Electromagnetic fields and thrust-ripples calculation for segmented-secondary linear flux-switching motors

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Abstract

Purpose – This paper aims to investigate analytical electromagnetic fields and thrust ripples representation of linear flux-switching motors with simple modulated secondary referred as segmented secondary linear flux-switching motor (SSLFSM).

Design/methodology/approach – SSLFSMs are applicable to transportation systems like Maglev due to their simple and consequently low-cost secondary structures and high force density. However, they have high thrust ripples that deteriorate a smooth motion in rail transportation systems. Therefore, derivation of accurate analytical models for thrust ripples minimization of the motor is essential, which is absent in the literature. In this paper, a two-dimensional analytical model is developed for this motor. The model is based on transfer relations and Fourier theory used for solving a two-dimensional boundary value problem. Certain model regions are determined by considering actual machine structure and observing specific rules. Analytical solution of Maxwell and Poison equations are then obtained in the regions.

Finding – Using the presented modeling method, the airgap electromagnetic field distribution and developed thrust of the motor are calculated for different positions of the motor as well as its thrust ripples. They are verified by the results obtained from finite element method. Also, the analytical results are compared with the presented experimental results.

Originality/value – This paper has analytically presented the airgap electromagnetic field distribution, thrust and thrust ripples of the SSLFSMs. This modeling is essential in thrust ripples minimization of the motor.

Keywords Electromagnetic fields, Applied electromagnetism, Finite element method, Magnetic force, Boundary value problems, Linear machine

Paper type Research paper

1. Introduction

A new type of linear flux-switching motors (LFSMs) referred as segmented secondary linear flux-switching motor (SSLFSM) is developed for rail transportation applications (Abdollahi and Vaez-Zadeh, 2014). SSLFSMs inherit most merits of LFSMs such as simple and robust structure of secondary and high force density (Shen et al., 2018; Cao et al., 2018). However,

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the segmented structure of the secondary in SSLFSMs makes them more applicable when the stator is very long, such as the one in the case of Maglev or underground train drives. Sinusoidal back-emf that allows the motor control system with less complexity is another advantage of SSLFSMs (Hosseini et al., 2018). Nevertheless, SSLFSMs have high thrust ripples caused by their segmented structure. Therefore, calculation of thrust ripples is crucial for design optimization and performance analysis of these motors. On the other hand, the thrust-ripples calculation in the motor faces extra difficulties due to the especial machine structure. The calculation has not been the focus of any work so far according to the best of authors’ knowledge (Wang et al., 2017).

Modeling of LFSMs by finite element method (FEM) is addressed in the literature (Cao et al., 2012). FEM-based modeling as an essential final stage of virtually all design procedures for electrical motors before the prototyping gives accurate results. However, it is a time-consuming method, which is hard to be used in the middle stages of the design procedures due to the need for numerous runs during these stages. Instead, magnetic equivalent circuits (MECs) with much faster calculations are usually used in the essential middle-stage modeling for many electrical machines including conventional LFSMs (Zhou et al., 2012). However, MEC-based modeling is very sensitive to small changes in geometrical parameters and provides approximated results unless certain enhancements are applied (Ghalavand et al., 2010).

Recently, a modeling method based on solving Maxwell equations in different regions of a rotary permanent-magnet FSM is addressed (Gysen et al., 2010a, 2010b). This model considers actual motor structure including slots and segments and gives almost accurate solution of field and force calculation. It is more time savings compared with FEM-based models and does not suffer from the deficiencies associated with MEC modeling. Nevertheless, the method depends on the machine structure and cannot be directly adapted to SSLFSMs. Specially that a SSLFSM has a big structure with no repetition.

Hence, in this paper, the analytical calculation of the electromagnetic field fluctuations, caused by motor structure saliency such as slots for SSLFSMs is proposed by setting and solving Maxwell equations in certain regions of the motors, taking into account the gaps between adjacent secondary segments. The regions are determined such that an accurate field calculation is achieved, leading to precise thrust-ripples calculation, despite the limited number of the regions, which keeps the calculation time manageable. As a result, the modeling can be effectively used in intermediate design stages of the motors, when low ripples are a major design constraint.

Motor topology and its operating principles are explained in Section 2. The 2D analytical model of the SSLFSM is presented in Section 3. The electromagnetic field distribution for different positions of the motor primary, beside its thrust and thrust ripples are obtained and verified by finite element method (FEM) in Section 4. In Section 5, some conclusions are extracted, which clarify that the presented modeling of SSLFSMs is effective in exactly presenting the motor field distribution, thrust and thrust ripples.

2. Operation principles and topology of the SSLFSM

Figure 1 shows the parameters and dimensions of a SSLFSM which are as follow: \( w_{se} \) is the secondary segment width, \( w_s \) is the primary slot width, \( w_{ft} \) is the field tooth width, \( w_{at} \) is the armature tooth width, \( h_s \) is the segment height, \( h_m \) is the tooth height, \( L \) is the motor width along the \( z \)-direction (depth of modeling) and \( \tau_p \) is twice the motor pole pitch, which is the distance between one secondary segment to the next one. As it is seen the primary of the motor which is its mover, includes both 3-ph armature winding and DC excitation field coils. These coils are of concentrated type. The polarization of each excitation winding should be
opposite to the next one and they are connected in series. Totally, there are seven field coils. The adjacent mover teeth are wounded alternatively with one field winding and one armature winding. Each armature phase has two coils connected in series. The secondary of SSLFSM which is composed of simple laminated segments is built from magnetic steel. This topology is low cost and very simple to be constructed in rail transportation systems due to employment of bare segments in the secondary side on the track (Abdollahi, 2014).

Opposite currents in DC windings beside secondary segments make electromagnetic poles. When the field coils are excited and the mover travels along the secondary segments, the polarity of the armature winding flux-linkage changes as well as its value. This is why these motors are referred as flux-switching motors. Due to this bipolar flux-linkage, a sinusoidal back-emf is induced in the armature coils. Also this bipolar flux-linkage leads to high thrust density of the SSLFSM (Abdollahi and Vaez-Zadeh, 2014).

3. The 2D analytical model

In this section, an analytical modeling method based on flowchart of Figure 2 is developed by setting and solving Maxwell and Poisson equations in certain regions of the SSLFSM. Motor dimensions and currents are inputs of this flowchart, and motor magnetic and electrical parameters are its outputs. In order to determine appropriate model regions in Cartesian coordinate system, the following rules are followed:

- Soft-magnetic materials are not divided into smaller regions.
- The motor airgap is considered as one region.
- Each region is considered to have a rectangle shape with parallel boundaries along x- and y-directions.
- For actual regions with ramped or curved shape boundaries, an approximation of vertical or horizontal boundaries is used.
- If the length of the ramped or curved boundaries of an actual region is considerably big against its straight dimensions, the region should be divided into smaller model regions.
- Rectangle regions should not be divided into small model regions unless they have different materials with different permeabilities or different current densities along y-direction.

