AN INTRODUCTION TO ZERO-DIVISOR GRAPHS OF A COMMUTATIVE MULTIPLICATIVE HYPERRING

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ABSTRACT

The purpose of this paper is the study of zero-divisor graphs of a commutative multiplicative hyperring, as a generalization of commutative rings. In this regards we consider a commutative multiplicative hyperring $(R,+,o)$, where $(R, +)$ is an abelian group, $(R, +)$ is a semihypergroup and for all $a,b,c \in R$, $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(a + b) \circ c \subseteq a \circ b + a \circ c$. For $a \in R$ a nonzero element $a \in R$ is said to be a zero-divisor of $a$, if $0 \in a \circ b$ and the set of zero-divisors of $R$ is denoted by $Z(R)$. We associative to $R$ a zero-divisor graph $\Gamma(R)$, whose vertices of $\Gamma(R)$ are the elements of $Z(R)^* (= Z(R) \setminus \{0\})$ and two distinct vertices of $\Gamma(R)$ are adjacent if they were in $Z(R)$. Finally, we obtain some properties of $\Gamma(R)$ and compare some of its properties to the zero-divisor graph of a classical commutative ring and show that almost all properties of zero-divisor graphs of a commutative ring can be extend to $\Gamma(R)$ while $R$ is a strongly distributive multiplicative hyperring.

Keywords: Multiplicative hyperring, zero-divisor graph, strongly distributive.

1. INTRODUCTION

The concept of the zero-divisor graph of a ring was raised by I. Beck when discussing the coloring of a commutative ring in [3] for the first time. Later D. F. Anderson and P. S. Livingston introduced the zero-divisor graph of a unitary commutative ring $R$, denoted by $\Gamma(R)$ in [2]. They considered the set of nonzero zero-divisor of as a vertex of $\Gamma(R)$ and assumed that two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Subsequently, they proved that if $R$ is a finite ring, then $\Gamma(R)$ is finite and connected and any two vertices can be joined by less than four edges. In particular, they were determined when $\Gamma(R)$ is a complete graph and a star graph.

In this paper we create a connection between the concept of the zero-divisor graph of commutative rings and commutative multiplicative hyperrings and generalize some results and properties of zero-divisor graph of a commutative ring to the strongly distributive multiplicative hyperrings.

In this section we will list some definitions, notions and results about commutative hyperrings from some references.
Definition 1.1. Let $H$ be a nonempty set and $P^*(H)$ denotes the set of all of nonempty subsets of $H$. A hyperoperation $o$ on $H$ is a mapping $o : H \times H \rightarrow P^*(H)$. A nonempty set $H$ together with a family of hyperoperation is a hyperstructure. A hyperstructure $(H, o)$ is a semihypergroup if for all $a, b, c \in H, (a \circ b \circ c) = a \circ b \circ c$ and $(a \circ b) \circ c = a \circ (b \circ c)$. (Associativity axiom). A hyperstructure $(H, o)$ is a quasihypergroup if for all $a \in H$, we have $a \circ o H = H = H \circ a$. In the other words for all $a, b, c \in H$ there exist $x, y \in H$ such that $a \in x \circ o b \cap b \circ o y$ (Reproduction axiom).

Definition 1.2. A hyperstructure $(H, o)$ which is the both semihypergroup and quasihypergroup is called a hypergroup.

Definition 1.3. A general hyperring is an algebraic hyperstructure $(R, +, o)$ that satisfies the following axioms:

1. $(R, +)$ is a hypergroup.
2. $(R, o)$ is a semihypergroup.
3. For all $a, b, c \in R, a \circ (b + c) = a \circ b + a \circ c$ and $(a + b) \circ c = a \circ c + b \circ c$.

A hyperring $(R, +, o)$ is commutative, if the both hyperoperations $+$ and $o$ are commutative. The hyperring $R$ is unitary if there exists an element $u \in R$ such that for all $a \in R, a \circ o u = u \circ o a = \{a\}$.

Definition 1.4. The unitary commutative hyperring $R$ is a hyperfield if for every non-zero element $a \in R$, there exists $b \in R$ such that $u \in a \circ o b$, where $u$ is an unit element of $R$.

