Static and free vibration analysis of Timoshenko beam based on combined peridynamic-classical theory besides FEM formulation

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ABSTRACT

In this paper, combination of classical and peridynamic theories is implemented to study static and free vibrational behavior of a Timoshenko beam. Employing Hamilton’s principle, the governing integro-differential equations are developed. Due to problematic analytical solution, FEM formulations are constructed and necessary computer codes are developed. Static and vibrational problems are numerically studied to reveal influential parameters. Mesh sensitivity analysis and comparison with relevant results reported in the open literature are done for verification of the methodology and ensuring reliable results. Effects of peridynamic parameters, boundary conditions, and the beam size are studied. Regarding each influencing factor, relevant illustration is presented and the results are discussed from the physical point of view.

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1. Introduction

In recent years, many researches have focused on studying behavior of non-conventional structures such as micro/nano in size, made of tissue materials and so on. This attraction is due to their unique physical, chemical, electrical and mechanical properties. For instance, dependency of the material properties to the structure size have been proved via experimental, molecular dynamics and molecular mechanics methods [1–8]. For instance, nanoindentation showed that the plate stiffness is more than what predicted by the classical beam theory [5]. As another example, small dislocations are important in plastic deformation and classical theories that neglect gradient effects lead to insufficient accuracy [8]. On the other hand, conventional theories such as Hook’s law in solid mechanics, Fourier law in heat conduction do not capture such dependency so that they bring inaccuracy in modeling the novel structures. Hence, development of adequate theories with capability to predict the behavior of non-conventional structures attracted a great deal of interests in recent researches. Different approaches have been introduced to account size effects for theoretical modeling in different fields of mechanics. In this regard, strain gradient is a non-classical theory that brings non-classical elastic properties to create relation between the stresses, strains and strain gradients [9–11]. A simplified version of that theory has also been presented in which there are three non-classical constants [12]. Couple stress is a more simplified version of the strain gradient theory that the number of non-classical constants is reduced to one [13]. Eringen proposed a non-local theory in which the stress at a point depends on the integration of the strain on whole domain of the body [14]. However, differential form of the Eringen theory has also been developed that is easier to be implemented [14]. According to both gradient and Eringen based theories, extra material constants are introduced in the constitutive equations that are generally called non-classical elastic constants. These non-classical material constants are recognized as non-local or material length scale in the literature. As a matter of fact, employing gradient based theories causes exhibition of non-classical boundary conditions that are hard to be interpreted practically [15]. As a matter of fact, no absolute consensus can be observed in the literature about applicability of these theories [16]. Therefore, more investigations on such theories are required to clarify the reliability of such theories especially by making comparison with experimental data.

On the other hand, instead of introducing non-classical constitutive equations, Silling proposed peridynamic theory in which the constitutive equations remained as classical versions. However, an extra effect is introduced that represents long-range (non-local) interactions between each pair of material points inside the body [17]. Therefore, the non-local effect takes role in the system of governing equations via the equilibrium equations; and constitutive equations and boundary conditions remain similar to the classical
ones. Although this methodology was originally developed to model discontinuities, its capability to capture the behavior of small-size structures has also been proved [18–24] and a few of the relevant references are explained in more detail hereafter.

Weckner and Abeyaratne used long-range interaction model and developed elastodynamic governing equations of a 1-D bar [20]. They compared dynamic behavior of the classical elastic bar with peridynamic bar. Parks and co-workers proved possibility of implementation of the peridynamic theory via molecular dynamics approach [25]. Silling and Lehoucq proved that if peridynamic characteristics approach to zero, the results of such non-local model converges to the classical results [26]. Di Paola and co-workers solved the governing equation of peridynamic 1-D bar by discretization of the domain [24]. They concluded that although for unbounded 1-D domain peridynamic model converges to the Eringen non-local theory, they are different in bounded domain. Chen and Gunzburger investigated into using FEM for numerical solution of peridynamic 1-D governing equation and studied effects of peridynamic parameters on the solution convergence [27]. Di Paola and co-workers studied static deflection of the beam structure based on peridynamic theory [28]. They insisted requirement of experimental data to set different peridynamic parameters for any specific structure. Zingales studied dynamic behavior of 1-D bar to observe influencing factors of the peridynamic model [29]. According to that reference, peridynamic model has the advantage that does not perturb static boundary conditions. Huang studied the vibration characteristics of a nanorod and showed that long-range interactions increase stiffness of the structure [30]. Alotta and co-workers developed FEM formulation for static analysis of the beam structure with peridynamic behavior [31]. They studied effects of different peridynamic factors on the beam deflection. O’Grady and Foster developed peridynamic governing equation of Euler–Bernoulli beam using the concept of conventional rotational spring at each point of the structure to model its bending stiffness [32]. They examined applicability of that approach to study plastic deformation of that structure. Diyaroglu and co-workers studied the beam behavior using peridynamic theory and extended it to plate structure [33]. In addition to the elasticity field, peridynamic model has been implemented in thermomechanics [34], fluid flow [35] and heat transfer [36–40]. Moreover, solution of the peridynamic governing equations especially efficiency of the numerical techniques have become an attractive research field [41–44]. This literature review shows importance of the issue and indicates that more investigations are required to enhance this research field.