It should be noted that it is essential to avoid unnecessary divisions of regions, as it leads to a complicated modeling with time-consuming solution. Different regions used for modeling of the developed SSLFSM are shown in Figure 3(a). Compared to Figure 1, wedge, slot-opening and inter segment regions are shown in detail in this figure. As it is seen actual trapezoidal regions are approximated with rectangle shape regions [Figure 3(b)]. In total,

![Figure 1. Parameters and dimensions of the SSLFSM](image-url)
12 slot regions, S1-S12, 12 wedge regions, W1-W12, 12 slot-opening regions, O1-O12, 1 airgap region, g, 8 intersegments up regions, I1-I1-8, eight intersegments middle regions, I2-I2-8, eight intersegments down regions, I3-I3-8 and 1 surrounding air region, Air, are different regions of the 12/8 SSLFSM. As it is seen between two dashed red lines is the period of solving the problem, which is equal to the motor length as long as 8 secondary segments or 12 primary slot-tooth regions. The direction of periodicity is the $x$-direction. $x=0$ is in the beginning of the first slot region and $y=0$ is in the middle of the airgap.

3.1 Assumptions
The main assumptions of the analytical model are as follows:

1. All materials in all regions are assumed to be isotropic and homogenous.
   - The relative permeability of the soft magnetic material is assumed to be infinite; hence, the magnetic field distribution is not calculated inside the soft-magnetic material but the magnetic field strength ($\vec{H}$) normal to its boundary is set to zero (Neumann boundary condition) (Gysen et al., 2010a, 2010b).
Therefore, the nonlinearity (saturation) and DC bias properties of the cores are ignored.

- The source regions (current carrying coils), are invariant in the $z$-direction.

(2) The motor is extended to infinity in $\pm x$-direction.

- The end effect and edge effect of the motor are ignored. As the end effect (Hosseini et al., 2018) and edge effect (Boldea, 2017) of the presented motor are negligible, this assumption is correct.

As the soft-magnetic materials are assumed to have infinite permeability, two different types of region can be assumed; source-free regions (air) and current carrying regions (coils). Every region should be enclosed by four boundaries, where each boundary is in parallel with $x$- or $y$-direction. For wedge and inter segment regions, which have two boundaries in ramp shape, two approximated boundaries parallel to $y$-direction are assumed in middle of each side. The number of regions determines the complexity of the problem and defines the form of the solution.

To simplify the magnetic field formulation, each region is assumed to have a local coordinate system. The main coordinate system is $(x,y)$, where the local coordinate system $(x_k,y)$ is defined as:

$$x_k = x - \Delta_k,$$  \hspace{1cm} (1)

T1-2 where the offset $\Delta_k$ is indicated as in Table I for a SSLFSM with dimensions of Table II. The factors 0.85, 0.9, 0.7 of Table I multiplied by $w_{se}$, respectively, indicate the distance between

**Notes:** (a) All regions; (b) closer view of regions illustrating approximations used in trapezoidal regions

Figure 3. Different regions of the SSLFSM

Thrust-ripples calculation
intersegment regions of I1-1, I1-2, I1-3 and \( x_0 \) along \( x \)-direction. \( \Delta_k \) presents two sets of fixed and moving regions, an increment of the parameter for all regions within the moving set results in a positive displacement. All regions, except for intersegment regions, are fixed and the motor movement is considered by \( x_0 \). Now, it is possible to calculate the field distribution for all position of the moving part. \( w_h \) is the tooth overhang width.

The main difference between the analytical model and FEM-based modeling is that using a rule-based division of motor regions, extra regions are avoided, especially soft magnetic materials are not divided at all. Also, in other parts of the motor, divided regions are considerably smaller even in comparison with the case of FEM analysis applying big meshes.

### 3.1.1 Maxwell equations

It is known that the magnetic flux density (\( \vec{B} \)) in regions presented in Table I can be written as:

\[
\vec{B} = \nabla \times \vec{A},
\]

where \( \vec{A} \) is the magnetic vector potential. The magnetostatic Maxwell equations lead to a Poisson equation, given by (Gysen et al., 2010a, 2010b):

\[
\nabla^2 \vec{A} = -\mu_0 \mu_r \vec{J},
\]

where \( \mu_r \) is the relative permeability of regions presented in Table I. As the current-density vector \( \vec{J} \) has only a component in the \( z \)-direction, \( \vec{A} \) also has a component in \( z \)-direction; therefore, equation (3) reduces to:

\[
\frac{\partial^2 A_z(x,y)}{\partial x^2} + \frac{\partial^2 A_z(x,y)}{\partial y^2} = -\mu_0 \mu_r J_z,
\]

For each region, the local coordinate system needs to be considered by replacing \( x \) by \( x_k \). Solving \( \vec{A} \) results in \( \vec{B} \) from (2) and then \( \vec{H} \) is given by:

\[
\vec{B} = \mu_0 \mu_r \vec{H}.
\]

To solve equation (4) in the 12 slot regions, it is necessary to describe current sources in term of Fourier series. Figure 4 shows current-density in \( i \)-th slot. Half of the slot has \( J_{1i} \) current-density and the rest \( J_{2i} \), which are in term of A/m². As the slot regions are governed by the Neumann boundary condition, the total source description is obtained by applying the imaging method.

<table>
<thead>
<tr>
<th>Region name</th>
<th>Symbol</th>
<th>Value/equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air gap region</td>
<td>( \Delta_g )</td>
<td>0</td>
</tr>
<tr>
<td>Air region</td>
<td>( \Delta_A )</td>
<td>0</td>
</tr>
<tr>
<td>First slot region</td>
<td>( \Delta_{s1} )</td>
<td>0</td>
</tr>
<tr>
<td>Other slots regions</td>
<td>( \Delta_{s2} ) for ( i = 2-12 )</td>
<td>( \Delta_{s(l)} + w_s + w_{ta} )</td>
</tr>
<tr>
<td>Wedge regions</td>
<td>( \Delta_{w} ) for ( i = 1-12 )</td>
<td>( \Delta_{s(r)} + w_h/2 )</td>
</tr>
<tr>
<td>Slot-opening regions</td>
<td>( \Delta_{o} ) for ( i = 1-12 )</td>
<td>( \Delta_{s(d)} + w_h )</td>
</tr>
<tr>
<td>intersegments up regions</td>
<td>( \Delta_{I1,j} ) for ( j = 1-8 )</td>
<td>( x_0 + 0.85w_{se} + (j-1)\tau_p )</td>
</tr>
<tr>
<td>intersegments middle regions</td>
<td>( \Delta_{I2,j} ) for ( j = 1-8 )</td>
<td>( x_0 + 0.9w_{se} + (j-1)\tau_p )</td>
</tr>
<tr>
<td>intersegments down regions</td>
<td>( \Delta_{I3,j} ) for ( j = 1-8 )</td>
<td>( x_0 + 0.7w_{se} + (j-1)\tau_p )</td>
</tr>
</tbody>
</table>