Definition 1.5. A commutative hyperring $R$ is a strong hyperdomain if for all $a, b \in R$, if $0 \in a \circ o b$ with $a \neq 0$ (or $b \neq 0$), then $b = 0$ (or $a = 0$). If $a \circ o b = \{0\}$ implies $a = 0$ or $b = 0$, we will talk about hyperdomain. Obviously, every strong hyperdomain is a hyperdomain and every hyperfield is a strong hyperdomain.

Definition 1.6. A nonempty subset $A$ of a hyperring $(R, +, o)$ is subhyperring of $R$ if $(A, +, o)$ is itself a hyperring, under the restriction of hyperoperation $+$ and $o$ to $A$.

Definition 1.7. Let $A$ is a subhyperring of a hyperring $R$. We say that $A$ is a left (right) hyperideal of $R$ if for all $r \in R$ and $a \in A, r \circ o a \in A(a \circ o r \in A)$. $A$ is called a hyperideal if $A$ is both a left and a right hyperideal. A hyperideal $P$ of a commutative hyperring $R$ is said to be prime if $P \neq R$ and for all $a, b \in R, a \circ o b \subseteq P$ implies $a \in P$ or $b \in P$. A hyperideal $P$ of $R$ is said to be strong prime if $a \circ o b \cap P \neq 0$ implies $a \in P$ or $b \in P$.

Definition 1.8. A triple $(R, +, o)$ is multiplicative if $+$ be a classical commutative operation and $o$ be a hyperoperation and following statements hold:

1. $(R, +)$ is an abelian group.
2. $(R, o)$ is a semihypergroup.
3. For all $a, b, c \in R, a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(a + b) \circ c \subseteq a \circ c + b \circ c$.
4. For all $a, b \in R, a \circ o (b - c) = (-a) \circ o b = -(a \circ o b)$.

If in (3) equality hold, then $R$ is a strongly distributive multiplicative hyperring (briefly, we say that $R$ is a $SDMH$).

Definition 1.9. A nonempty subset $S$ of a commutative multiplicative hyperring $(R, +, o)$ is a subhyperring of $R$ if $(S, +, o)$ is a multiplicative hyperring. In other words, $S$ is a subhyperring of $R$ if $(S, +)$ is a subgroup of $(R, +)$ (i.e., $S - S \subseteq S$) and for all $r, s \in S, r \circ o s \subseteq S$.

Definition 1.10. A nonempty subset $I$ of a multiplicative hyperring $(R, +, o)$ is a hyperideal if following axioms hold:

1. $(I, +)$ is a subgroup of $(R, +)$.
2. $(I \circ o R) \cup (R \circ o I) \subseteq I$.

By this definition clearly, every hyperideal is a subhyperring.
Let \((R, +, o)\) be a multiplicative hyperring and \(I\) is a hyperideal of \(R\). Let \(R/I\) be the set of all cosets of \(R\) with restrict to \(I\), \(R/I = \{a + I \mid a \in R\}\). We define a hyperoperation \(*\) on \(R/I\) by
\[
(a + I) * (b + I) = \{c + I \mid c \in a \circ b\}.
\]

Then \((R/I, +, *)\) is a multiplicative hyperring, moreover if \(R\) is a SDMH, so is \(R/I\).

**Theorem 1.11.** A strongly distributive hyperring \((R, +, o)\) is a ring if and only if there exists \(a, b \in R\), such that \(|a \circ o b| = 1\).

**Proof.** Corollary 4.1.6 [5]. □

**Theorem 1.12.** If \(I\) is a hyperideal of a commutative multiplicative hyperring \((R, +, o)\), then for every element \(a + I \in R/I\), we have \(|(a + I) * (0 + I)| = 1\). In other words, if \(R\) is a SDMH, then \(R/I\) is a ring.

**Proof.** According to Theorem 4.3.5 [5] and Theorem 1.11. □

**Theorem 1.13.** Let \((R, +, o)\) is a SDMH, then for all \(a, b \in R\), we have:

1. \(0 \in a \circ o 0\) and \(0 \in 0 \circ a\).
2. For all \(x, y \in a \circ o 0, x - y \in a \circ o 0\). (i.e., \(a \circ o 0\) is a subgroup of \(R\).)
3. \(a \circ o b\) is a cosets of \(0 \circ o 0\).
4. \(0 \circ o 0 \circ o 0 = 0 \circ o 0\).
5. For all \(s \in 0 \circ o 0\) and \(r \in R, s \circ o r = 0 \circ o 0\).
6. If \(0 \in a \circ o b\) then \(a \circ o b = 0 \circ o 0\).