In this work, elastodynamic governing equations of Timoshenko beam, considering internal long-range body (peridynamic) forces are developed. The integro-differential governing equations are solved based on a finite element formulation of the problem. Due to existence of integral terms in the governing equations, development of FEM equations is a novelty of the current work. Moreover, it is proved that convergence analysis requires careful attention. In the part of numerical studies, influence of peridynamic parameters, boundary conditions and the beam size on static deflection and natural frequencies are discussed in detail.

2. Basic formulation and peridynamic effect

In the classical continuum theory, each volume element is under direct forces exerted by adjacent elements, external body forces and surface loads. On the other hand, based on the peridynamic theory, internal body forces exerted by nonadjacent volume elements take role in addition to classical interactions. To illustrate this point, a target element with volume $dV$ and position vector of $\mathbf{x} = [xyz]^T$ in a Cartesian coordinate system $o(x,y,z)$, is shown in Fig. 1, which is under classical external force $\mathbf{F}$ and surface traction $T^{(x)}(\mathbf{x})dA_x$, and peridynamic long-range body force $\mathbf{f}$. Bold symbols are used to show vectors and matrices however, index method is also used to represent tensors.

Components of the surface traction (stress vector) with unit normal vector $\mathbf{n}$, are related to the components of stress tensor $\sigma_{ij}$ via the well-known Cauchy equation $T_{rns} = \sigma_{ij}n_i$. As depicted in Fig. 2, the introduced peridynamic body force $\mathbf{f}$ is resultant of vectors $\mathbf{q}(\mathbf{x},\mathbf{x}')$ that are exerted by any typical volume element located at $\mathbf{x}' = [x'y'z']^T$ on the target element located at $\mathbf{x} = [xyz]^T$.

Based on the peridynamic model, the internal body force between each pair of volume elements located at $\mathbf{x}$ and $\mathbf{x}'$, depends on their relative displacement. In other words, this force is related to the variation of the distance between the interacting elements as reflected in the following equation. Actually, this force, $\mathbf{q}$, is per unit the product of volume of the element at $\mathbf{x}'$ and the volume of the element at $\mathbf{x}$, so that its dimension becomes in (force)/(volume)$^2$.

$$\mathbf{q}(\mathbf{x},\mathbf{x}') = g(\mathbf{x},\mathbf{x}')\{\eta(\mathbf{x},\mathbf{x}') \cdot \Gamma(\mathbf{x},\mathbf{x}')\} \Gamma(\mathbf{x},\mathbf{x}')$$

(1)

Here, $\Gamma(\mathbf{x},\mathbf{x}')$ is a unit vector positively oriented from point located at $\mathbf{x}$ toward the point located at $\mathbf{x}'$ obtained as follows

$$\Gamma(\mathbf{x},\mathbf{x}') = \frac{\mathbf{x}' - \mathbf{x}}{\|\mathbf{x}' - \mathbf{x}\|}$$

(2)

$\eta(\mathbf{x},\mathbf{x}')$ indicates the vector of relative displacement between the points, defined by

$$\eta(\mathbf{x},\mathbf{x}') = \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) = (u_x(\mathbf{x}') - u_x(\mathbf{x}))\hat{e}_x + (u_y(\mathbf{x}') - u_y(\mathbf{x}))\hat{e}_y$$

$$+ (u_z(\mathbf{x}') - u_z(\mathbf{x}))\hat{e}_z$$

(3)

(\mathbf{u}$ being the displacement vector), and “.” is used to show inner product of two vectors. The function $g(\mathbf{x},\mathbf{x}')$ in Eq. (1) is the equivalent stiffness of a virtual spring between $\mathbf{x}$ and $\mathbf{x}'$. This function is called the peridynamic kernel and plays a main role in this theory. In comparison with the classical stiffness matrix, which is defined by numeral components, the peridynamic stiffness is a function depending on the distance between the interacting points. Similar to any other physical characteristics, this property should be measured experimentally. Nevertheless, to the best of the authors'
knowledge, no such data has been reported in the literature. As a matter of fact, selection of the kernel function is a challenging issue in this field; however, decaying with distance is a logical form for this function, which is adopted here.