**Table I.**  
Definition for different region’s offsets (\( \Delta_k \))
(Hague, 1929), where the source is mirrored around its tangential direction \( (y\)-direction) as indicated in Figure 4. Consequently, by applying this imaging method, the sine terms will become zero. Hence, by using general Fourier theory, the current sources can be described as:

\[
J_i = J_0i \sum_{n=1}^{\infty} \left( J_{si} \sin(\omega_s x_{si}) + J_{ci} \cos(\omega_s x_{si}) \right),
\]

\[
\begin{aligned}
J_0i &= \frac{1}{2\tau_s} \int_{0}^{2\tau_s} J_i dx_{si} = \frac{J_1i + J_2i}{2}, \\
J_{si}(n) &= \frac{1}{\tau_s} \int_{0}^{2\tau_s} J_i \sin(\omega_s x_{si}) dx_{si} = 0, \\
J_{ci}(n) &= \frac{1}{\tau_s} \int_{0}^{2\tau_s} J_i \cos(\omega_s x_{si}) dx_{si} \\
&= \frac{2}{n\pi} \sin \frac{n\pi}{2} \left( J_1i + (-1)^n J_{2i} \right),
\end{aligned}
\] (6) (7)

Table II.
Geometric parameters of the SSLFSM

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary slot height</td>
<td>mm</td>
<td>70</td>
<td>( h_s )</td>
</tr>
<tr>
<td>Primary yoke height</td>
<td>mm</td>
<td>13</td>
<td>( h_y )</td>
</tr>
<tr>
<td>Airgap</td>
<td>mm</td>
<td>1</td>
<td>( g )</td>
</tr>
<tr>
<td>Segment pitch</td>
<td>mm</td>
<td>52</td>
<td>( \tau_p )</td>
</tr>
<tr>
<td>Segment width</td>
<td>mm</td>
<td>42</td>
<td>( w_{se} )</td>
</tr>
<tr>
<td>Segment height</td>
<td>mm</td>
<td>18</td>
<td>( h_s )</td>
</tr>
<tr>
<td>Segment factor</td>
<td>–</td>
<td>0.8</td>
<td>( S_f )</td>
</tr>
<tr>
<td>Slot width</td>
<td>mm</td>
<td>20</td>
<td>( w_s )</td>
</tr>
<tr>
<td>Slot opening width</td>
<td>mm</td>
<td>14</td>
<td>( w_{so} )</td>
</tr>
<tr>
<td>Field tooth width</td>
<td>mm</td>
<td>15</td>
<td>( w_{ft} )</td>
</tr>
<tr>
<td>Armature tooth width</td>
<td>mm</td>
<td>15</td>
<td>( w_{at} )</td>
</tr>
<tr>
<td>Motor width</td>
<td>mm</td>
<td>100</td>
<td>( L )</td>
</tr>
<tr>
<td>Tooth overhang width</td>
<td>mm</td>
<td>2.5</td>
<td>( w_{to} )</td>
</tr>
<tr>
<td>Number of armature winding turns</td>
<td>–</td>
<td>100</td>
<td>( N_a )</td>
</tr>
<tr>
<td>Number of field winding turns</td>
<td>–</td>
<td>100</td>
<td>( N_f )</td>
</tr>
</tbody>
</table>

Figure 4. Current density distribution in i-th slot and its waveforms using imaging method
where \( t_s \) is the width of the slot regions, S1-S12, and \( v_s \) is the spatial frequency of the region given by:

\[
v_s(n) = \frac{n\pi}{t_s}.
\] (8)

### 3.2 Field solution in each region

The solution for \( \vec{A} \) is written in terms of Fourier components as the following:

\[
\vec{A} = A(x,y)\hat{z} = \left[ A_0(y) + \sum_{n=1}^{\infty} (A_s(y)\sin(\omega_kx_k) + A_c(y)\cos(\omega_kx_k)) \right] \hat{z},
\] (9)

where \( \omega_k \) is the spatial frequency of the region \( k \), defined as:

\[
\omega_k = \frac{n\pi}{t_k}.
\] (10)

Hence, the expressions for \( \vec{B} \) in different regions are given as:

\[
\vec{B} = \left( \sum_{n=1}^{\infty} (B_{ys}(y)\sin(\omega_kx_k) + B_{xc}(y)\cos(\omega_kx_k)) + B_{x0}(y) \right) \hat{x}
\] + \[
\left( \sum_{n=1}^{\infty} (B_{ys}(y)\sin(\omega_kx_k) + B_{yc}(y)\cos(\omega_kx_k)) \right) \hat{y},
\] (11)

where the functions \( B_{ys}, B_{xc}, B_{x0}, B_{ys} \) and \( B_{yc} \) can be obtained by considering the transfer relations (Gysen et al., 2010a, 2010b) and are given by:

\[
\begin{align*}
B_{ys} &= a_n e^{\omega_ky} + b_n e^{-\omega_ky} + \frac{\mu J_c}{\omega_k}, \\
B_{yc} &= -c_n e^{\omega_ky} - d_n e^{-\omega_ky} - \frac{\mu J_s}{\omega_k}, \\
B_{xs} &= c_n e^{\omega_ky} - d_n e^{-\omega_ky}, \\
B_{xc} &= a_n e^{\omega_ky} - b_n e^{-\omega_ky}, \\
B_{x0} &= -\mu J_0y + B_0.
\end{align*}
\] (12)

The set of unknowns \( a_n, b_n, c_n, d_n, \) and \( B_0 \) for every region is obtained by solving equation (12), considering the boundary conditions in \( y \)-direction, which will be discussed in the following sub-section.

### 3.3 Boundary conditions

The boundaries of a region parallel to the \( y \)-direction should be periodic or Neumann. The boundaries parallel to \( x \)-direction can either be Neumann or continuous or a combination of both (Gysen et al., 2010a, 2010b). In the presented SSLFSM, air and airgap regions have periodic boundary conditions, and all other regions are with Neumann boundary conditions.
on boundaries parallel to \( y \)-direction. For regions with Neumann boundary conditions, \( B_x \) only contains sine terms (\( B_{xc} = 0 \)) and consequently \( B_{ys} = 0 \).

