Full length article

Finite element vibration analysis of multi-scale hybrid nanocomposite beams via a refined beam theory

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**ABSTRACT**

Present manuscript is mainly arranged to take into consider the influences of nanofillers' aggregation, beam's shear deformation and various boundary conditions on the vibration frequency of multi-scale hybrid nanocomposites in the framework of finite element based Rayleigh-Ritz method. The constituent material is made from three phases, namely polymer matrix, macro-scale carbon fibers and nano-scale carbon nanotubes. Homogenization procedure is procured based on Eshelby-Mori-Tanaka approach incorporated with a micromechanical scheme to obtain the effective material properties via a two-step method. In addition, a new refined higher-order beam theory is introduced to govern shear stress and strain through the thickness direction. Furthermore, influences of different boundary conditions are included, too. The accuracy of the presented finite element formulations is examined by setting a comparison between the dimensionless frequency of multi-scale hybrid nanocomposite beams via both analytical and finite element solutions. Afterwards, parametric studies are adopted to put emphasize on the influence of various terms on the vibrational behaviors of nanocomposite beams. It is reported that influence of different parameters deeply depends on the magnitude of volume fraction of nanofillers inside the inclusions.

1. Introduction

Laminated composites are one of the well-known kinds of composite materials which are fabricated by inserting each ply with a desirable orientation angle to achieve pre-defined mechanical properties. Due to their remarkable characteristics, such composites have gained the attention of a large number of authors to employ these materials in the design procedures and scientific researches. In the early 2000s, Zenkour and Fares [1] presented a general mechanical analysis on the laminated composite shells independent from any additional shear correction coefficient. Static and dynamic answers of laminated plates via a meshless method are explored by Wang et al. [2] with respect to various edge conditions (ECs) of the structure. Patel et al. [3] used a finite element method (FEM) to estimate the hygro-thermo-elastic buckling and free vibration behaviors of laminated composite plate via a \( C^0 \) continuous element. Later, Ferreira et al. [4] developed a grid-free approximation function to probe the free vibration problem of a laminated plate in the framework of Mindlin's plate theory. Aydogdu [5] could study thermo-elastic stability limits of multi-layered composite beams employing a higher-order beam theorem. In another attempt, the vibration and buckling problems of cylindrical laminates are solved [6].

Chai and Yap [7] developed a closed form FE model to derive the lateral modulus of laminated composite beams and showed its applicability in static and dynamic problems of Euler-Bernoulli beams. Another study is performed by Ćetković and Vuksonavić [8] to present a layerwise displacement field for mechanical behaviors of laminates. Putting emphasize on the crucial role of choosing a proper shape function in higher-order theories, Aydogdu [9] introduced a new shear deformable model for laminated composite plates and showed its application in the cases of vibration, bending and buckling problems. Afterwards, Carrera's unified formulation (CUF) is utilized by Fazzolari and Carrera [10] for the goal of solving the vibration and stability problems of laminated composite plates on the basis of various approximation techniques. As same as previous work, CUF is again used to be combined with a zig-zag kinematic theorem and a finite difference approach in order to reach both static and dynamic answers of composite plates [11]. Shishesaz et al. [12] considered the delamination of a laminated circular plates while investigating the stability behaviors of such structures. In some other numerical studies, authors could present a FE-based isogeometric analysis (IGA) for mechanical response of laminated elements [13–20].

Besides, another type of famous composites which is used in designs
and analyses is fiber reinforced composites (FRCs). In this type, the desirable properties can be generated by inserting a group of fibers such as glass- or carbon-fibers into a matrix with a designed orientation angle. Just as same as laminated composites, FRCs are widely employed as constituent material of a continua which is seemed to be analyzed. Aref and Alampalli [21] implemented fiber reinforced polymers (FRPs) to achieve the vibrational mode shapes of bridges consisted from FRPs. Later, both experimental and theoretical ways are implemented by Tata et al. [22] in order to examine the dynamic behaviors of FRC beams. Zenkour [23] performed a viscoelastic stability analysis on the FRC plates using both classical and higher-order plate theories. Qiao and Yang [24] studied the mechanical impact responses of a sandwich beam made of FRPs by the means of ABAQUS commercial software. Moreover, a genetic algorithm (GA) based model is introduced by Roy and Chakraborty [25] to optimize the vibration control of FRP shells. In another research, Mareishi et al. [26] investigated the electromechanical nonlinear mechanical static and dynamic behaviors of fiber reinforced piezoelectric beams. Sepahvand [27] carried out a FEM based stochastic dynamic study on FRCs.