To account for the total forces exerted by all elements \( dV(x') \) inside the body, on the target element \( dV(x) \), a summation is applied using the following integral on the whole domain \( V \) of the medium

\[
\mathbf{f}(x) = \int_V \mathbf{q}(x,x')dV(x')
\]

Therefore, \( \mathbf{f}(x) \) represents vector of total peridynamic force per unit volume of the element that is located at \( x \) and hence its dimension becomes \( \text{(force)/(volume)} \). On the other hand, regarding the classical portion of the system behavior, the well-known Hook’s constitutive equation is employed that relates stress and strain tensors. Hereafter, governing equations of a deformable solid can be developed by incorporating equilibrium, constitutive and kinematic equations. This procedure is followed in the rest of the paper to develop the beam governing equations. It is worth mentioning that total elastic energy consisting both classical and peridynamic effects, indicated with superscripts \( cl \) and \( pd \), respectively, is as follows:

\[
U = U(\mathbf{e}, \mathbf{\eta}) = U^{cl}(\mathbf{e}) + U^{pd}(\mathbf{\eta});
\]

\[
U^{cl}(\mathbf{e}) = \frac{1}{2} \int_0^L \sigma_{pq}(x)\epsilon_{pq}(x)dV(x);
\]

\[
U^{pd}(\mathbf{\eta}) = \int_0^L \int_0^L \mathbf{q}(x,x') \cdot \mathbf{\eta}(x,x')dV(x')dV(x)
\]

In the above, the work of peridynamic force (inner product of force and displacement vectors) is considered equivalent to its elastic energy.

### 3. Development of the governing equations

An initially straight beam with length \( L \), width \( b \), thickness \( h \) and cross section area \( A \) is considered in Cartesian coordinate system, as depicted in Fig. 3. \( F_z \) and \( F_x \) are distributed external forces along \( z \) and \( x \) directions, respectively. In addition, \( N_0 \), \( T_0 \), and \( M_0 \) are concentrated axial force, shear force and bending moment at the beam ends \( (i = 0, L) \) respectively.

Based on the Timoshenko beam theory, at any time \( t \), displacement components of an arbitrary point of the beam is as follows \([45]\):

\[
\mathbf{u}_x = \mathbf{u}(x,t) - z\mathbf{\varphi}(x,t), \quad \mathbf{u}_y = \mathbf{0}, \quad \mathbf{u}_z = \mathbf{w}(x,t)
\]

In which, \( \mathbf{u}_x, \mathbf{u}_y \), and \( \mathbf{u}_z \) are components of the displacement vector of any typical point, along \( x, y \) and \( z \) directions, respectively. Moreover, \( \mathbf{u}(x,t) \) and \( \mathbf{w}(x,t) \) represent the axial and lateral displacement (deflection) of the central axis, respectively, and \( \mathbf{\varphi}(x,t) \) is rotation of the cross section about \( y \) axis. On the other hand, introducing Eq. (3) into (6) gives the components of relative displacement as follows (it should be noted that for a straight beam, position vector of any point on the central axis has only \( x \) (and \( x' \)) component, assisting to simplification of the equations)

\[
\eta_x(x,x',t) = \mathbf{u}_x(x,x',t) - \mathbf{u}_x(x,t)
\]

\[
\eta_y(x,x',t) = \mathbf{u}_y(x,x',t) - \mathbf{u}_y(x,t)
\]

\[
\eta_z(x,x',t) = \mathbf{u}_z(x,x',t) - \mathbf{u}_z(x,t)
\]

\[
\eta_x(x,x',t) = \mathbf{u}_x(x,x',t) - \mathbf{u}_x(x,t) = \mathbf{u}(x',t) - \mathbf{u}(x,t) - \{z\mathbf{\varphi}(x,t) - z\mathbf{\varphi}(x,x',t)\}
\]

\[
\eta_y(x,x',t) = \mathbf{u}_y(x,x',t) - \mathbf{u}_y(x,t) = \mathbf{0}
\]

\[
\eta_z(x,x',t) = \mathbf{u}_z(x,x',t) - \mathbf{u}_z(x,t) = \mathbf{0}
\]

\[
\eta_x(x,x',t) = \mathbf{u}_x(x,x',t) - \mathbf{u}_x(x,t) = \mathbf{u}(x',t) - \mathbf{u}(x,t) - \{z\mathbf{\varphi}(x,t) - z\mathbf{\varphi}(x,x',t)\}
\]

\[
\eta_y(x,x',t) = \mathbf{u}_y(x,x',t) - \mathbf{u}_y(x,t) = \mathbf{0}
\]

\[
\eta_z(x,x',t) = \mathbf{u}_z(x,x',t) - \mathbf{u}_z(x,t) = \mathbf{0}
\]

\[
\eta_x(x,x',t) = \mathbf{u}_x(x,x',t) - \mathbf{u}_x(x,t) = \mathbf{u}(x',t) - \mathbf{u}(x,t) - \{z\mathbf{\varphi}(x,t) - z\mathbf{\varphi}(x,x',t)\}
\]

\[
\eta_y(x,x',t) = \mathbf{u}_y(x,x',t) - \mathbf{u}_y(x,t) = \mathbf{0}
\]

\[
\eta_z(x,x',t) = \mathbf{u}_z(x,x',t) - \mathbf{u}_z(x,t) = \mathbf{0}
\]

\[
\eta_x(x,x',t) = \mathbf{u}_x(x,x',t) - \mathbf{u}_x(x,t) = \mathbf{u}(x',t) - \mathbf{u}(x,t) - \{z\mathbf{\varphi}(x,t) - z\mathbf{\varphi}(x,x',t)\}
\]