Boundary conditions for different \( y \) positions of the regions are as follows:

For \( y = h_1 = h_s + h_w + h_0 + g/2 \),

\[
B_x \big|_{y=h_1} = 0,
\]

(13)

for \( y = h_2 = h_w + h_0 + g/2 \),

\[
B_{ys} = B_{ys} \big|_{y=h_2} \quad 0 \leq x_{oi} \leq \tau_w \rightarrow \text{for } i = 1 - 12,
\]

(14)

\[
H_{ys} = \begin{cases} 
H_{ys} \big|_{y=h_2} \quad 0 \leq x_{oi} \leq \tau_w \big|_{y=h_2} \quad \text{for } i = 1 - 12, \\
0 \quad \text{else}
\end{cases}
\]

(15)

for \( y = h_3 = h_o + g/2 \),

\[
B_{ys} = B_{ys} \big|_{y=h_3} \quad 0 \leq x_{oi} \leq \tau_o \quad \text{for } i = 1 - 12,
\]

(16)

\[
H_{ys} = \begin{cases} 
H_{ys} \big|_{y=h_3} \quad 0 \leq x_{oi} \leq \tau_o \big|_{y=h_3} \quad \text{for } i = 1 - 12, \\
0 \quad \text{else}
\end{cases}
\]

(17)

for \( y = h_4 = g/2 \),

\[
B_{ys} = B_{ys} \big|_{y=h_4} \quad 0 \leq x_{oi} \leq \tau_o \quad \text{for } i = 1 - 12,
\]

(18)

\[
H_{ys} = \begin{cases} 
H_{ys} \big|_{y=h_4} \quad 0 \leq x_{oi} \leq \tau_o \big|_{y=h_4} \quad \text{for } i = 1 - 12, \\
0 \quad \text{else}
\end{cases}
\]

(19)

for \( y = h_5 = -g/2 \),

\[
B_{ys} = B_{ys} \big|_{y=h_5} \quad 0 \leq x_{I1-i} \leq \tau_{I1} \quad \text{for } i = 1 - 8,
\]

(20)

\[
H_{ys} = \begin{cases} 
H_{ys} \big|_{y=h_5} \quad 0 \leq x_{I1-i} \leq \tau_{I1} \big|_{y=h_5} \quad \text{for } i = 1 - 8, \\
0 \quad \text{else}
\end{cases}
\]

(21)

for \( y = h_6 = -g/2 - h_{I1} \),

\[
B_{y_{n-i}} = B_{y_{n-i}} \big|_{y=h_6} \quad 0 \leq x_{I2-i} \leq \tau_{I2} \quad \text{for } i = 1 - 8,
\]

(22)

\[
H_{y_{n-i}} = \begin{cases} 
H_{y_{n-i}} \big|_{y=h_6} \quad 0 \leq x_{I2-i} \leq \tau_{I2} \big|_{y=h_6} \quad \text{for } i = 1 - 8, \\
0 \quad \text{else}
\end{cases}
\]

(23)
for $y = h_7 = -g/2 - h_{I1} - h_{I2}$,

$$B_{y_{I1-i}} = B_{y_{I2-i} \mid y = h_7} \quad 0 \leq x_{I2-i} \leq \tau_{I2} \quad \text{for } i = 1 - 8,$$

$$H_{y_{I3-i}} = \begin{cases} H_{y_{I2-i}} & 0 \leq x_{I2-i} \leq \tau_{I2} \\ 0 & \text{else} \end{cases} \quad y = h_7 \quad \text{for } i = 1 - 8,$$

for $y = h_8 = -g/2 - h_{I1} - h_{I2} - h_{I3}$

$$B_{y_{I3-i}} = B_{y_{I1-i} \mid y = h_7} \quad 0 \leq x_{I1-i} \leq \tau_{I1} \quad \text{for } i = 1 - 8,$$

$$H_{y_{I3-i}} = \begin{cases} H_{y_{I2-i}} & 0 \leq x_{I2-i} \leq \tau_{I2} \\ 0 & \text{else} \end{cases} \quad y = h_8 \quad \text{for } i = 1 - 8,$$

for $y = \infty$

$$B_x \bigg|_{y = \infty} = B_y \bigg|_{y = \infty} = 0,$$

where $h_{I1}$, $\tau_{I1}$, $h_{I2}$, $\tau_{I2}$, $h_{I3}$, $\tau_{I3}$, $h_{A}$, $\tau_{A}$ are, respectively, the height and width of wedge, slot, slot-opening, up, middle and down intersegment regions. Also, $\tau_{A}$, $\tau_{y}$ are half of the width of air and airgap regions, respectively.

### 3.4 Exact field calculation in regions

For every region the field equations are as the following ones:

**I-slot regions by considering equation (13):**

$$\vec{B} = \left( \sum_{n=1}^{n_1} \left[ \cos(\omega_s x_{Ii}) \left( b_{n-sI} \left( e^{\omega_s(y-2h_1)} - e^{-\omega_s y} \right) \right) + \mu_0 J_0 (h_1 - y) \right] \right) \hat{x}$$

$$+ \left( \sum_{n=1}^{n_1} \left[ \sin(\omega_s x_{Ii}) \left( b_{n-sI} \left( e^{\omega_s(y-2h_1)} + e^{-\omega_s y} \right) \right) + \mu_0 J_{ci-n} \frac{1}{\omega_s} \right] \right) \hat{y} \quad \text{for } i = 1 - 12.$$

where $J_{ci-n}$ is $n$-th harmonic of cosine-component of Fourier series for current-density of $i$-th slot and $J_0$ is its DC component.

**II-wedge regions:**

$$\vec{B} = \left( \sum_{n=1}^{n_w} \left[ \cos(\omega_w x_{wi}) \left( a_{n-w1} e^{\omega_w y} - b_{n-w1} e^{-\omega_w y} \right) \right] + \frac{\tau_s}{\tau_w} \mu_0 J_0 \right) \hat{x}$$

$$+ \left( \sum_{n=1}^{n_w} \left[ \sin(\omega_w x_{wi}) \left( a_{n-w1} e^{\omega_w y} + b_{n-w1} e^{-\omega_w y} \right) \right] \right) \hat{y} \quad \text{for } i = 1 - 12.$$

**III-slot-opening regions:**
\[
\vec{B} = \left( \sum_{n=1}^{n_k} \left[ \cos(\omega_n \alpha) (a_n - b_n e^{-\omega_n y}) - (a_n - b_n e^{\omega_n y}) \right] \right) \hat{x} \\
+ \left( \sum_{n=1}^{n_k} \left[ \sin(\omega_n \alpha) (a_n - b_n e^{-\omega_n y}) + (a_n - b_n e^{\omega_n y}) \right] \right) \hat{y} \quad \text{for } i = 1 \rightarrow 12. 
\]

**IV-airgap region:**

\[
\vec{B} = B_x \hat{x} + B_y \hat{y} \\
= \left( \sum_{n=1}^{n_k} \left[ \sin(\omega_n x) (c_n g - d_n e^{-\omega_n y}) + \cos(\omega_n x) (a_n g - b_n e^{-\omega_n y}) \right] \right) \hat{x} \\
+ \left( \sum_{n=1}^{n_k} \left[ \sin(\omega_n x) (a_n g + b_n e^{-\omega_n y}) + \cos(\omega_n x) (-c_n g e^{\omega_n y} - d_n e^{-\omega_n y}) \right] \right) \hat{y}. 
\]

**V-intersegment regions:**

\[
\vec{B} = \sum_{n=1}^{n_k} \left[ \cos(\omega_n x) (a_n - b_n e^{-\omega_n y}) - (a_n - b_n e^{\omega_n y}) \right] \hat{x} \\
+ \sum_{n=1}^{n_k} \left[ \sin(\omega_n x) (a_n - b_n e^{\omega_n y}) + (a_n - b_n e^{-\omega_n y}) \right] \hat{y}. \quad \text{for } i = 1 \rightarrow 8 \text{ & } j = 1 \rightarrow 3. 
\]

**VI-Air region by considering boundary condition of (28):**

\[
\vec{B} = \sum_{n=1}^{n_A} \left[ \sin(\omega_A x) (-d_n A e^{-\omega_A y}) + \cos(\omega_A x) (-b_n A e^{-\omega_A y}) \right] \hat{x} \\
+ \sum_{n=1}^{n_A} \left[ \sin(\omega_A x) (b_n A e^{-\omega_A y}) + \cos(\omega_A x) (-d_n A e^{-\omega_A y}) \right] \hat{y}. 
\]

In equations (29)-(34), \(n_k\) is the number of harmonics in region \(k\). A very small \(n_k\) deteriorates the model accuracy and a very large one makes the solving time so long.