Furthermore, once elements with at least one dimension in nano scale are selected as reinforcements, the composite is named a nanocomposite. Indeed, the outstanding mechanical properties of nanocomposites were appealing enough in the engineers’ opinion to be employed as reinforcement in composites. One of the most famous nano size reinforcing elements is carbon nanotube (CNT). In the 2010s, with the increasing application of nano structures in mechanical analyses of continuums, many researchers devoted their field of interest to investigate the static and dynamic behaviors of CNT reinforced (CNTR) nanocomposites. For example, Ke et al. [28] utilized Timoshenko beam model incorporated with von Kármán relations to survey nonlinear vibrational behaviors of CNTR nanocomposite beams. In another attempt, Zhu et al. [29] used Mindlin plate model coupled with FEM to study the static and dynamic responses of CNTR nanocomposite plates. Shen and Xiang [29] investigated the thermo-elastic postbuckling problem of a cylindrical CNTR nanocomposite panel once an axial compression is applied. All of the aforementioned valuable researches can provide reliable results about the mechanical behaviors of CNTR nanocomposite structures. However, some practical phenomena which usually occur in such nanocomposites are not included in these researches. Actually, due to the tiny dimensions of the CNTs, they do not behave as same as macroscale structural elements. In the nano dimension, the influences of the van der Waals (vdW) attractive forces are of great significance. Usually, the nanoparticles can be absorbed by the vdW potential of another nanosize element and this phenomenon happens many times once a CNTR nanocomposite material is going to be manufactured. Thus, some of the CNTs may attract each other in a particular part of the nanocomposite. This phenomenon which has destructive effects on the stiffness of a nanocomposite is named agglomeration. In fact, in such cases, the appropriate uniform distribution of the nanofilbers in the matrix which is the major goal of the engineers cannot be satisfied. Henceforward, it is of high importance to earn as more as possible data about the mechanical behaviors of the CNTR nanocomposites in the presence of the nanofilbers’ agglomeration. Due to this fact, some micromechanical models are presented that are powerful enough to estimate the influences of the CNTs’ agglomeration on the equivalent stiffness of the nanocomposite. Heshmati et al. [30] could probe the dynamic characteristics of a CNTR beam with respect to the influences of CNTs’ agglomeration and waviness. Lei et al. [31] utilized a kp-Ritz method in order to survey vibration analysis of nanocomposite plates reinforced with single-walled CNTs (SWCNTs) in the framework of Mindlin plate theory. Besides, Wattanasakulpong and Chaikittiratana [32] could present an efficient model for bending, buckling and vibration behaviors of CNTR nanocomposite plates based on a higher-order plate model. Zhang and Liew [33] taken into consideration the geometrical nonlinearity in their research dealing with nonlinear large deflection behaviors of nanocomposite skew plates rested on a two-parameter elastic substrate. Also, Jam and Kiani [34] could present a solution for low-velocity impact problem of a nanocomposite beam while the structure is supposed to be affected by thermal loading. An Isogeometric analysis (IGA) in conducted by Phung-Van et al. [35] for both stability and vibration problems of CNTR nanocomposite plates on the basis of Reddy’s plate model. Thereafter, Song et al. [36] employed a higher-order plate theory to consider impact responses of CNTR nanocomposite structures with respect to different distributions of CNTs in the initial matrix. The agglomeration effects are regarded by Tornabene et al. [37] analyzing the vibrational characteristics of nanocomposite shells containing CNTs as reinforcements. Lei et al. [38] developed a parametric study for dynamic behaviors of rotating cylindrical panels reinforced with CNTs on the basis of an element free kp-Ritz method. In addition, the postbuckling analysis of laminated nanocomposite plates reinforced with CNTs subjected to a bi-directional compression is performed by Zhang et al. [39]. Moreover, the low-velocity impact analysis of nanocomposite plates in thermal environments is carried out by Ebrahimi and Habibi [40]. An Estebly-Mori-Tanaka based homogenization model for CNTR nanocomposite panels is developed by García-Macías et al. [41] for the goal of investigating the postbuckling characteristics of such structures under axial compression with respect to waviness and agglomeration effects. Another remarkable paper in this field of interest is arranged by Ansari et al. [42] dealing with the vibration and stability responses of circular sector nanocomposite plates reinforced with CNTs. Zarei et al. [43] could numerically solve the low-velocity impact problem of a nanocomposite plate with arbitrary boundary conditions (BCs) by considering both shear deformation and thermal effects. Fantuzzi et al. [44] presented a non-uniform rational B-splines curve based model for vibration problem of arbitrary shaped CNTR plates by considering agglomeration influences. Ebrahimi and Farazmandnia [45] examined the thermally affected mechanical responses of sandwich beams made of CNTR nanocomposites. Besides, Arefi et al. [46] employed a CNTR nanocomposite material for the purpose of analyzing the thermally affected static responses of nanocomposite pressure vessels. On the other hand, CNTs are not the only nano size reinforcement which is used in the nanocomposites. Nano fillers consisted of other carbon based materials are utilized in nanocomposites, too. For instance, graphene platelets (GPLs) are recently employed by researchers to design and analyze novel nanocomposites. Song et al. [47] investigated both free and forced vibrational characteristics of GPL reinforced (GPLR) plates. Stability problem of graphene reinforced laminated nanocomposite plates is solved by Shen et al. [48] based upon a two-step perturbation technique. Besides, Feng et al. [49] carried out the nonlinear bending analysis of a GPLR beam with respect to the distribution of the reinforcing elements in the cross-section area of the structure. Yang et al. [50] could explore the stability behaviors of multi-layered nanocomposite beams reinforced with GPLs. Also, the issue of postbuckling analysis of a porous GPLR beam is undertaken and studied by Barati and Zenkour [51] considering geometrical imperfection. Fan et al. [52] successfully probed the nonlinear low-velocity impact behaviors of laminated nanocomposite plates reinforced with graphene whenever the structure is embedded on a visco-elastic foundation. Also, thermally affected nonlinear bending analysis of graphene reinforced nanocomposite cylindrical panels is studied by Shen et al. [53]. García-Macías et al. [54] surveyed the frequency and deflection characteristics of nanocomposite plates reinforced with both graphene and CNT. Furthermore, Liu et al. [55] examined the 3-D stability and oscillation behaviors of graphene reinforced cylindrical shells whenever the structure is subjected to pre-stress. The vibrational responses of GPLR nanocomposite nanoplates are investigated by Arefi et al. [56] with respect to various distributions of the nanoparticles in the media. The nonlinearity effects are included in another attempt performed by Kiani and Mirzaei [57] relating to the thermal postbuckling of a laminated nanocomposite beam reinforced with graphene. Song et al. [58] highlighted the effects of GPLs as reinforcements in the bending and...
buckling responses of nanocomposite plates on the basis of Mindlin plate theory. Moreover, Barati and Zenkour [59] surveyed the vibration problem of a porous nanocomposite shell reinforced with GPLs. In another scientific endeavor, Arefi et al. [60] implemented the first-order shear deformation theory to probe the bending characteristics of tiny beams made from GPLR nanocomposites.

Even though the aforementioned nanocomposites possess lots of merits to be selected as the material for mechanical elements, a novel type of nanocomposites have been recently found which is able to exhibit a combined behavior of both macro- and nano-composites. These nanocomposites are made from three phases: a primary matrix, macro scale reinforcement and nano scale reinforcement. Due to this mixture, these nanocomposites are named multi-scale hybrid nanocomposites. As a matter of fact, utilization of multi-scale hybrid nanocomposites empowers the structure to support higher critical stability limit, natural frequency and also lower deflection. Thus, it is of high importance to gain adequate knowledge about the mechanical behavior of structures consisted of this type of nanocomposites. In the recent years, some of the authors made their effort to study the mechanical responses of multi-scale hybrid nanocomposites. Rafiee et al. [61] surveyed nonlinear dynamic characteristics of piezoelectric laminated plates made from multi-scale hybrid nanocomposites. He et al. [62] explored the large amplitude nonlinear free and forced vibrational responses of multi-scale hybrid nanocomposite beams. Later, Rafiee et al. [63] investigated static and dynamic responses of thin-walled rotating multi-scale hybrid nanocomposite beams. Also, Ghorbanpour Arani et al. [64] studied the vibrational responses of double-layered sandwich beams made from a smart core and facesheets made from multi-scale hybrid CNT/glass fiber reinforced nanocomposites. Ebrahimi and Habibi [65] tried to determine the behaviors of multi-scale hybrid nanocomposite plates in a hygrothermal environment once the structure is subjected to a low-velocity impactor. They considered for kinematical nonlinearities on the basis of von-Karman theory. In another attempt, Gholami and Ansari [66] surveyed the nonlinear deflection problem of a multi-scale hybrid nanocomposite plate. Most recently, Ebrahimi and Dabbagh [67] analyzed the vibrational behaviors of multi-scale hybrid nanocomposite beams in thermal environments. According to the literature, it can be declared that although valuable researches have been performed in the recent years about the static and dynamic responses of the novel three-phase nanocomposite (multi-scale hybrid nanocomposite) structures, many problems about the mechanical behaviors of continuous systems made from these new nanocomposites can be defined which have never been answered yet. Present paper is basically arranged to cover lack of a numerical study on the vibrational behaviors of multi-scale hybrid nanocomposite beams regarding for the practical effect of the CNTs’ agglomeration which has never been studied hitherto.