\[
\eta_y(x,x',t) = \mathbf{u}_y(x,x',t) - \mathbf{u}_y(x,t) = \mathbf{0}
\]

\[
\eta_z(x,x',t) = \mathbf{u}_z(x,x',t) - \mathbf{u}_z(x,t) = \mathbf{0}
\]
respectively. For a specific cross section with two axis of symmetry, these parameters become

\[
R_1(x, t) = \int_0^L \{ l_1(x, x') [\bar{u}(x', t) - \bar{u}(x, t)] \} \, dx' \tag{13a}
\]

\[
R_2(x, t) = \int_0^L \{- I_2(x, x') \varphi(x', t) + I_3(x, x') \varphi(x, t) + I_4(x, x') \{ w(x', t) - w(x, t) \} \} \, dx' \tag{13b}
\]

\[
R_3(x, t) = \int_0^L \{ l_5(x, x') \varphi(x', t) - l_6(x, x') \varphi(x, t) - I_3(x, x') \{ w(x', t) - w(x, t) \} \} \, dx' \tag{13c}
\]

in which, set of variables \( l_i (x, x'), i = 1, 2, \ldots, 6 \), are defined as follows.

\[
I_1 = \int_A \int_A g(x, x') \bar{\Gamma}_2(x, x') \, dA(x) \, dA(x) \tag{14a}
\]

\[
I_2 = \int_A \int_A g(x, x') \bar{\Gamma}_1(x, x') \, dA(x) \, dA(x) \tag{14b}
\]

\[
I_3 = \int_A \int_A g(x, x') \bar{\Gamma}_1(x, x') \, dA(x) \, dA(x) \tag{14c}
\]

\[
I_4 = \int_A \int_A g(x, x') \bar{\Gamma}_2(x, x') \, dA(x) \, dA(x) \tag{14d}
\]

\[
I_5 = \int_A \int_A g(x, x') \bar{\Gamma}_1(x, x') \, dA(x) \, dA(x) \tag{14e}
\]

\[
I_6 = \int_A \int_A g(x, x') \bar{\Gamma}_1(x, x') \, dA(x) \, dA(x) \tag{14f}
\]

Finally, in Eq. (12), the virtual displacements \( \delta u, \delta w \) and \( \delta \varphi \) are arbitrary, and therefore, their multipliers in each integral must be zero independently, leading to the following governing equations. In the following set of equations, the first one is pertaining to axial motion, and the other two explain shear and bending behavior, respectively.

\[
EA \frac{\partial^2 \bar{u}(x, t)}{\partial t^2} + R_1(x, t) + F_3(x) = \rho A \frac{\partial^2 \bar{u}(x, t)}{\partial t^2} \tag{15a}
\]

\[
K_G A \left( \frac{\partial^2 w(x, t)}{\partial x^2} - \frac{\partial \varphi(x, t)}{\partial x} \right) + R_2(x, t) + F_2(x) = \rho A \frac{\partial^2 w(x, t)}{\partial t^2} \tag{15b}
\]

\[
EI \frac{\partial^2 \varphi(x, t)}{\partial x^2} + K_G A \left( \frac{\partial w(x, t)}{\partial x} - \varphi(x, t) \right) + R_3(x, t) = \rho I \frac{\partial^2 \varphi(x, t)}{\partial t^2} \tag{15c}
\]

Next, the non-integral terms in Eq. (12) lead to the following boundary conditions.

\[
\bar{u}(0, t) = \bar{u}_0 \quad \text{or} \quad EA \frac{\partial \bar{u}(x, t)}{\partial x} \bigg|_{x=0} = -N_0 \tag{16a}
\]

\[
\bar{u}(L, t) = \bar{u}_L \quad \text{or} \quad EA \frac{\partial \bar{u}(x, t)}{\partial x} \bigg|_{x=L} = N_L \tag{16b}
\]

\[
w(0, t) = w_0 \quad \text{or} \quad K_G A \left( \frac{\partial w(x, t)}{\partial x} - \varphi(x, t) \right) \bigg|_{x=0} = -T_0 \tag{16c}
\]

\[
w(L, t) = w_L \quad \text{or} \quad K_G A \left( \frac{\partial w(x, t)}{\partial x} - \varphi(x, t) \right) \bigg|_{x=L} = T_L \tag{16d}
\]

\[
\varphi(0, t) = \varphi_0 \quad \text{or} \quad E I \frac{\partial \varphi(x, t)}{\partial x} \bigg|_{x=0} = -M_0 \tag{16e}
\]

\[
\varphi(L, t) = \varphi_L \quad \text{or} \quad E I \frac{\partial \varphi(x, t)}{\partial x} \bigg|_{x=L} = M_L \tag{16f}
\]

It is observed from Eq. (15) that governing equations, considering the effect of peridynamic body forces are in integro-differential form; and by elimination of the kernel function, \( g(x, x') \), the well-known classical differential equation is obtained.