### 3.5 Thrust calculation

To calculate the motor thrust, virtual work method is used. A differential variation of coenergy along the x-direction equals to the thrust force, when motor currents are constant, given by (Woodson and Melcher, 1968):

\[
F(x) = \frac{dW_{co}^e(x)}{dx} \bigg|_{i_{dc}=\text{constant}}. 
\]
Due to the magnetic linearity of the motor, its coenergy \( W^{co}(x) \) equals to its energy \( W(x) \):

\[
W^{co}(x) = W(x).
\]  

(36)

On the other hand, magnetic energy which is mainly stored in the airgap is calculated as (Cheng, 1989):

\[
W(x) = \int_{0+\epsilon}^{M+x} \frac{B^2(x',y)}{2\mu_0} \times L \times g \times dx',
\]

(37)

where \( M \) is the primary length and \( B(x,y) \) is the airgap flux density produced by both field excitation and armature currents given by:

\[
B^2(x,y) = B_x^2(x,y) + B_y^2(x,y),
\]

(38)

where \( B_x(x,y) \) and \( B_y(x,y) \) are obtained from equation (32). Considering equations (35), (36) and (37), thrust is obtained as:

\[
F(x) = \frac{dW^{co}(x)}{dx} \bigg|_{i_{a,b,c} \text{ and } i_{d}=constant}
\]

\[
= \left. \frac{d}{dx} \left( \int_{0+\epsilon}^{M+x} \frac{B^2(x',y)}{2\mu_0} \times L \times g \times dx' \right) \right|_{i_{a,b,c} \text{ and } i_{d}=constant}.
\]

(39)

In the case of small airgap, the flux density variation in the airgap along the \( y \)-axis is not considerable. Therefore, middle of the airgap \( y = g/2 \) is assumed as a reference point. Thrust-ripples equation is as follows:

\[
\text{Thrust} - \text{ripple} = \frac{F_{max} - F_{min}}{2 \times F_{ave}} \times 100.
\]

(40)

where \( F_{max} \), \( F_{min} \) and \( F_{ave} \) are, respectively, the maximum, minimum and average values of thrust force obtained from equation (39).

4. Comparison and evaluation

To validate the mathematical model of SSLFSMs, 2D-FEM results are drawn in this section. A prototype 12/8 SSLFSM (12 primary teeth and 8 secondary segments) with the geometric parameters as listed in Table II is considered. The rated voltage, power and speed of the motor are 380V, 15kVA and 20m/s, respectively. The SSLFSM modeling by 2D-FEM is depicted in Figure 5. For different positions of the motor, tangential and normal open circuit flux density distribution in the airgap \( (B_x \text{ and } B_y) \) for 15A DC excitation are obtained by FEM analysis, and the results are compared with the analytical ones (Figures 6 and 7). It is seen that the flux density distribution obtained by the analytical method agrees well with the FEM results for \( B_x \) and \( B_y \), which are essential components in calculating thrust ripples. As the model considers all the slots, openings and inter segment regions, flux density variations are carefully modeled. However, the little difference seen between the results are due to the limited number of harmonics used in the modeling of each region and the fact that iron parts permeability are considered as infinity while it is an approximation of actual condition. Each
secondary segment makes an electromagnetic path from one armature tooth to its adjacent field tooth. Depending on the position of the secondary segment, the electromagnetic overlap differs. Therefore, the flux lines density under every segment pitch will be different. The airgap flux density has one positive and one negative cycle under every segment pitch. The airgap flux density distribution is obtained analytically from equation (32).

Harmonic numbers considered for different regions are given in Table III. The motor thrust and thrust ripples for 15A DC excitation and 20A effective armature AC current are shown in Figure 8, obtained by both analytical and 2D-FEM models. Average thrust obtained analytically from (39) is 599N, while by 2D-FEM is 598N. Therefore, only 0.17 per cent error is occurred. Thrust ripples which is determined from equation (40), is 77.96 per cent while that one from 2D-FEM is 77.34 per cent with only 0.8 per cent error. The analytical model is almost four times faster than 2D-FEM model and can be directly used in design and optimization procedures in contrast to FEM-based modeling.

To further verify the analytical model in practice, the experimental results presented in Hosseini et al. (2018) are compared with 2D-FEM and analytical ones in Figure 9. It is seen an acceptable agreement between FEM, analytical and experimental results are achieved.

5. Conclusion
In this paper, a 2D analytical model is developed for SSLFSMs to calculate the motor electromagnetic fields, thrust and thrust ripples considering primary slotting effect and secondary segmentation. The model is based on transfer relations and Fourier theory used for solving a 2D boundary value problem. A limited number of rule-based selected regions of

Notes: (a) Structure and mesh of the motor; (b) B-H curve of the core employed in 2D-FEM modeling
Figure 6.
Normal open circuit flux density distribution in the airgap by $I_{dc} = 15\,\text{A}$ for different positions of the secondary...
Figure 7.
Tangential open circuit flux density distribution in the airgap by $I_{dc} = 15$ A for different positions of the secondary

Notes: (a) $x_0 = 0$; (b) $x_0 = 20$ mm; (c) $x_0 = 40$ mm
the motor are considered, and analytical results are obtained by solving Maxwell and Poisson equations in the regions.

To verify the analytical model, a 2D-FEM-based model is developed. The results of analytical and FEM models are compared. Electromagnetic field distribution for different positions of the motor beside its thrust and thrust ripples obtained by the proposed analytical model are in close agreement with those obtained by FEM. The errors in calculating of average thrust and thrust ripples are 0.17 and 0.8 per cent, respectively. Also, the results are compared with those obtained from experiments and acceptable accuracy of the motor modeling is confirmed.