**Proof.** (1) \(0 \circ o a = (a - a) \circ o a = a \circ o a - a \circ o a\). Since \(0 \in a \circ o a - a \circ o a\), then \(0 \in 0 \circ o a\) and similarly \(0 \in a \circ o 0\).

(2) \(a \circ o 0 = a \circ o(0 - 0) = a \circ o 0 - a \circ o 0\). Then for all \(x, y \in a \circ o 0, x - y \in a \circ o 0\).

(3) Let \(c \in a \circ o b\). For all \(x \in a \circ o b\), we have \(x - c \in a \circ o b - a \circ o b = a \circ o (b - b) = a \circ o 0\).

This means that \(x + a \circ o 0 = c + a \circ o 0\). Thus \(a \circ o b = a \circ o (b + 0) = a \circ o b + a \circ o 0 = \cup_{x \in a \circ o b} x + a \circ o 0 = c + a \circ o 0\). Similarly, \(a \circ o b\) is a coset of \(a \circ o 0\). Since \(a \circ o 0\) and \(0 \circ o b\) are cosets of \(0 \circ o 0\), therefore \(a \circ o b\) is a coset of \(0 \circ o 0\).

(4) \(0 \circ o 0 \circ o 0 = 0 \circ o (0 \circ o 0) = \cup_{x \in a \circ o 0} 0 \circ o a = \cup_{a \in \delta o 0} 0 \circ o 0 = 0 \circ o 0\). Similarly, \(a \circ o 0\) is a coset of \(0 \circ o 0\).

(5) Suppose \(s \in 0 \circ o 0\) and \(r \in R\), then \(s \circ o r \subseteq 0 \circ o 0 \circ o r = 0 \circ o (0 \circ o r) = 0 \circ o 0\). Since \(s \circ o r\) is a coset of \(0 \circ o 0\) then \(s \circ o r = 0 \circ o 0\).

(6) Suppose \(0 \in a \circ o b\), then for \(c \in a \circ o b\), we have \(0 \in c + 0 \circ o 0\). Thus there exists \(m \in 0 \circ o 0\) such that \(0 = c + m\). It follow that \(c \in 0 \circ o 0\). Thus \(a \circ o b \subseteq 0 \circ o 0\), and Since \(a \circ o b\) is a coset of \(0 \circ o 0\), therefore \(a \circ o b = 0 \circ o 0\). □

**Corollary 1.14.** We denote \(0 \circ o 0\) by \(\Omega\), then by Theorem 1.12 clearly if \(R\) is a SDMH, \(\Omega\) is a hyperideal of \(R\). Moreover, \(R/\Omega\) is a ring.

2. THE ZERO-DIVISOR GRAPH OF A SDMH WHEN \(Z(R)^* \cap \Omega = \emptyset\)

In this section, we investigate zero-divisor graph of a strongly distributive multiplicative hyperring and compare their properties with zero-divisor graph of a classical commutative ring.

Let \((R, +, o)\) be a commutative multiplicative hyperring. An element \(0 \neq b\) of \(R\) is said to be a zero-divisor of \(a \in R\), if \(0 \in a \circ o b\). The set of zero-divisors of \(R\) denote by \(Z(R)\). The zero-divisor graph of \(R\) is a graph with elements of \(Z(R)^* = Z(R) \setminus \{0\}\) as vertices and two distinct vertices \(a, b\) are adjacent if and only if \(0 \in a \circ o b\). This graph denote by \(\Gamma(R)\). By definition 1.5, \(R\) is a strongly distributive if and only if \(Z(R) = \{0\}\), and if \(R\) is a strong hyperdomain then \(\Gamma(R) = \emptyset\). An element \(0 \neq a\) of \(R\) is regular if \(a \notin Z(R)\). The set of regular elements of \(R\) denote by \(Reg(R)\).
The zero-divisor graph $\Gamma(R)$ is connected if there exists a path between any two distinct vertices. $\Gamma(R)$ is a complete graph if any two distinct vertices of $\Gamma(R)$ are adjacent. $\Gamma(R)$ is a star graph if there exists an unique vertex of $\Gamma(R)$, which is adjacent to every other vertex.