In this article, a blend of Eshelby-Mori-Tanaka micromechanical model and rule of mixture is utilized to homogenize the constituent material. Here, a new refined higher-order beam theory is extended. Furthermore, the natural frequency is going to be reached by calculating the original form of strain energy and kinetic energy for linear elastic solids. At the end, Rayleigh-Ritz well-known FE based approach is developed for the present problem. Also, the problem is solved via a Navier type solution to show the applicability of the presented FE formulations.

2. Theory and formulation

2.1. Micromechanical homogenization scheme

There are some available methods which can be employed for the goal of homogenization of composites. These techniques can be generally distinguished from each other by paying attention to the estimation coefficients which are used in them. Moreover, in some of them, maybe a practical phenomenon is covered via a mathematical formulation. For instance, the difference between Voigt-Reuss and Halpin-Tsai models can be roughly summarized in approximation functions which are utilized in aforementioned theorems in order to guess axial and flexural Young modules. Also, it is of great significance to point that both of the above discussed methods are majorly implemented for a lamina, not for a 3-D structure. On the other hand, one can select each of the Modified mixture law or Cox model once a 3-D problem is seemed to be solved. Furthermore, 3-D experimental analyses have proven that Cox model is the best micromechanical method which can be used to reach the equivalent material properties of a 3-D CNTR composite [68]. Moreover, Mori and Tanaka [69] developed a new method which is able to capture the inclusion effects while evaluating the average stress in a media. However, the influences of nanofillers’ aggregation cannot be included in the aforementioned homogenization approaches.

In this section, the homogenization process is explained presenting Eshelby-Mori-Tanaka model in order to capture the effect of CNTs’ agglomeration while reaching the effective mechanical properties of multi-scale hybrid nanocomposites [70]. Furthermore, the rule of mixture is employed in order to account for the dispersion of CFs in the nanocomposite. First, the effective properties of CF reinforced (CFR) composites are going to be discussed as follows:

\[
E_{11} = V_FE_{NCM}^F + V_NCMMNC, \]

\[
\frac{1}{G_{12}} = \frac{V_FE_{NCM}^F}{G_{NCM}^F} + \frac{V_NCMMNC}{G_{NCM}^M}, \]

\[
\rho = V_F\rho_F^F + V_NCMMNC, \]

\[
v_2 = V_Fv_F^F + V_NCMMNC, \]

where \(E, G, \nu, \) and \(\rho\) stand for Young’s modulus, shear modulus, Poisson’s ratio and mass density, respectively. Also, the superscripts \(F\) and \(M\) denote fiber and nanocomposite matrix, respectively. Evidently, \(V_F\) and \(V_NCMMNC\) are volume fractions of fiber and nanocomposite matrix, respectively. Obviously, the aforementioned volume fractions can be related to each other by:

\[
V_F + V_NCMMNC = 1 \]

Next, it is turn to investigate the effect of adding nanoparticles to the media. CNTs, which are employed as the nano scale reinforcements in this article, possess a remarkable stiffness incorporated with a high slenderness ratio. Due to these features, sometimes CNTs do not follow the initial uniform distribution inside the matrix. In other words, in some regions inside the continua some spherical inclusions can be found which are filled with a set of CNTs. Thus, CNTs’ concentration can be different from a region to another one. This effect is of high significance whenever the mechanical behavior of a nanocomposite is supposed to be analyzed. in this situation, the total volume of CNTs can be divided in two parts, one of them is related to the CNTs inside the inclusions and another one corresponds with CNTs which are inserted in the matrix. The volume of CNTs inside the inclusions (clusters) \(W_{in}\) and the volume of CNTs inside the matrix \(W_{out}\) can be related to each other as:

\[
W = W_{in} + W_{out} \]

\[
W = W_{in} + W_{out} \]

Now, it is turn to relate the volume of CNTs to the entire volume of the structure as follows:

\[
W = W_{in} + W_{out} \]

where \(W_{in}\) is the volume of the matrix which CNTs are dispersed in it. In this problem, a polymeric matrix is employed. Also, \(W\) is the volume of CNTs. Dividing these volumes to the total volume (\(W\)), the volume fraction of each part can be written as:

\[
V_{in} = \frac{W_{in}}{W}, \quad V_{out} = \frac{W_{out}}{W} \]
As same as the volume of CNTs (W), the volume fraction of CNTs in the matrix can be divided in two parts of inside the clusters and outside of clusters. To this reason, two new parameters are introduced to formulate this issue in the following form:

\[ \mu = \frac{W_{in}}{W}, \quad \eta = \frac{W_{in}^{n}}{W_{in}} \]  

(9)

where \( \mu \) indicates on the volume fraction of clusters and \( \eta \) stands for the volume fraction of CNTs inside the clusters. It should be regarded that \( \mu \leq \eta \) is a limitation for this methodology.

One should be aware of the particular cases which can be generated by changing agglomeration parameters. For instance, once \( \mu = 1 \), the entire matrix can be considered as a big cluster which contains all of the nanoparticles, henceforward, aggregation of nanofillers cannot be observed. However, full accumulation can occur in the situation that \( \eta = 1 \) (fully agglomerated CNTs). In another condition (\( \mu < \eta, \eta \neq 1 \)), some of the nanofillers are placed inside the clusters and the others are scattered in the matrix free from any membrane (partially agglomerated CNTs).