As the presented motor has large thrust ripples, it should be first modeled appropriately and then optimized. Multilevel design optimization can be used in accompany with the presented analytical model to speed-up the motor design optimization process, which may be carried out as a future work. Due to considerably less computational regions, the

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>$n_s$</td>
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</tr>
<tr>
<td>$n_w$</td>
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</tr>
<tr>
<td>$n_o$</td>
<td>20</td>
</tr>
<tr>
<td>$n_B$</td>
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<tr>
<td>$n_{II}$</td>
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</tr>
<tr>
<td>$n_{III}$</td>
<td>20</td>
</tr>
<tr>
<td>$n_{IV}$</td>
<td>20</td>
</tr>
<tr>
<td>$n_A$</td>
<td>40</td>
</tr>
</tbody>
</table>

Table III. The selected harmonic number for each region

Figure 8. Motor thrust for $I_{dc} = 15\text{A}$ and $I_{ac} = 20\text{A}$ effective during one electrical period

Figure 9. Static torque for $I_{ac} = 4\text{A}$ and $I_{dc} = 10\text{A}$ obtained from the presented analytical model, 2D-FEM and experiments (Hosseini et al., 2018)
analytical model is also faster than FEM-based modeling and can be directly used in optimization procedures. The presented model can be easily extended to any other motor with any structure especially with no repetition which is a challenge in modeling electrical motors. Back-emf, normal force and other characteristics of electrical motors may be calculated by using the presented method.

References

Appendix

Here the set of unknowns $a_n, b_n, c_n, d_n$ are obtained considering the boundary conditions and the field equations of every regions.

At boundary $y = h_2$, substituting equations (29)-(30) into equation (14) leads to:

$$
\sum_{m=1}^{n_w} \left[ \sin(\omega_s x_s) \left( b_{m-S} \left( e^{\omega_s(h_2-2h_1)} + e^{-\omega_s h_2} \right) + \frac{\mu_0 J_{ci-n}}{\omega_s} \right) \right]
= \sum_{n=1}^{n_w} \left[ \sin(\omega_w x_w) \left( a_n e^{\omega_w h_2} + b_n e^{-\omega_w h_2} \right) \right] \quad 0 \leq x \leq \tau_w \quad \text{for } i = 1 - 12.
$$

Multiplying both sides of above equation by $\sin(\omega_w x_w)$ and integrating over $[0, \tau_w]$ gives $12 \times n_w$ equations as:

$$
E_{1s}(m,n,i) = a_n e^{\omega_w h_2} + b_n e^{-\omega_w h_2}, \quad \text{for } i = 1 - 12 \text{ and } n = 1 - n_w,
$$

$$
E_{1s}(m,n,i) = \frac{2}{\tau_w} \int_0^{\tau_w} \sin(\omega_s x_s) \sin(\omega_w x_w) dx_w = \frac{2m \tau_s^2 (\sin(\omega_s (\Delta_w - \Delta_s)) - \cos(m \pi) \sin(\omega_s ((\Delta_w - \Delta_s) + \tau_w)))}{\pi (m^2 \tau_s^2 - n^2 \tau_w^2)}.
$$

Substituting equations (29)-(30) into equation (15), multiplying by $\cos(\omega_w x_w)$ and integrating over $[0, \tau_w]$ leads to:

$$
b_{m-S} \left( e^{\omega_s(h_2-2h_1)} - e^{-\omega_s h_2} \right) = \sum_{n=1}^{n_w} \left( (a_n e^{\omega_w h_2} - b_n e^{-\omega_w h_2}) \times S_{1s}(n,m,i) \right.
+ \left. \frac{\tau_s \mu_0 J_{0} h_s}{\tau_w} \times S_{1o}(m,i) \right) \quad \text{for } i = 1 - 12 \text{ and } m = 1 - n_s,
$$

$$
S_{1s}(n,m,i) = \frac{2}{\tau_s} \int_0^{\tau_w} \cos(\omega_w x_w) \cos(\omega_s x_s) dx_s
= \frac{2n \tau_s^2 \sin(\omega_s (\Delta_w - \Delta_s)) - \cos(m \pi) \sin(\omega_s ((\Delta_w - \Delta_s) + \tau_w)))}{\pi (m^2 \tau_s^2 - n^2 \tau_w^2)};
$$

$$
S_{1o}(m,i) = \frac{2}{\tau_s} \int_0^{\tau_w} \cos(\omega_w x_w) dx_s = \frac{-2 \sin(\omega_s (\Delta_w - \Delta_s)) \sin(\omega_s ((\Delta_w - \Delta_s) + \tau_w)))}{m \pi}.
$$

Following the same procedure for other boundaries leads to adequate number of equations which give all unknowns as:
At $y = h_3$:

$$
\sum_{n=1}^{n_w} (a_{m-w}e^{\omega_u h_3} + b_{m-w}e^{-\omega_u h_3}) \times E_2^s(m, n, i) = a_{n-o}e^{\omega_v h_3} + b_{n-o}e^{-\omega_v h_3} \text{ for } i = 1 - 12 \text{ and } n = 1 - n_o,
$$

$$
E_2^s(m, n, i) = \frac{2}{\tau_0} \int_0^{\tau_0} \sin(\omega_w x_0) \sin(\omega_w x_0) dx_0
$$

$$
= \frac{2m \tau_w^2 (\sin(\omega_w (\Delta_o - \Delta_w)) - \cos(m \pi) \sin(\omega_w ((\Delta_o - \Delta_w) + \tau_o)))}{\pi (m^2 \tau_w^2 - n^2 \tau_o^2)},
$$

$$
a_{m-w}e^{\omega_u h_3} + b_{m-w}e^{-\omega_u h_3} + \frac{\tau_s}{\tau_w} \mu_o J_0 h_3 \times S_2^o(m, i)
$$

$$
= \sum_{n=1}^{n_w} \left( (a_{n-o}e^{\omega_v h_3} - b_{n-o}e^{-\omega_v h_3}) \times S_2^c(n, m, i) \right) \text{ for } i = 1 - 12 \text{ and } m = 1 - n_w,
$$

$$
S_2^o(m, i) = \frac{2}{\tau_w} \int_0^{\tau_0} \cos(\omega_w x_0) dx_0 = \frac{-2(\sin(\omega_w (\Delta_o - \Delta_w)) - \sin(\omega_w ((\Delta_o - \Delta_w) + \tau_o)))}{m \pi},
$$

$$
S_2^c(n, m, i) = \frac{2}{\tau_w} \int_0^{\tau_0} \cos(\omega_w x_0) \cos(\omega_w x_0) dx_0
$$

$$
= \frac{2m \tau_w^2 (\sin(\omega_w (\Delta_o - \Delta_w)) - \cos(m \pi) \sin(\omega_w ((\Delta_o - \Delta_w) + \tau_o)))}{\pi (m^2 \tau_w^2 - n^2 \tau_o^2)}.
$$

At $y = h_3$:

$$
a_{n-o}e^{\omega_v h_3} + b_{n-o}e^{-\omega_v h_3} = \sum_{m=1}^{n_g} \left( (a_{m-g}e^{\omega_v h_3} + b_{m-g}e^{-\omega_v h_3}) \times E_3^s(m, n, i) \right.
$$

$$
+ \left( -c_{m-g}e^{\omega_v h_3} - d_{m-g}e^{-\omega_v h_3} \right) \times E_1^c(m, n, i) \right) \text{ for } i = 1 - 12 \text{ and } n = 1 - n_o,
$$

$$
E_3^s(m, n, i) = \frac{2}{\tau_0} \int_0^{\tau_0} \sin(\omega_w x_0) \sin(\omega_w x_0) dx_0
$$

$$
= \frac{2m \tau_w^2 (\sin(\omega_w (\Delta_o - \Delta_g)) - \cos(m \pi) \sin(\omega_w ((\Delta_o - \Delta_g) + \tau_o)))}{\pi (m^2 \tau_w^2 - n^2 \tau_o^2)},
$$