Let $d(a, b)$ be the length of the shortest path from $a$ to $b$ in $\Gamma(R)$. The diameter of $\Gamma(R)$ is denoted by $diam(\Gamma(R))$, is equal to $\sup\{d(a, b) \mid a, b$ are distinct vertices of $\Gamma(R)\}$. The girth of $\Gamma(R)$ is denoted by $gr((R))$, is defined as the length of the shortest cycle in $\Gamma(R)$. $d(a, b) = \infty$ if there is no such path and $gr(\Gamma(R)) = \infty$ if $\Gamma(R)$ contains no cycles.

In the following statements we will generalize some Theorems and results about zero-divisor graph of a commutative ring that were obtained by D. F. Anderson and P. S. Livingston in [2].

**Theorem 2.1.** Let $(R, +, o)$ be a $SDMH$. Then $\Gamma(R)$ is finite if and only if either $R$ is finite or a strong hyperdomain. In particular, if $1 \leq |\Gamma(R)| < \infty$, then $R$ is finite and not a hyperfield.

**Proof.** Suppose that $\Gamma(R) = (Z(R))^*$ is finite and nonempty. Then there are nonzero $a, b \in R$ such that $0 \neq a \circ o b$. Let $A = \{r \in R \mid 0 \circ o r\}$. Then $A \subseteq Z(R)$ is finite and for all $r \in R$, $0 \circ o r \subseteq (a \circ o b) \circ o r = a \circ o (b \circ o r)$ since $0 \neq 0 \circ o r$, therefore $b \circ o r \subseteq A$. Let $R$ be infinite. Since $A$ is finite, then there are $a_1, a_2, \ldots, a_n \in A$ such that $B = \{r \in R \mid b \circ o r \subseteq \{a_1, a_2, \ldots, a_n\}\}$ is infinite. So for all $r, s \in B$, $0 \neq b \circ o (r - s)$. If $\mathcal{C} = \{r \in R \mid 0 \neq b \circ o r\}$, then $\mathcal{C} \subseteq Z(R)$ is infinite, that is a contradiction. Thus $R$ must be finite. Converse is obviously. □

In this part for determining the zero-divisor graph, we suppose that $(R, +, o)$ is a $SDMH$, $\Omega = 0 \circ o 0$ and $Z(R)^* \cap \Omega = \emptyset$. According to Corollary 1.14, $R/\Omega$ is a ring. We denoted $R/\Omega$ by $\bar{R}$ and the element $a + \Omega$ of $R/\Omega$ by $\bar{a}$. Here, we state a useful theorem that helps us to determine zero-divisor graph and their properties for a $SDMH$.

**Theorem 2.2.** If $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then there exists an one-to-one correspondence between the set of zero-divisors of $R$ and the set of zero-divisors of ring $\bar{R}$.

**Proof.** If $a \in Z(R)$, there exists $0 \neq b \in R$ such that $0 \neq a \circ o b$. According to Theorem 1.13(6), $a \circ o b = \Omega$. Since $Z(R)^* \cap \Omega = \emptyset$, then $\bar{a}, \bar{b} \in \bar{R}$ are nonzero and $\bar{a}\bar{b} = a \circ o b + \Omega = \Omega$. Therefore $\bar{a} \in Z(\bar{R})$. Conversely, suppose $\bar{a} \in Z(\bar{R})$. There exists $\bar{0} \neq \bar{b} \in \bar{R}$ such that $\bar{a}\bar{b} = (a + \Omega) o (b + \Omega) = \Omega$. It means that $a \circ o b + \Omega = \Omega$ and hence $a \circ o b = \Omega$. Since $0 \in \Omega$, hence $0 \in a \circ o b$. Then we have $a \in Z(R)$. This complete the proof. □

This results immediately follow from Theorem 2.2:

**Corollary 2.3.** If $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. There exists an one-to-one correspondence between the set of $Reg(R)$ and the set of $Reg(\bar{R})$.

**Corollary 2.4.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then $\Gamma(R)$ is isomorphic to $\Gamma(\bar{R})$. In other words, $\bar{a}$ and $\bar{b}$ are adjacent in $\Gamma(\bar{R})$ if and only if $a$ and $b$ are adjacent in $\Gamma(R)$. Hence $\Gamma(\bar{R})$ is connected if and only if $\Gamma(R)$ is so.