4. Development of FEM formulations

The developed integro-differential governing equations cannot be solved analytically, even under static conditions, and hence finite element method (FEM) is implemented here. The first step for FEM formulation is defining the element geometry and interpolation functions. Fig. 4 indicates the target element with length of \( L_e \) between two nodes \( i \) and \( j \) and natural coordinate system \( (\vec{x}, \vec{z}) \). On the other hand, to model the interacting peridynamic force, another element is presented at location \( x' \) with natural coordinate system \( (\vec{x}', \vec{z}') \). The following equations relate the global and natural coordinate systems.

\[
\vec{x} = x - x_i, \quad \vec{x}' = x' - x_j; \quad x_i \leq x \leq x_j, \quad x'_i \leq x' \leq x'_j; \quad 0 \leq \vec{x}, \quad \vec{x'} \leq L_e \tag{17a}
\]

\[
\vec{z} = 2z - h, \quad \vec{z'} = 2z' - h; \quad 0 \leq z, \quad z' \leq h, \quad -h \leq \vec{z}, \quad -h \leq \vec{z'} \leq h \tag{17b}
\]

The following functions are considered for deflection and rotation of the \( n \)th element. It should be noted that axial displacement of the central axis (\( u \)) is neglected and no axial load is applied on the beam. Hence, lateral deflection and rotation are remained as independent degree of freedoms (DOF) at each point of the beam. Moreover, there is no displacement along y direction.

\[
\begin{cases}
\bar{w}_n(x, t) = N^{w(n)}_n(x)\bar{d}_n(t) \\
\varphi_n(x, t) = N^{\varphi(n)}_n(x)\bar{d}_n(t)
\end{cases} \tag{18}
\]

In the above, \( \bar{d}_n(t) = [w_1(t)\varphi_1(t)w_2(t)\varphi_2(t)]^T \) is vector of nodal values of displacement (and rotation) for the \( n \)th element, in which indices \( i \) and \( j \) pertain two element nodes depicted in Fig. 4. As there are two unknown variables at each node, four unknowns are exhibited for each element; hence, interpolation functions for displacement and rotation are defined as follows, respectively.

\[
N^{w(n)}_n = \begin{bmatrix} N^{w(n)}_1 & N^{w(n)}_2 & N^{w(n)}_3 & N^{w(n)}_4 \end{bmatrix} \tag{19a}
\]

\[
N^{\varphi(n)}_n = \begin{bmatrix} N^{\varphi(n)}_1 & N^{\varphi(n)}_2 & N^{\varphi(n)}_3 & N^{\varphi(n)}_4 \end{bmatrix} \tag{19b}
\]

Here, cubic polynomial shape functions are considered for deflection and quadratic shape functions are used for rotation as follows. In fact, such selections satisfy both of the beam bending governing Eqs. (15b) and (15c). However, shape functions with lower degree can be used leading to less accuracy [46].

\[
N^{w(n)}_n(\vec{x}) = \frac{1}{4} \left[ 1 - 3(\vec{x}/L_e)^2 + 2(\vec{x}/L_e)^3 + (1 - (\vec{x}/L_e))\psi \right] / (1 + \psi) \tag{20a}
\]
In the above equations, the subscript \( n \) was dropped and variable \( \psi = 12EI/(K_GC^2) \) was employed to condense the expressions.

According to Eqs. (1) and (2), peridynamic force on an element depends on relative displacement between that element and the other elements in the system. Hence, in spite of the classical theory, displacement field of all nodes is incorporated into the peridynamic force on each element. To implement such procedure, nodal vectors of all elements are assembled to form total nodal vector as follows.

\[
d(t) = \left[ d_1^T(t) d_2^T(t) ... d_n^T(t) \right]^T
\]

(22)

In the above, the superscript \( T \) stands for transpose and \( N \) is the total number of elements. Therefore, nodal vector of each element can be extracted from the total vector using the following equation.

\[
d_n = C_n d \quad n = 1, 2, \ldots, N
\]

(23)

In the above, \( C_n \) is a Boolean matrix that connects the matrices of the \( n \)th element to the overall matrices of the entire system [47]. Here, FEM formulations are developed employing the approach of minimization of total potential energy. In this regard, kinetic energy, elastic energy and work of external loads should be introduced in terms of the nodal vector. Kinetic energy of the beam is introduced as summation on all elements as follows.

\[
KE = \sum_{n=1}^{N} KE_n
\]

(24a)

\[
KE_n = \frac{1}{2} d_n^T M_n d_n
\]

(24b)

In the above, \( KE_n \) and \( M_n \) are kinetic energy and mass matrix of \( n \)th element; and dot is used to show differentiation with respect to time. Using the definition of the Boolean matrix, kinetic energy of each element becomes as follows.

\[
KE_n = \frac{1}{2} \int dx A_n \begin{bmatrix} \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} \\ \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_n^{(w)}}{dx} \\ \frac{dN_n^{(g)}}{dx} \end{bmatrix} + l_n \begin{bmatrix} \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \\ \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_n^{(g)}}{dx} \\ \frac{dN_n^{(w)}}{dx} \end{bmatrix} \right) \right] dx
\]

(25)