Incorporating Eqs. (8) and (9) yields in:

\[ \frac{W_{in}^{n}}{W_{in}} = V_{r} \varphi. \]  

(10)

\[ \frac{W_{in}^{M}}{W_{in}^{M}} = V_{r} (1 - \eta) \frac{1}{1 - \mu}. \]  

(11)

Also, the variation of the \( V_{r} \) with respect to the thickness direction produces mechanical properties as a function of \( z \). The volume fraction of nanofillers in the matrix can be expresses as:

\[ V_{r}(z) = \left[ \frac{\varphi_{C} - \varphi_{M}}{\varphi_{C} - \varphi_{M}} \right]^{1} \left( \frac{z}{h} + \frac{1}{2} \right)^{P} \]  

(12)

in which \( \varphi_{C} \) and \( \varphi_{M} \) are mass densities of CNT and matrix, respectively. Also, \( P \) is gradient index which governs the volume fraction \( V_{r}(z) \) through the thickness direction. In addition, \( w_{r} \) is the mass fraction of nanofillers and can be calculated by:

\[ w_{r} = \frac{M_{r}}{M_{r} + M_{M}} \]  

(13)

where \( M_{r} \) and \( M_{M} \) are related to the mass of CNTs and matrix, respectively. It is worth mentioning that two versions of \( V_{r} \) can be defined in the problems of which agglomeration phenomenon is studied. The main differences between these two types are about the position of agglomerated nanoparticles and the matrix. In this case, the matrix is seemed to be in the bottom and the agglomerated CNTs are assumed to be at the top of the structure. To gain more information about this issue, researchers are highly advised to read [44].

Now, the effective material properties can be reached following the relations of Eshelby-Mori-Tanaka micromechanical model [44,70]. According to this model, the bulk moduli of inclusions can be written as:

\[ K_{in}(z) = K_{M} + \frac{V_{r}(\xi_{C} - 3K_{M}\xi_{C})}{3(1 - \mu + V_{r}\xi_{C} + V_{r}\eta\xi_{C})} \]  

(14)

where \( K_{M} \) is the bulk moduli of matrix. Moreover, the shear moduli of inclusions can be introduced as:

\[ G_{in}(z) = G_{M} + \frac{V_{r}(\eta_{C} - 2G_{M}\eta_{C})}{2(\mu + V_{r}\eta + V_{r}\eta\xi_{C})} \]  

(15)

where \( G_{M} \) is the shear moduli of matrix. Next, the bulk and shear moduli of the remnant parts can be formulated as:

\[ K_{out}(z) = K_{M} + \frac{V_{r}(1 - \eta)(\xi_{C} - 3K_{M}\xi_{C})}{3(1 - \mu - V_{r}(1 - \eta) + V_{r}(1 - \eta)\xi_{C})}. \]  

(16)

\[ G_{out}(z) = G_{M} + \frac{V_{r}(1 - \eta)(\eta_{C} - 2G_{M}\eta_{C})}{3(1 - \mu - V_{r}(1 - \eta) + V_{r}(1 - \eta)\xi_{C})}. \]  

(17)

In Eqs. (14) – (17), the mechanical terms \( \alpha_{r}, \beta_{r}, \delta_{r} \) and \( \eta_{r} \) can be calculated as:

\[ \alpha_{r} = \frac{3(K_{M} + G_{M}) + k_{r} + l_{r}}{3(G_{M} + k_{r})}, \]  

(18.a)

\[ \beta_{r} = \frac{1}{5} \left[ 4G_{M} + 2k_{r} + l_{r} + \frac{4G_{M}}{G_{M} + p_{r}} \right] + \frac{2(G_{M}(3K_{M} + G_{M}) + G_{M}(3K_{M} + 7G_{M}))}{G_{M}(3K_{M} + G_{M}) + m_{r}(3K_{M} + 7G_{M})} \]  

(18.b)

\[ \delta_{r} = \frac{1}{3} \left( n_{r} + 2l_{r} + (2k_{r} + l_{r})(3K_{M} + G_{M} - l_{r}) \right). \]  

(18.c)

\[ \eta_{r} = \frac{1}{5} \left( \frac{2}{3} (n_{r} - l_{r}) + \frac{8G_{M}p_{r}}{G_{M} + p_{r}} + \frac{8G_{M}p_{r}}{G_{M} + p_{r}} + \frac{8G_{M}p_{r}}{G_{M} + p_{r}} \right) \]  

(18.d)

where \( k_{r}, l_{r}, m_{r}, n_{r}, \) and \( p_{r} \) are the elastic Hill’s coefficients of CNTs which can be different for each type of CNTs with respect to the chirality of the CNT. In this manuscript, the Hill’s constants are employed for SWCNTs with chirality of (10,10) from reference [71].

Based on the implemented homogenization scheme, the equivalent bulk moduli of the nanocomposite can be computed using the following formula:

\[ K(z) = K_{out} \left( 1 + \frac{\mu (K_{in} - K_{out})}{1 + (1 - \mu)(K_{in} - K_{out})} \right) \]  

(19)

where \( K_{out} \) is the Poisson’s ratio of the matrix and can be defined as:

\[ K_{out} = \frac{3K_{out} - 2G_{out}}{6K_{out} + 2G_{out}} \]  

(20)

Also, the equivalent shear moduli can be computed as:

\[ G(z) = G_{out} \left( 1 + \frac{\mu (G_{in} - G_{out})}{1 + (1 - \mu)(G_{in} - G_{out})} \right) \]  

(21)

Finally, the equivalent Young moduli and Poisson’s ratio of CNTR nanocomposites can be written in the following form:

\[ E(z) = \frac{9K(z) \times G(z)}{3K(z) + G(z)} \]  

(22)

\[ \nu(z) = \frac{3K(z) - 2G(z)}{6K(z) + 2G(z)} \]  

(23)

Moreover, the equivalent density of the CNTR nanocomposite can be formulated utilizing the fundamentals of mixture’s rule as:

\[ \rho(z) = (\rho_{C} - \rho_{M})V_{r} + \rho_{M} \]  

(24)

Here, the influences of gradient index (\( P \)) on the variation of the stiffness and mass density of the multi-scale hybrid nanocomposite material will be shown to earn a better understanding about the physical effect of this parameter. Fig. 1 illustrates the variation of the material’s longitudinal elastic modulus across the thickness for different gradient indices. As can be seen in this diagram, the more is the gradient index, the slower is the speed of the stiffness enhancement of the nanocomposite across the thickness. In other words, the elastic modulus of the hybrid nanocomposite reaches its maximum amount with a delay once a big gradient index is selected in comparison with the case of employing a small gradient index. This phenomenon can show its
effect once calculating the system's stress resultants (i.e., the integrals which must be calculated by integrating across the thickness of the beam). Therefore, it can be found that the stiffness of the system reduces whenever a big gradient index is implemented.