Thrust-ripples calculation
COMPEL

\[ E_{1c}(m, n, i) = \frac{2}{\tau_o} \int_0^{\tau_o} \cos(\omega_x x_0) \sin(\omega_x x_0) \, dx_0 \]
\[ = \frac{2m n_2^2 (\cos(\omega_x (\Delta_o - \Delta_g)) - \cos(m \pi) \cos(\omega_x ((\Delta_o - \Delta_g) + \tau_o)))}{\pi(m^2 \tau_g^2 - n^2 \tau_o^2)} \]

\[ c_{m-g} e^{i \omega_x h_k} - d_{m-g} e^{-i \omega_x h_k} = \sum_{i=1}^{n_k} \sum_{n=1}^{n_k} \left( (a_{n-o} e^{i \omega_x h_k} - b_{n-o} e^{-i \omega_x h_k}) \times k1_c(n, m, i) \right) \]

for \( m = 1 - n_g \),

\[ k1_c(n, m, i) = \frac{1}{\tau_g} \int_0^{\tau_g} \cos(\omega_x x_0) \sin(\omega_x x_0) \, dx_0 \]
\[ = - \frac{n n_2^2 (\cos(\omega_x (\Delta_o - \Delta_g)) - \cos(m \pi) \cos(\omega_x ((\Delta_o - \Delta_g) + \tau_o)))}{\pi(m^2 \tau_g^2 - n^2 \tau_o^2)} \]

\[ a_{m-g} e^{i \omega_x h_k} - b_{m-g} e^{-i \omega_x h_k} = \sum_{i=1}^{n_k} \sum_{n=1}^{n_k} \left( (a_{n-o} e^{i \omega_x h_k} - b_{n-o} e^{-i \omega_x h_k}) \times S3_c(n, m, i) \right) \]

for \( m = 1 - n_g \),

\[ S3_c(n, m, i) = \frac{1}{\tau_g} \int_0^{\tau_g} \cos(\omega_x x_0) \cos(\omega_x x_0) \, dx_0 \]
\[ = \frac{n n_2^2 (\cos(\omega_x (\Delta_o - \Delta_g)) - \cos(m \pi) \cos(\omega_x ((\Delta_o - \Delta_g) + \tau_o)))}{\pi(m^2 \tau_g^2 - n^2 \tau_o^2)} \]

At \( y = h_x \):

\[ a_{n-I1} e^{i \omega_x h_k} + b_{n-I1} e^{-i \omega_x h_k} = \sum_{m=1}^{n_k} \left( (a_{m-g} e^{i \omega_x h_k} + b_{m-g} e^{-i \omega_x h_k}) \times E4_k(m, n, i) \right) + \left( -c_{m-g} e^{i \omega_x h_k} - d_{m-g} e^{-i \omega_x h_k} \times E2_c(m, n, i) \right) \]