Therefore, total kinetic energy is related to the total mass matrix \( M \) and total nodal vector via the following equation.

\[
KE = \frac{1}{2} d^T M d
\]

(26)

\[
M = \sum_{n=1}^{N} M_n = \sum_{n=1}^{N} \int dx \begin{bmatrix} A_n \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} + I_n \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} + l_n \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_n^{(w)}}{dx} \\ \frac{dN_n^{(g)}}{dx} \end{bmatrix} \right] dx
\]

(27)

Regarding the elastic energy, as mentioned previously, it is divided into classical and peridynamic portions as follows.

\[
U = \sum_{n=1}^{N} U_n = \sum_{n=1}^{N} \int dx \left[ E \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} \right] + \sum_{n=1}^{N} \int dx \left[ K G \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \right]
\]

(28)

Skipping the intermediate steps, the classical and peridynamic parts of the energy are stated as

\[
U_n^{(c)} = \int dx \left[ E \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} \right] + \sum_{n=1}^{N} \int dx \left[ K G \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \right]
\]

(29)

In the above, \( K_n^{(c)} \) and \( K_n^{(pd)} \) are classical and peridynamic parts of the stiffness matrix for the \( n \)th element, respectively. Using the following well-known definition \( U = \frac{1}{2} d^T K d \), the total stiffness matrix \( K \) is obtained as follows.

\[
K = \sum_{n=1}^{N} (K_n^{(c)} + K_n^{(pd)}) = \sum_{n=1}^{N} \left[ E I \frac{dN_n^{(w)}}{dx} \frac{dN_n^{(w)}}{dx} \right] + \sum_{n=1}^{N} \left[ K G \frac{dN_n^{(g)}}{dx} \frac{dN_n^{(g)}}{dx} \right]
\]

(30)

The work of external forces is determined by inner product of each external force and associated displacement leading to the following result.

\[
W^{n,c} = \sum_{n=1}^{N} W_n^{n,c} = d^T \left[ \sum_{n=1}^{N} \left[ C_n^{(w)} \frac{dN_n^{(w)}}{dx} \right] \right] T_0 + C_n^{(g)} \frac{dN_n^{(g)}}{dx} \right] T_1 + C_n^{(g)} \frac{dN_n^{(g)}}{dx} \right] M_0
\]

(32)

Therefore, the vector of external loads and natural boundary conditions can be defined as follows.
\[ F^I = \sum_{n=1}^{N} \left\{ \int_{0}^{L} \left[ C_n^I \left(N_n^i(0)\right)^\top T_0 - C_n^I \left(N_n^i(0)\right)^\top M_0 \right. \\
\left. + C_n^I \left(N_n^i(L)\right)^\top T_1 + C_n^I \left(N_n^i(L)\right)^\top M_1 \right] dx \right\} \]

Finally, applying minimization of the total potential energy gives FEM formulation as follows.

\[ \ddot{M} + Kd = F \]

5. Numerical results and discussion

In a numerical analysis, verification is required to be sure about the accuracy of the results. In our case, two factors, namely FE technique and numerical integration might bring errors into the results. Therefore, some static problems are examined first to observe the sensitivity to the two mentioned factors. Afterwards, other examples will be presented related to vibrational analysis.

5.1. Static analysis

In this part, a simply supported beam with length of \( L \), width \( b \) and height of \( h \) under uniform lateral load, \( q \), is studied. Here, shear factor \( K_s = 5/6 \), \( v = 0.38 \) and Young’s modulus \( E \) are attributed to the beam. It should be noted that the deflection is normalized by dividing over the value \( 5qL^4/384EI \), which is the classical deflection at mid span of the corresponding Bernoulli-Euler beam. Therefore, the values of Young’s modulus and applied load are not needed. In this subsection, the values \( L = 10 \) nm, and \( b = h = 1 \) nm are used to provide possibility of comparison with the results reported in [28]. Here, the following peridynamic kernel function (i.e. non-local elastic property) is used as in [28].

\[ g(x, x') = \frac{(1 - \beta_1)E}{2A^2b_0} \exp \left( -\frac{||x - x'||}{b_0} \right) \]

As mentioned previously, \( 0 \leq \beta_1 \leq 1 \) is a weight factor to control the shares of classical and peridynamic parts. The kernel function is plotted in Fig. 5 for some values of the internal length scale, \( b_0 \), which acts as a decaying factor. It is seen that by increasing this factor, the function’s decay is slower, leading to increase of the range of non-locality, while the function peak is decreases. Because of the exponential form, the kernel becomes negligible at a distance from \( x \). Therefore, as depicted in the figure, in the numerical procedure, this function is cut after a distance that is called cutoff radius, \( R_{cut} \). In other words, the kernel function is taken to be zero beyond this distance. Here, numerical studies were conducted to reveal the effect of the cutoff radius on the results. In this regard, no considerable change was observed, when \( R_{cut} \geq 7b_0 \), however, \( R_{cut} = 10b_0 \) was used to ensure insensitivity to this parameter.