**Corollary 2.5.** As another proof of Theorem 2.1, if $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$, $\Gamma(R)$ is finite if and only if $\Gamma(\bar{R})$ is so. According to Theorem 2.2 [2], $\Gamma(\bar{R})$ is finite if and only if $\bar{R}$ is finite or a domain. Also $\bar{R}$ is finite if and only if $R$ is finite. Moreover, since $Z(R)^* \cap \Omega = \emptyset$, $\bar{R}$ is a domain if and only if $R$ is a strong hyperdomain.

**Theorem 2.6.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contain a cycle, then $gr(\Gamma(R)) \leq 7$.

**Proof.** According to Theorem 2.3 [2], $\Gamma(\bar{R})$ is connected and $diam(\Gamma(\bar{R})) \leq 3$, furthermore, if $\Gamma(\bar{R})$ contain a cycle, then $gr(\Gamma(\bar{R})) \leq 2diam(\Gamma(\bar{R})) + 1$. Therefore, according to Theorem 2.2, $\Gamma(\bar{R})$ is so. □

**Theorem 2.7.** Let $R$ is a finite $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. If $\Gamma(R)$ contains a cycle then $gr(\Gamma(R)) \leq 4$. 
**Proof.** Since \( R \) is finite if and only if \( \bar{R} = R/\Omega \) is finite and \( \Gamma'(R) \cong \Gamma(\bar{R}) \), if \( \Gamma'(R) \) contains a cycle then \( \Gamma'(\bar{R}) \) is so. By Theorem 2.4 [2], \( gr(\Gamma(R)) \leq 4 \). □

**Definition 2.8.** A hyperideal \( I \) is an annihilator hyperideal if and only if for all \( a \in I \) and for all \( r \in R, 0 \neq r a a r 0 \in a o r \).

**Theorem 2.9.** Let \( R \) is a SDMH and \( Z(R)^* \cap \Omega = \emptyset \). There exists a vertex of \( \Gamma(R) \) which is adjacent to every other vertex if and only if either \( R/\Omega \cong Z_2 \times A \), where \( A \) is an integral domain, or \( Z(R) \) is an annihilator hyperideal.

**Proof.** If \( \Gamma(R) \) contains a vertex which is adjacent with other vertices, then \( \Gamma'(R) \) is so. By Theorem 2.5 [4], we have \( \bar{R} = R/\Omega \cong Z_2 \times A \), where \( A \) is an integral domain, or \( Z(R/\Omega) \) is an annihilator ideal. If \( Z(R/\Omega) \) is an annihilator ideal then for all \( \bar{a} \in Z(R/\Omega) \) and for all \( \bar{r} \in R/\Omega, \bar{a}\bar{r} = a o r + \Omega = \Omega \). Since \( Z(R)^* \cap \Omega = \emptyset \), then \( a o r = \Omega \). Since \( 0 \in \Omega \), then \( 0 \in a o r \). Therefore \( Z(R) \) is an annihilator hyperideal. □

**Theorem 2.10.** Let \( R \) is a SDMH and \( Z(R)^* \cap \Omega = \emptyset \). Then \( \Gamma(R) \) is a complete graph if and only if \( \bar{R} \cong Z_2 \times Z_2 \) or \( x o y = \Omega \) for all \( x, y \in Z(R)^* \).

**Proof.** Let \( \Gamma(R) \) is a complete graph then \( \Gamma'(R) \) is so. According to theorem 2.6 [2], \( \Gamma'(R) \) is complete graph if and only if \( \bar{R} \cong Z_2 \times Z_2 \) or \( \bar{x}\bar{y} = \Omega \), for all \( \bar{x}, \bar{y} \in Z(R)^* \). If \( \bar{x}\bar{y} = \Omega \), according to theorem 2.2, for all \( x, y \in Z(R)^* \), \( 0 \in x o y \). Then \( x o y = \Omega \). Converse is obviously. □

**Corollary 2.11.** Let \( R \) is a SDMH and \( Z(R)^* \cap \Omega = \emptyset \). For \( x, y \in Z(R) \), define \( x \sim y \) if \( 0 \in x o y \) or \( x = y \). Then relation \( \sim \) is an equivalence relation if and only if \( \Gamma(R) \) is a complete graph.