Similar with the previous diagram, in Fig. 2, the effect of adding the gradient index on the variation of the nanocomposite's density is shown across the thickness. Again, it can be observed that the density of the material possesses its maximum amount later if a big gradient index is chosen. Thus, the density of the hybrid nanocomposite material can be lessened in any desired dimensionless thickness once a higher gradient index is implemented instead of a smaller one. Actually, this phenomenon can be better realized once computing the beam's mass resultants which will be calculated by integrating over the thickness of the beam.

2.2. Refined higher-order beam theory

In this part, the kinematic formulation of beam-type elements will be discussed to provide a set of primary equations which are used in order to reach the governing equations of a beam. One knows that the frequently used classical theory of beams, known as Euler-Bernoulli beam theory, disregards the deflection generated by shear effect in its mathematical formulation and only covers bending deflection. However, this is not a realistic assumption in fact; thus, it is better to employ higher-order beam theories which are powerful enough to account for shear deformation effects. Up to know, several famous theories are developed to cover this deficiency [72–75]. Also, many of the researchers have tried to implement higher-order kinematic theories in order to present a more realistic analysis about the chosen structure which is assumed to be studied. It is of importance to compare the shear approximation shape functions of higher-order theories. For the sake of brevity, this issue is not more expanded in the present part; however, readers are highly recommended to review the details of papers dealing with either static or dynamic behaviors of continuous systems by considering the influences of shear deformation [76–81].

Herein, we are about to introduce a new refined higher-order beam theory and show its application in the vibration problem of a nanocomposite beam. Based on a refined theory, the displacement field of a beam can be written in the following form:

\[
\begin{align*}
  u(x, z, t) &= u(x, t) - z \frac{\partial w_0(x, t)}{\partial x} - f(z) \frac{\partial w_1(x, t)}{\partial x}, \\
  w_b(x, z, t) &= w_b(x, t) + f(z) w_1(x, t)
\end{align*}
\]

where \( u \) is longitudinal displacement; \( w_b \) and \( w_s \) are bending and shear deflections, respectively. Besides, \( f(z) \) is a shape function which covers the shear strain and stress towards thickness via a particular profile. In this research, the following shape function is implemented [82]:

\[
\begin{align*}
  f(z) &= \frac{bc^2}{h^2 + \pi^2} \left( \frac{\pi}{h} \sin \left( \frac{\pi z}{h} \right) + h \cos \left( \frac{\pi z}{h} \right) \right) - \frac{bc^2}{h^2 + \pi^2} z \frac{\partial w_0(x, t)}{\partial x} - f(z) \frac{\partial w_1(x, t)}{\partial x}.
\end{align*}
\]

Thereafter, it is necessary to reach expressions for the nonzero strains of a beam via aforementioned displacement field. In order to achieve the strain components, the concept of infinitesimal strain in continuum mechanics is used. The strain tensor of a continua can be calculated as follows:

\[
\begin{align*}
  \varepsilon &= \frac{1}{2} (\nabla u + (\nabla u)^T)
\end{align*}
\]

where \( \varepsilon \) and \( u \) are strain tensor and displacement vector, respectively. Based on the above relation, the nonzero strains of the beam can be written as:

\[
\begin{align*}
  \varepsilon_{xx} &= \frac{\partial u}{\partial x} + z \frac{\partial^2 w_0}{\partial x \partial z} - f(z) \frac{\partial^2 w_1}{\partial x \partial z}, \\
  \gamma_{xz} &= g(z) \frac{\partial w_0}{\partial x}
\end{align*}
\]

in which

\[
\begin{align*}
  g(z) = 1 - \frac{df(z)}{dz}
\end{align*}
\]

2.3. Derivation of strain energy

The strain energy of a structure can be calculated via following definition:

\[
U = \frac{1}{2} \int \varepsilon u_s \, dV
\]
where $\sigma_i$ and $\epsilon_i$ are components of Cauchy stress and strain, respectively. The constitutive relation between stress and strain can be formulated as follows for linear elastic solids:

$$\sigma_{ii} = E_{ii} \epsilon_{ii}, \quad \sigma_{zz} = G_{12} \gamma_{zz}$$  \hspace{1cm} (32)

Now, the strain energy can be reached once inserting Eqs. (29) and (32) in Eq. (31) as:

$$U = \frac{1}{2} \int_0^L \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 + \frac{d}{dx} \left( \frac{3u_z}{x^2} \right)^2
+ \frac{d}{dx} \left( \frac{3u_w}{x^3} \right)^2
+ G_{12} \gamma_{zz} \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2
$$

$$+ \frac{1}{2} \int_0^L \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 - 2G_{12} \gamma_{zz} \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 - \frac{1}{2} \left( \frac{3u_z}{x^2} \right)^2
- \frac{1}{2} \left( \frac{3u_w}{x^3} \right)^2$$

$$+ \frac{1}{2} \int_0^L \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 - 2G_{12} \gamma_{zz} \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 - \frac{1}{2} \left( \frac{3u_z}{x^2} \right)^2 + A_{12} \left( \frac{3u_x}{x} \right)^2$$

$$+ G_{12} \gamma_{zz} \frac{d}{dx} \left( \frac{3u_x}{x} \right)^2 \right) dAdx$$  \hspace{1cm} (33)

2.4. The above equation can be rewritten like:

$$U = \frac{1}{2} \int_0^L \left[ A_{xx} \left( \frac{3u_x}{x} \right)^2 - 2B_{xx} \frac{4u_x}{x^2} \frac{3u_z}{x^3} - 2B_{xx} \frac{4u_z}{x^2} \frac{3u_w}{x^3} - D_{xx} \left( \frac{3u_x}{x} \right)^2 \right]
- 2D_{xx} \frac{4u_z}{x^2} \frac{3u_w}{x^3} - H_{xx} \left( \frac{3u_z}{x^2} \right)^2 + A_{12} \left( \frac{3u_x}{x} \right)^2 dx$$

in which cross-sectional rigidities can be expressed as:

$$[A_{xx}, B_{xx}, D_{xx}, D_{xx}, H_{xx}] = \int_{A} \left[ 1, z, z^2, f(z), f^2(z) \right] E_{ii}(x) dA$$

$$A_{12} = \int_{A} \left( \frac{3u_x}{x} \right)^2 G_{12} dA$$

(34)