As mentioned previously, the accuracy of the numerical results depend on the elements size and the numerical integration pertaining to the peridynamic part. Gauss integration technique is implemented to evaluate the integrals in Eq. (31) in the FEM formulation. The accuracy of such evaluation depends on the number of Gaussian points along the length, \( x \) and \( x' \). \( \beta_1 \) and \( \beta_2 \) represent number of Gaussian points along the length \( x \) and \( x' \) respectively. For accuracy analysis, different numbers were assigned to the Gaussian points besides using various meshing indicated by \( N \) (number of elements). Interestingly, in all cases, no considerable variation was observed in the results when the number of Gaussian points along the length goes upper than four \( (\beta_1 = 4) \), whereas, the results were sensitive to a wide range of the number of Gaussian points along the height, \( \beta_2 \). For example, in the specific case of \( \beta_1 = 0.5 \) and \( b_0 = 0.08 \) nm, dependency of the beam normalized deflection on the mesh number is shown in Fig. 6, when \( G_a = 4, G_b = 12 \). It is seen that by increasing the mesh number the graphs converge to a specific form that is expected to be the accurate result.

To save the space, presentation of the beam deflection curve for different conditions is neglected and only the convergence graphs related to the midspan of the beam for three cases of: \( \beta_1 = 0.9, \beta_1 = 0.5 \) and \( \beta_1 = 0.1 \) are plotted in Fig. 7 while \( b_0 = 0.08 \) nm. In fact, the final convergence is achieved when no considerable variation is happened by increasing number of both elements and Gaussian points. It is observed that for each case, by increasing the Gaussian points and meshing number the results converge to a specific value. However, it is observed that when the weight factor is increased, more Gaussian points and finer meshing is required to achieve accurate results so that when \( \beta_1 = 0.9 \), convergence is achieved easily while, large deviations exist between the graphs for case \( \beta_1 = 0.5 \) and deviations are wider in case of
According to the figure, in case of $b_1 = 0.9$, convergence is achieved when $G_a = 4$, $G_b = 10$ and $N = 30$ while for case $b_1 = 0.5$ values of $G_a = 4$, $G_b = 16$, $N = 100$ are required for case values of $b_1 = 0.1$. $G_a = 4$, $G_b = 18$, $N = 140$ are required.

In brief, comparing the graphs in Fig. 7(a), (b) and (c) pertaining to cases $b_1 = 0.9$, 0.5 and 0.1, respectively, reveals that by decreasing the weight factor, $b_1$, more efforts is required to achieve the convergence. In physical viewpoint, decreasing the weight factor, $b_1$, means decreasing the classical share and increasing the peridynamic effect. Therefore, it is concluded that by increasing the peridynamic effect, achieving the convergence becomes more difficult. In mathematical viewpoint, by increasing the peridynamic effect, the share of integrals in the governing integro-differential equation is intensified and the numerical solution should be done more accurately. In other words, a small error in evaluating the integral terms can bring significant errors in the results, while sensitivity to the differential terms related to the classical effect is not so substantial. Moreover, similar convergence analyses were conducted in case of other values of the internal length scale, $l_0$, however, they are not presented here to avoid lengthening the article. Actually, it was observed that by decreasing this factor, the convergence issue became more difficult. To explain such behavior, we should see the impact of this factor on the peridynamic effect and consequently the integral terms. Referring to Fig. 5, by decreasing the internal length scale, the kernel function becomes sharper while is attenuated in smaller distance. Actually, when the function becomes sharper, it becomes more concentrated and therefore, finer meshing are required to calculate the integrals accurately.

To see the effect of the weight factor and internal length scale simultaneously, the normalized deflection at center of the simply supported beam is presented in Fig. 8. Moreover, the results reported in [28] are presented for comparison. Comparing the results shows that for large values of the weight factor, the results are in good agreement, however, for low weight factors (high non-local effect), the difference is considerable. This could be attributed to the fact that finite difference method was used in [28], and no accuracy analysis was performed in that reference. It should be noted that, very few works do an accuracy analysis to ensure the reliability of their results. But in the current study, adequate accuracy analyses were done to give this insurance to the reader.

According to Fig. 8, for any specific value of the weight factor, by increasing the internal length scale, the deflection is decreased. This suggests that this factor might have stiffening effect on the mechanical behavior. In physical viewpoint this can be due to the fact that increasing the non-local factor means farther material points help to hold point $x$, leading to stiffening, like an object that is held in place by many strings. On the other hand, it is observed that by increasing the weight factor, the normalized deflection decreases and approaches asymptotically to unity. To explain such behavior, it should be noted that, although increasing the weight factor enhances the non-local effect (leading to stiffening), it causes degrading the local effect (leading to softening). Therefore, it can be concluded that resultant of these two opposite effects lead to overall softening. Nevertheless, it should be emphasized that such conclusion pertains to the current example and the values of the parameters can affect this conclusion.