### 3. THE ZERO-DIVISOR GRAPH OF A SDMH WHEN \( Z(R)^* \cap \Omega \neq \emptyset \)

In this section, we suppose that \( R \) is a SDMH and \( Z(R)^* \cap \Omega \neq \emptyset \). According to Theorem 1.13, for every \( a \in Z(R)^* \cap \Omega \), all of elements of \( R \) are adjacent to \( a \). In this case, \( \Gamma(R) \) is connected. But \( \Gamma(R) \) and \( \Gamma'(R) \) are not isomorphic necessarily.

In the following example we prove that if \( R \) is a SDMH and \( Z(R)^* \cap \Omega \neq \emptyset \), \( \Gamma(R) \) is not isomorphic to \( \Gamma(R) \).

**Example 3.1.** Let \( (R, +, \cdot) \) is a ring and \( \emptyset \neq P \) be a prime ideal of ring. We define \( a o_p b = ab + P \), for \( a, b \in R \). Obviously \( (R, +, o_p) \) is a SDMH and \( \Omega = 0 o_p 0 = P \). According to Corollary 1.14, \( \bar{R} = R/P = \{ r + P \mid r \in R \} \) is a ring. Let \( a, b \in \Gamma(R) \), are adjacent. Then \( 0 \in a o_p b \). Hence \( a o_p b = ab + P = P \) and \( ab \in P \). Since \( P \) is a prime ideal of \( R \), \( a \in P \) or \( b \in P \). Therefore \( a \notin Z(\bar{R})^* \) or \( b \notin Z(\bar{R})^* \).

**Theorem 3.2.** Let \( R \) is a SDMH and \( Z(R)^* \cap \Omega \neq \emptyset \). Then \( \Gamma(R) \) is connected and \( diam(\Gamma'(R)) \leq 2 \). Moreover, if \( \Gamma(R) \) contains a cycle, then \( gr(\Gamma(R)) \leq 5 \).

**Proof.** If \( Z(R)^* \cap \Omega \neq \emptyset \), then by theorem 1.13, for all \( a \in Z(R)^* \cap \Omega \), and for all \( b \in R \), \( a o b = \Omega \). Since \( 0 \in a o b \), Then \( a \) is adjacent to all of elements of \( R \), and \( \Gamma'(R) \) is connected and \( d(a,b) = 1 \). Now, we suppose that \( a, b \in Z(R)^* \cap \Omega \). If \( 0 \in a o b \), obviously \( \Gamma'(R) \) is connected and \( d(a,b) = 1 \). Otherwise, there exist \( x \in Z(R)^* \cap \Omega \) such that \( 0 \in a o x \) and \( 0 \in x o b \). Then \( a - x - b \) is a path of length 2 and consequently \( \Gamma(R) \) is connected and \( diam(\Gamma'(\bar{R})) \leq 2 \). □

**Theorem 3.3.** Let \( R \) is a SDMH and \( Z(R)^* \cap \Omega \neq \emptyset \). If \( \Gamma(R) \) contains a cycle, then \( gr(\Gamma(R)) \leq 3 \).

**Proof.** If \( \Gamma'(R) \) contains a cycle, then there exist \( a, b \in Z(R)^* \cap \Omega \) such that \( 0 \in a o b \). On the other hand, for all \( x \in Z(R)^* \cap \Omega \), we have \( 0 \in a o x \) and \( 0 \in x o b \). Then \( a - x - b - a \) is a triangle. □
By Theorem 3.3, if $Z(R)^* \cap \Omega \neq \emptyset$, we have seen that $\Gamma(R)$ can be a triangle. But $\Gamma(R)$ cannot be an $n$-gon for any $n \geq 4$.

**Theorem 3.4.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. Then there is always at least one vertex of $\Gamma(R)$ which is adjacent to every other vertex.

**Proof.** According to Theorem 1.13(5). □

**Theorem 3.5.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. Then $\Gamma(R)$ is complete if and only if for all $x, y \in Z(R)^* \setminus \Omega$, $x \circ y = \Omega$.

**Proof.** The proof is obviously. □

**Corollary 3.6.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. For $x, y \in Z(R)$, define $x \sim y$ if $0 \in x \circ y$ or $x = y$. Then relation $\sim$ is an equivalence relation if and only if $\Gamma(R)$ is a complete graph.

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