2.5. Derivation of kinetic energy

The kinetic energy of the beam can be obtained based on the following equation:

$$T = \frac{1}{2} \int \rho(z) \left[ \left( \frac{d\gamma}{dt} \right)^2 + \left( \frac{d\gamma}{dt} \right)^2 \right] dV$$

(36)

Now, the expression of kinetic energy can be extended by substituting Eqs. (25) and (26) in Eq. (36):

$$T = \frac{1}{2} \int_0^L \rho(z) \left[ \left( \frac{d\gamma}{dt} \right)^2 + \left( \frac{d\gamma}{dt} \right)^2 \right] dV$$

$$+ \frac{1}{2} \int_0^L \rho(z) \left[ \left( \frac{d\gamma}{dt} \right)^2 + \left( \frac{d\gamma}{dt} \right)^2 \right] dV$$

(37)

Eq. (37) can be expressed as below:

$$T = \frac{1}{2} \int_0^L \left[ \frac{d}{d\gamma} \left( \frac{d\gamma}{dt} \right)^2 + \frac{d}{d\gamma} \left( \frac{d\gamma}{dt} \right)^2 \right] dV$$

(38)

where the utilized mass moment of inertias can be calculated via following equation:

$$[I_0, I_1, I_2, I_3, I_4] = \int_A \left[ 1, z, f(z), z^2, f(z), f^2(z) \right] \rho(z) dA$$

(39)

2.6. Finite element solution

Present part is allocated to present Rayleigh-Ritz method in order to achieve the natural frequency of nanocomposite plates. On the basis of this method, the solution functions of displacement components can be presumed to be made from two completely independent terms. In other words, this method is developed based on the separation of variables. According to this method, the displacement components of refined higher-order beams can be formulated as:

$$u(x, t) = U(x)e^{i\omega t}, \quad w_0(x, t) = W_0(x)e^{i\omega t}, \quad w_1(x, t) = W_1(x)e^{i\omega t}$$

(40)

As a matter of fact, the terms $U(x)$, $W_0(x)$ and $W_1(x)$ are the vibration amplitudes axial displacement, bending deflection and shear deflections, respectively. Furthermore, the harmonic type response is presented in the framework of an exponential term which contains the natural frequency parameter ($\omega$). Now, the maximum amounts of strain and kinetic energies can be reached whenever Eq. (40) is inserted in Eqs. (34) and (38) as follows:

$$U_{max} = \frac{1}{2} \int_0^L \left[ A_{xx} \left( \frac{d\gamma}{dt} \right)^2 - 2B_{xx} \frac{4u_x}{x^2} \frac{3u_z}{x^3} - 2B_{xx} \frac{4u_z}{x^2} \frac{3u_w}{x^3} - D_{xx} \left( \frac{3u_x}{x} \right)^2 \right] dx$$

(41)

$$T_{max} = \frac{1}{2} \int_0^L \left[ I_0 \left( \frac{d\gamma}{dt} \right)^2 + 2I_2 \left( \frac{d\gamma}{dt} \right)^2 - 2I_2 \left( \frac{d\gamma}{dt} \right)^2 - K_0 \left( \frac{3u_x}{x} \right)^2 \right] dx$$

(42)

In the utilized Rayleigh-Ritz method, the vibration amplitudes are seemed to be as summation of a set of polynomials of order $n$. Therefore, axial and transverse vibration amplitudes of a refined higher-order beam can be written in the following form:

$$U(x) = \sum_{i=1}^n c_i U_i, \quad W_0(x) = \sum_{j=1}^n d^0_j W^0_j, \quad W_1(x) = \sum_{k=1}^n d^1_k W^1_k$$

(43)

In which $c_i$, $d^0_j$ and $d^1_k$ are unknown coefficients which should be determined. The vibration amplitudes cannot be determined unless the polynomials $U_i$, $W^0_j$ and $W^1_k$ are found. The admissible polynomials for Rayleigh-Ritz method can be defined as follows:

$$U_i = x^i (L - x)^i x^{-i}, \quad W^0_j = x^j (L - x)^j x^{-j}, \quad W^1_k = x^k (L - x)^k x^{-k}$$

(44)

where $i$, $j$ and $k$ can be from 1 to $n$. In addition, $p$ and $q$ are the power exponents that are presented to determine the Eqs. at both ends. The corresponding $p$ and $q$ values for various BCs can be observed in Table 1. Thereafter, substituting Eq. (43) in Eqs. (41) and (42) reveals:

$$U_{max} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^L \left[ c_i c_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} + 2c_i c_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} + 2c_i c_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} - 2c_i c_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} \right] dx$$

$$= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^L \left[ c_i d^0_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} + 2c_i d^0_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} - 2d^0_j A_{xx} \frac{4u_i}{x^2} \frac{3u_j}{x^3} \right] dx$$

(45)

Table 1

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<td>C-C</td>
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\[ T_{\text{max}} = \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{L} \left[ K_{ij} \frac{d^{2}u_{ij}}{dx^{2}} + d^{2}w_{ij} \right] \]

The above equations can be rewritten in the new form as:

\[ U_{\text{max}} = \frac{\alpha}{2} \sum_{i=1}^{n} \sum_{j=1}^{L} \left[ K_{ij} \frac{d^{2}u_{ij}}{dx^{2}} + d^{2}w_{ij} \right] \]

in which

\[ K_{ij} = A_{0} \int_{0}^{L} \frac{\partial^{4} d_{ij}}{\partial x^{4}} dx, \quad K_{ij} = -2B_{0} \int_{0}^{L} \frac{\partial^{4} d_{ij}}{\partial x^{4}} dx, \]

\[ M_{ij} = I_{0} \int_{0}^{L} \frac{d^{4} u_{ij}}{dx^{4}} dx, \quad M_{ij} = -2I_{0} \int_{0}^{L} \frac{d^{4} u_{ij}}{dx^{4}} dx, \]

Next, the Rayleigh parameter should be calculated via below definition:

\[ R = U_{\text{max}} - T_{\text{max}} \]