5.2. Free vibration analysis

In free vibration analysis, the external load is set to zero leading to identification of the natural frequencies and mode shapes.
this case, the principle of synchronous motion \( \mathbf{d} = \mathbf{d}^{\text{exc}} \) is applied; in which \( \omega \) and \( \mathbf{d} \) are natural frequency and its relevant mode shape, respectively. Substitution of this form into the governing Eq. (34), leads to the following eigenvalue problem.

\[
(K - M\omega^2)\mathbf{d} = 0
\]  

(36)

Solution of the above equation, considering the prescribed essential boundary conditions, gives natural frequencies and mode shapes. In this part, \( h = b = 1 \, \text{nm} \), shear factor \( K_s = 5/6 \) and \( \nu = 0.38 \) for the beam are considered. To study the effects of the beam aspect ratio and non-local parameter, different values were attributed to \( L, l_0 \) and \( \beta_1 \). The beam is studied under two boundary conditions consisting simply supported and clamped-clamped that are labeled by symbols S-S and C-C, respectively. To provide possibility of making comparison with classical case, the results were divided by the values \( 9.87216\sqrt{EI/qL^4} \) and \( 22.3729\sqrt{EI/qL^4} \) to obtain dimensionless first natural frequency for S-S and C-C conditions, respectively. These values are classical first natural frequency of Bernoulli-Euler beam under S-S and C-C boundary conditions, respectively. In order to get the classical results, the value \( \beta_1 = 1 \) is used. As mentioned previously, accuracy of the numerical results depends on the meshing and integration procedure. Hence, adequate convergence analyses were conducted to ensure acceptable accuracy.

Variations of the first natural frequency versus the weight factor \( \beta_1 \) are plotted in Fig. 9. It is seen that for both C-C and S-S conditions, by increasing the beam length, the deviations between the graphs corresponding to different values of the internal length scale are increased. This means that by increasing the beam length, the role of both the weight factor and the internal length scale become more significant. To illustrate such behavior, it should be noted that when the beam length is increased, a larger region is affected by the long-range interactions. Comparing the graphs belonged to S-S and C-C boundary conditions shows that dependencies to both the weight factor and the internal length scale in S-S condition are more significant than C-C condition. Therefore, it can be concluded that when the beam is more constrained (C-C condition), its behavior is less affected by the non-local elastic characteristics.

In order to elaborate more on this issue, dependency on the parameters is studied in a dimensionless way, which provides a better physical sense. To see effects of the ratio \( l_0/h \), beam aspect ratio, \( L/h \), and weight factor, \( \beta_1 \), several graphs are presented in Fig. 10. It is seen that for all values of \( \beta_1 \), and for both boundary...
conditions, by increasing the ratio $l_0/h$, the dimensionless frequency is increased. Physically, this means that while $h$, $L$, and $b_1$ remain constant, by increasing the internal length scale, $l_0$, the beam becomes stiffer. To explain such dependency we should refer back to Fig. 5 in which for any specific value of the weight factor, by increasing the internal length scale, the peridynamic force goes to zero in a longer distance leading to stiffening the beam consequently. Furthermore, the graphs in Fig. 10 show that by increasing the weight factor, dependency to the ratio, $l_0/h$, is weakened. This is a logical trend, because according to aforementioned arguments, by increasing the weight factor, Young’s modulus increases toward its classical value and the peridynamic share goes to zero concurrently. Regarding the beam length, it is seen that for all values of $b_1$, by increasing the beam length, variations of the graphs become steeper. In other words, by increasing the beam length, dependency to the peridynamic effect is intensified. This indicates that by increasing the structure length, the impact of the long range interactions (due to the peridynamic effect) is increased.

In terms of the boundary conditions, it is seen that the graphs for the S-S condition are steeper than those belonged to C-C condition. This can be due to the fact that C-C condition puts more constraint onto the beam than S-S condition. Physically, this means that in C-C condition a larger portion of the beam is constrained by means of the supports and the region that is affected by the peridynamic effect is shortened.

6. Conclusion

Elastodynamic governing equation of the Timoshenko beam was developed based on combined classical and peridynamic effects. As a part of this research, a standard formulation was constructed to implement FEM approach suitable for computer programming. To verify the methodology and numerical results, mesh sensitivity analysis was done and comparison was made with reported data in the relevant references. According to the numerical results, values of the peridynamic parameters affect the numerical convergence significantly. In this regard, it was seen that by increasing the peridynamic effect, finer mesh was required to achieve the convergence. In other words, in the numerical solution, integral terms required more efforts than differential terms.

In addition to the static analysis, free vibrational behavior of the beam was studied. It was proved that by increasing the internal length scale, the natural frequency is increased. This means that the internal length scale brings stiffening effect. On the other hand, it was proved that the weight factor might bring either stiffening or softening effects. This can be due to the fact that although this parameter weakens the peridynamic effect, it enhances the classical stiffness. Therefore, its overall effect depends on other characteristics of the beam. Numerical results indicated that by increasing the beam length, the impact of the peridynamic effect was increased. Regarding the boundary conditions, it was seen that when the beam is constrained by means of firmer supports, its overall rigidity is increased and dependence to the peridynamic parameters is decreased.

References
