 \leq n < i \leq b(G)$. Then

$$T_{i-n}(G) = \frac{\lambda^{i-n}(G) + \tau(G - \lambda^{i-n}(G))}{(i-n)+(i-n)}$$ for some $c, i-n \leq c \leq p - 1.$

$$T_{i-(n+1)}(G) = \frac{\lambda^{i-(n+1)}(G) + \tau(G - \lambda^{i-(n+1)}(G))}{m+1-(i-(n+1))}$$ for some $m, i-n-1 \leq m \leq p - 1.$

Suppose

$$\frac{T_i(G)}{T_{i-n}(G)} > \frac{p+n-i}{p-i}.$$

Now we must show that

$$\frac{T_{i-1}(G)}{T_{i-(n+1)}(G)} > \frac{p+n-i}{p-i}.$$ By the definition of $T_{i-(n+1)}(G)$, we have

$$\frac{\lambda^{i-(n+1)}(G) + \tau(G - \lambda^{i-(n+1)}(G))}{m+1-(i-(n+1))} \geq \frac{\lambda^{i-n}(G) + \tau(G - \lambda^{i-n}(G))}{(i-n)+(i-n)} = T_{i-n}(G).$$

Thus

$$T_{i-n}(G) = \frac{\lambda^{i-n}(G) + \tau(G - \lambda^{i-n}(G))}{(i-n)+(i-n)} \geq \frac{\lambda^{i-(n+1)}(G) + \tau(G - \lambda^{i-(n+1)}(G))}{m+1-(i-(n+1))}.$$

By considering (1) and (2), we have

$$T_i(G) > \frac{p+n-i}{p-i}T_{i-n}(G) > \frac{p+n-i}{p-i}T_{i-(n+1)}(G).$$

as required.
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References