Now, it is time to calculate the partial derivatives of Rayleigh parameter with respect to the unknown coefficients introduced in Eq. (43) and set the other side of the equation to zero. Once the aforementioned mathematical operations are performed, an eigenvalue vibration problem in its famous form can be reached:

\[ (K) - \omega^{2}(M) \begin{bmatrix} U_{n} \\ W_{n} \end{bmatrix} = 0 \]

Again, solving the above eigenvalue problem for \( \omega \), the natural frequency of a multi-scale hybrid nanocomposite beam can be achieved. The Eigen functions for S-S and C-C BCs can be seen as follows:

S–S: \( X_{n}(x) = \sin \left( \frac{\pi x}{L} \right) \),

C–C: \( X_{n}(x) = \sin \left( \frac{n \pi x}{L} \right) \)

2.8. Numerical results and discussion

This section is dedicated to investigate the influences of a wide range of variants on the vibrational characteristics of multi-scale hybrid nanocomposite beams. The geometry and cross-section of the analyzed structure can be seen in Fig. 3. The results of the problem are computed via both Galerkin and Rayleigh-Ritz solutions and these obtained results are compared with each other to put emphasize on the accuracy of the introduced methodology. The model is able to capture the influences of various supports at the ends. Besides, Convergence of the utilized finite element solution is shown, too. Prior to any discussion, the dimensionless form of natural frequency should be presented for the sake of simplicity. Thus, the dimensionless frequency can be reached in the following form:

\[ \Omega = \omega L^{2} \sqrt{\frac{E_{M}}{E_{F}}} \]

The validity of the presented method is proven in Table 2 by comparing the results of the present work with those reported by Yas and Samadi [86]; Wattanasakulpong and Ungbhakorn [85]; and Ebrahimi and Dabbagh [67]. Based on this comparison study, it can be concluded that the differences are negligible and both of the analytical and numerical methods that are employed in the present work can estimate the natural frequency behaviors of the CNTR nanocomposite beams accurately.

Besides, the convergence study is performed in the framework of Table 3 by applying different number of polynomials for both S-S and C-
C BCs in order to reach converged answers. It can be observed that mechanical response of C-C beams can be converged by using a large number of polynomials of various degrees. However, the frequency of S-S nanocomposite beams can be converged rapidly. Also, it is of high importance to point that the presented model is an efficient model which is able to estimate a reliable response once only about 5 polynomials are selected. In other words, the maximum error can be 3.63% by choosing the first five sentences of the admissible shape functions. However, dimensionless frequency can be more converged by adding the number of admissible shape functions. On the other hand, it is crucial to employ an appropriate number of admissible shape functions in order to achieve to major goals: 1st to enrich more reliable frequency responses and 2nd to lessen the run time of the developed MATLAB code and avoid unnecessary computational costs. Henceforward, in the following numerical examples, n = 15 is employed.

Moreover, the presented results are verified by an exact analytical solution in the framework of Tables 4–6. According to the referenced tables, the dimensionless frequency which is reached from the Rayleigh-Ritz method is in a very good agreement with that obtained from the Galerkin's method. Also, the decreasing influence of gradient index and volume fraction of CNTs inside the clusters can be easily seen. As shown in Fig. 1, the stiffness of the nanocomposite reduces while a big value is assigned to the gradient index. So, it is naturals to observe a decrease in the frequency of the beam in the case of choosing a great gradient index. In addition, the lessening influence of adding the volume fraction of the nanofillers inside the clusters can be physically realized once remembering the definition of the studied parameter. Indeed, the bigger is the volume fraction of the CNTs inside the clusters, the worst is the distribution of the nanoparticles in the media. In other words, increasing the volume fraction of the CNTs in the clusters results in possessing a larger number of the CNTs which are covered by the clusters and as we know this is not a suitable phenomenon and can directly reduce the nanocomposite's stiffness due to its destroying impact on the desired uniform distribution of the CNTs in the matrix. Moreover, it can be figured out that dimensionless frequency can be aggravated while either mass fraction of nanofillers or volume fraction of inclusions are increased. It is not strange to see an increasing effect whenever the mass fraction of the CNTs is intensified. Actually, in this condition, a larger amount of the CNTs which are covered by the clusters is increased. Also, whenever the volume fraction of the clusters grows, the nanocomposite moves toward making a giant cluster which covers the nanoparticles and in this case the nanoparticles are not completely free from the gradient index. In other words, the decreasing or increasing being of their impact can be changed if \( \eta \) parameter. In other words, the decreasing or increasing being of their impact can be changed if \( \eta \) parameter. It can be concluded that dimensionless frequency can be lessened once volume fraction of CNTs inside the clusters is added. The main reason of this phenomenon is that in higher amounts of \( \eta \), a greater number of nanofillers are aggregated, henceforward, the uniform distribution of nanofillers can be more dismissed which is not a favorable feature in the designers' point of view.

On the other hand, combined influences of boundary condition, gradient index and CEs volume fraction on the first dimensionless frequency of multi-scale hybrid nanocomposites are covered within Tables 7 and 8 using various shear strain approximation functions. Based on the tables, it is obvious that the presented \( f(z) \) is powerful enough to consider for shear deformation up to high orders. As you see, the
responses of the presented higher-order theory are equal with those of formerly developed exact theories. In addition to this issue, the influence of boundary condition can be seen, too. As estimated before, C-C beams possess a greater mechanical frequency in comparison with S-S ones. This phenomenon can be justified once comparing the stiffness of the structure in both S-S and C-C BCs. In fact, the structural stiffness of a beam-type element is larger once he C-C supports are chosen. Implementation of the S-S supports results in possessing a more flexible structure and due to this effect, the structural stiffness reduces. Thus, it is natural to have a nanocomposite beam that is able to endure higher frequencies whenever both ends are clamped. Also, one should pay attention to the increasing impact of CFs volume fraction on the dimensionless frequency of multi-scale hybrid nanocomposites. Actually, the nanocomposite material can be more strengthened in the cases of implementing higher volume fractions for the reinforcing phases. Thus, it is natural to see an improvement in the frequency responses while the volume fraction of the macroscopic CFs is increased. As same as previous tables, the results show that the more is the gradient index, the lower is the supported natural frequency.