for \( i = 1 - 8 \) and \( n = 1 - n_{I1} \),

\[ E4_k(m, n, i) = \frac{2}{\tau_{I1}} \int_0^{\tau_{I1}} \sin(\omega_x x_0) \sin(\omega_{I1} x_{I1}) \, dx_{I1} \]
\[ = \left( 2m n_2^2 (\cos(\omega_x (\Delta_{I1} - \Delta_g)) - \cos(m \pi) \cos(\omega_x ((\Delta_{I1} - \Delta_g) + \tau_{I1}))) \right) / \]
\[ \left( \pi(m^2 \tau_g^2 - n^2 \tau_{I1}^2) \right) \],
\[ E_2(m, n, i) = \frac{2}{\tau_{11}} \int_0^{\tau_{11}} \cos(\omega_g x_g) \sin(\omega_{11} x_{11}) dx_{11} \]
\[ = \left( 2m \tau_{11}^2 \left( \cos(\omega_g (\Delta_{11} - \Delta_g)) - \cos(m \pi) \cos(\omega_g ((\Delta_{11} - \Delta_g) + \tau_{11})) \right) \right) / \left( \pi \left( m^2 \tau_{11}^2 - n^2 \tau_{11}^2 \right) \right), \]
\[ c_{m-g} e^{\omega x_{h5}} - d_{m-g} e^{-\omega x_{h5}} = \sum_{i=1}^{8} \left\{ \sum_{n=1}^{n_{11}} \left( (a_{n-11} e^{\omega_{11} x_{h5}} - b_{n-11} e^{-\omega_{11} x_{h5}}) \times k_2(n, m, i) \right) \right\} \]
\[ \text{for } m = 1 - n_g, \]
\[ k_2(n, m, i) = \frac{1}{\tau_{11}} \int_0^{\tau_{11}} \cos(\omega_{11} x_{11}) \sin(\omega_g x_g) dx_g \]
\[ = -n \tau_{11}^2 \left( \cos(\omega_g (\Delta_{11} - \Delta_g)) - \cos(m \pi) \cos(\omega_g ((\Delta_{11} - \Delta_g) + \tau_{11})) \right) / \left( \pi \left( m^2 \tau_{11}^2 - n^2 \tau_{11}^2 \right) \right), \]
\[ a_{m-g} e^{\omega x_{h5}} - b_{m-g} e^{-\omega x_{h5}} = \sum_{i=1}^{8} \left\{ \sum_{n=1}^{n_{11}} \left( (a_{n-11} e^{\omega_{11} x_{h5}} - b_{n-11} e^{-\omega_{11} x_{h5}}) \times S_4(n, m, i) \right) \right\} \]
\[ \text{for } m = 1 - n_g, \]
\[ S_4(n, m, i) = \frac{1}{\tau_{11}} \int_0^{\tau_{11}} \cos(\omega_{11} x_{11}) \cos(\omega_g x_g) dx_g \]
\[ = n \tau_{11}^2 \left( \sin(\omega_g (\Delta_{11} - \Delta_g)) - \cos(m \pi) \sin(\omega_g ((\Delta_{11} - \Delta_g) + \tau_{11})) \right) / \left( \pi \left( m^2 \tau_{11}^2 - n^2 \tau_{11}^2 \right) \right), \]
\[ \text{At } y = h_6, \]
\[ a_{n-12} e^{\omega_{12} x_{h6}} + b_{n-12} e^{-\omega_{12} x_{h6}} = \sum_{m=1}^{n_{11}} \left[ \left( a_{m-11} e^{\omega_{11} x_{h6}} + b_{m-11} e^{-\omega_{11} x_{h6}} \right) \times E_5(n, m, i) \right] \]
\[ \text{for } i = 1 - 8 \& n = 1 - n_{12}, \]
\[ E_5(n, m, i) = \frac{2}{\tau_{12}} \int_0^{\tau_{12}} \sin(\omega_{11} x_{11}) \sin(\omega_{12} x_{12}) dx_{12} \]
\[ = \left[ 2m \tau_{12}^2 \left( \sin(\omega_{11} (\Delta_{12} - \Delta_{11})) - \cos(m \pi) \sin(\omega_{11} ((\Delta_{12} - \Delta_{11}) + \tau_{12})) \right) \right] / \left[ \pi \left( m^2 \tau_{12}^2 - n^2 \tau_{12}^2 \right) \right], \]
\[
\begin{align*}
\text{COMPEL} & \quad a_{m-1}e^{\omega_{11}h_6} - b_{m-1}e^{-\omega_{11}h_6} = \sum_{n=1}^{n_{12}} \left( (a_{n-12}e^{\omega_{12}h_6} - b_{n-12}e^{-\omega_{12}h_6}) \times S\delta_c(n,m,i) \right) \text{ for } i = 1 - 8 \text{ and } m = 1 - n_{11}, \\
& \quad S\delta_c(n,m,i) = \frac{2}{\tau_{12}} \int_0^{\tau_{12}} \cos(\omega_{12}x_{12})\cos(\omega_{11}x_{11})dx_{11} \\
& \quad = \left[ 2n\tau_{12}^2(\sin(\omega_{11}(\Delta_{12} - \Delta_{11})) - \cos(m\pi)\sin(\omega_{11}((\Delta_{12} - \Delta_{11}) + \tau_{12})) \right] / \\
& \quad \left[ \pi\left( m^2\tau_{12}^2 - n^2\tau_{12}^2 \right) \right]. \\
\text{At } y = h_5: & \quad a_{n-12}e^{\omega_{12}h_7} + b_{n-12}e^{-\omega_{12}h_7} = \sum_{m=1}^{n_{12}} \left( (a_{n-12}e^{\omega_{12}h_6} + b_{n-12}e^{-\omega_{12}h_6}) \times E\delta_5(m,n,i) \right) \text{ for } i = 1 - 8 \text{ and } n = 1 - n_{12}, \\
& \quad E\delta_5(m,n,i) = \frac{2}{\tau_{12}} \int_0^{\tau_{12}} \sin(\omega_{12}x_{12})\sin(\omega_{11}x_{11})dx_{12} \\
& \quad = \left[ 2m\tau_{12}^2(\sin(\omega_{12}(\Delta_{12} - \Delta_{12})) + \cos(m\pi)\sin(\omega_{12}((\Delta_{12} - \Delta_{12}) + \tau_{12})) \right] / \\
& \quad \left[ \pi\left( m^2\tau_{12}^2 - n^2\tau_{12}^2 \right) \right], \\
\text{At } y = h_6: & \quad a_{m-13}e^{\omega_{13}h_7} - b_{m-13}e^{-\omega_{13}h_7} = \sum_{n=1}^{n_{12}} \left( (a_{n-12}e^{\omega_{12}h_6} - b_{n-12}e^{-\omega_{12}h_6}) \times S\delta_6(n,m,i) \right) \text{ for } i = 1 - 8 \text{ and } m = 1 - n_{13}, \\
& \quad S\delta_6(n,m,i) = \frac{2}{\tau_{13}} \int_0^{\tau_{12}} \cos(\omega_{12}x_{12})\cos(\omega_{13}x_{13})dx_{13} \\
& \quad = \left[ 2n\tau_{12}^2(\sin(\omega_{13}(\Delta_{12} - \Delta_{13})) - \cos(m\pi)\sin(\omega_{13}((\Delta_{12} - \Delta_{13}) + \tau_{12})) \right] / \\
& \quad \left[ \pi\left( m^2\tau_{12}^2 - n^2\tau_{12}^2 \right) \right]. \\
\end{align*}
\]
\[
E7_n(m, n, i) = \frac{2}{\tau_{13}} \int_{0}^{\tau_{13}} \sin(\omega_A x_A) \sin(\omega_{13} x_{13}) \, dx_{13} \\
= \frac{2m \tau_A^2}{\tau_{13}} \left( \sin(\omega_A (\Delta_{13} - \Delta_A)) - \cos(m \pi) \sin(\omega_A ((\Delta_{13} - \Delta_A) + \tau_{13})) \right) \\
\frac{\pi (m^2 \tau_A^2 - n^2 \tau_{13}^2)}{,}
\]

\[
E3_c(m, n, i) = \frac{2}{\tau_{13}} \int_{0}^{\tau_{13}} \cos(\omega_A x_A) \sin(\omega_{13} x_{13}) \, dx_{13} \\
= \frac{2m \tau_A^2}{\tau_{13}} \left( \cos(\omega_A (\Delta_{13} - \Delta_A)) - \cos(m \pi) \cos(\omega_A ((\Delta_{13} - \Delta_A) + \tau_{13})) \right) \\
\frac{\pi (m^2 \tau_A^2 - n^2 \tau_{13}^2)}{,}
\]

\[-d_{m-A} e^{-\omega_A \Delta \Delta} = \sum_{i=1}^{8} \sum_{n=1}^{n_{13}} \left( a_{n-I3} e^{\omega_{13} \Delta \Delta} - b_{n-I3} e^{-\omega_{13} \Delta \Delta} \right) \times k3_c(n, m, i) \]

\[
\text{for } i = 1 - 8 \text{ & } m = 1 - n_A,
\]

\[
k3_c(n, m, i) = \frac{1}{\tau_{13}} \int_{0}^{\tau_{13}} \cos(\omega_{13} x_{13}) \sin(\omega_A x_A) \, dx_A \\
= -n \tau_{13}^2 \left( \cos(\omega_A (\Delta_{13} - \Delta_A)) - \cos(m \pi) \cos(\omega_A ((\Delta_{13} - \Delta_A) + \tau_{13})) \right) \\
\frac{\pi (m^2 \tau_A^2 - n^2 \tau_{13}^2)}{,}
\]

\[-b_{m-A} e^{-\omega_A \Delta \Delta} = \sum_{i=1}^{8} \sum_{n=1}^{n_{13}} \left( a_{n-I3} e^{\omega_{13} \Delta \Delta} - b_{n-I3} e^{-\omega_{13} \Delta \Delta} \right) \times S7_e(n, m, i) \]

\[
\text{for } i = 1 - 8 \text{ & } m = 1 - n_A,
\]

\[
S7_e(n, m, i) = \frac{1}{\tau_{13}} \int_{0}^{\tau_{13}} \cos(\omega_{13} x_{13}) \cos(\omega_A x_A) \, dx_A \\
= n \tau_{13}^2 \left( \sin(\omega_A (\Delta_{13} - \Delta_A)) - \cos(m \pi) \sin(\omega_A ((\Delta_{13} - \Delta_A) + \tau_{13})) \right) \\
\frac{\pi (m^2 \tau_A^2 - n^2 \tau_{13}^2)}{.}
\]

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