Afterwards, Fig. 4 is plotted to show the frequency profile once the volume fraction of CNTs inside the clusters is added. According to the figure, previous interpretations can be verified another time. In fact, it is obvious that dimensionless frequency goes through a decreasing path as volume fraction of CNTs inside the clusters increases. But, the influence of parameter \( \eta \) cannot be summarized in this decreasing impact. In other words, variation of this coefficient can highly affect the influence of gradient index. As \( \eta \) raises, gradient index possesses both amplifying and lessening effects on the mechanical response of the system. For instance, in Fig. 4a, an increase in the amount of gradient index can be resulted in a decrease in the dimensionless frequency of nanocomposite beams unless \( \eta \) is considered to be greater than nearby 0.9. Similarly, this trend can be observed in Fig. 4b, too. However, the critical value of \( \eta \) which leads to this change is a bit different. The physical reason of this phenomenon is that in high values of parameter \( \eta \), the influence of agglomeration of the nanopillars is too powerful so that it can defeat the stiffness enhancement which can be obtained by scattering CNTs in the media. Hence, after a particular amount of \( \eta \), an increase in the gradient index can result in an inverse change in the stiffness of the nanocomposite compared with the conditions of which \( \eta \) is lower than that specific value. So, it is natural to observe a dual behavior from the nanocomposite as the term \( \eta \) varies. It is worth mentioning that the boundary value of \( \eta \) which causes this dual behavior can be different with respect to the volume fraction of the CFs. Indeed, the higher is the volume fraction of the CFs, the lower is the amount of the CNTs in the matrix; henceforward, a smaller number of clusters can cover the aggregated CNTs and due to this reality, the critical value of the term \( \eta \) shifts to the left and becomes smaller. Besides, Fig. 5 is presented to study the coupled effects of nanopillars’ mass fraction and edge conditions on the variation of dimensionless frequency versus volume fraction of CNTs inside the clusters. As expressed in the above discussions, the highest value of dimensionless frequency can be obtained in C-C nanocomposite beams followed by S-S ones. Also, as same as previous diagram, the dimensionless frequency becomes smaller whenever the \( \eta \) coefficient is added. Moreover, it is clear that in \( \eta < 0.9 \) dimensionless frequency can be aggravated by employing higher mass fractions of CNTs. However, this increasing trend will be changed into a decreasing one once \( \eta \) exceeds 0.9. Similar with the former illustration, the physical reason of the dual behavior of the nanocomposite structure is related to the fact that as the term \( \eta \) grows, in a particular critical value, the
disadvantages of the nanofillers are more than their advantages. So, for the amounts of $\eta$ larger than this certain value, all of the parameters will show an inverse effect compared with their influence in the cases of $\eta$ is smaller than the aforementioned specific value. In this figure, due to the fact that the material properties are the same for both plots and the difference of the plots relates to the beam's support at the ends, there is no change in the amount of this critical value for the volume fraction of the CNTs inside the clusters.

In another illustration, variation of natural frequency against the clusters' volume fraction is plotted for different nanofillers' mass fractions. Fig. 6a and b are dedicated to partially agglomerated situations, whereas, Fig. 6c is presented for the cases that all of the CNTs are inside the inclusions. Similarly, the influence of changing different mass fractions for nanotubes depends on the value of $\eta$ coefficient. Also, it can be perceived that natural frequency can be intensified whenever the volume fraction of clusters is added. This continuous growing impact can be observed in all of the $\eta$ domain. The physical interpretation of the observed trends was previously discussed in the above paragraphs, so, no more discussion is presented here for the sake of brevity.

Finally, Fig. 7 is depicted to put emphasize on the effects of CFs

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volume fraction and clusters’ volume fraction on the dimensionless frequency of S-S nanocomposite beams in both partially and fully agglomerated conditions for nanofillers. It is obvious that the mechanical vibration can be amplified by utilizing a higher value for the volume fraction of CFs because of the stiffness enhancement which can be achieved by adding the amount of the reinforcing phase in the composition of the nanocomposite material. Furthermore, according to the diagram, the vibrational response of the system can be amplified by implementing higher volume fractions for the clusters. On the other hand, as estimated before, the dimensionless frequency is bigger in the case of partially agglomerated CNTs in comparison with the condition that all of the nanofillers are inside the clusters. The reason of these phenomena can be understood by reading the above paragraphs.

3. Conclusion

Present paper involves both analytical and numerical solutions for vibration problem of a multi-scale hybrid nanocomposite beam on the basis of a new refined higher-order shear deformable beam theory. The homogenization of the nanocomposite was performed in the framework of a two-step method which is designed to include the influence of adding the nanofillers followed by that of the macroscale carbon fibers. The strain-displacement relations were obtained employing the infinitesimal strains incorporated with a refined shear deformation beam

![Graphs](image-url)
hypothesis. On the basis of the Rayleigh-Ritz method, the maximum amounts of strain energy and kinetic energy were developed, thereafter, the Rayleigh parameter was obtained and by differentiating with respect to the unknown coefficients and finally the corresponding eigenvalue problem was achieved. Now, the most important highlights of this paper are going to be reviewed as follows:

1. The natural frequency can be decreased by either using higher gradient indices or implementing greater volume fractions of CNTs inside the inclusions. The main reason of reducing effect in the stiffness softening which can be found in the nanocomposite by adding the gradient index or increasing the volume fraction of the CNTs inside the inclusions. Actually, the higher is the volume fraction of the nanofillers inside the inclusions, the more is the destroying influence of the agglomeration on the stiffness of the nanocomposite compared with the situation which nanofillers are distributed in a uniform manner.
2. The dimensionless frequency can be amplified once higher volume fractions are employed for clusters. Indeed, once this term is strengthened, the nanocomposite moves toward making a unique inclusion which is able to cover all of the nanofillers and due to this fact, the uniform distribution can be better satisfied compared with the case of having a group of discrete inclusions which are dispersed in the media.
3. The natural frequency can be added once mass fraction of CNTs is increased in the \( \eta < \eta_{\text{critical}} \) domain. In this domain, an increase in the mass fraction of the nanofillers can result in a stiffness enhancement and due to this fact the frequency of the beam can be intensified. After this value, the disadvantages of the existence of the nanofillers in the matrix will be more than their advantages and this

![Graphs showing coupled effects of agglomeration coefficients (\( \mu, \eta \)) and CNTs mass fraction on the dimensionless frequency of multi-scale hybrid nanocomposite beams with S-S EC (\( L/h = 25, P = 2, V_F = 0.2 \)).](image)
fact results in observing completely inverse effects as the involved variants are changed.

4. The presented shear deformable theorem is efficient enough to estimate the mechanical vibration responses of beam type nano-composite elements.

Conflicts of interest

The authors declare no conflict of interest in preparing this article.

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References


