A fast algorithm for regularized focused 3D inversion of gravity data using randomized singular-value decomposition

Saeed Vatankhah1, Rosemary Anne Renaut2, and Vahid Ebrahimzadeh Ardestani1

ABSTRACT

We develop a fast algorithm for solving the under-determined 3D linear gravity inverse problem based on randomized singular-value decomposition (RSVD). The algorithm combines an iteratively reweighted approach for L1-norm regularization with the RSVD methodology in which the large-scale linear system at each iteration is replaced with a much smaller linear system. Although the optimal choice for the low-rank approximation of the system matrix with \( m \) rows is \( q = m \), acceptable results are achievable with \( q \ll m \). In contrast to the use of the iterative LSQR algorithm for the solution of linear systems at each iteration, the singular values generated using RSVD yield a good approximation of the dominant singular values of the large-scale system matrix. Thus, the regularization parameter found for the small system at each iteration is dependent on the dominant singular values of the large-scale system matrix and appropriately regularizes the dominant singular space of the large-scale problem. The results achieved are comparable with those obtained using the LSQR algorithm for solving each linear system, but they are obtained at a reduced computational cost. The method has been tested on synthetic models along with real gravity data from the Morro do Engenho complex in central Brazil.

INTRODUCTION

It is well known that the gravity inverse problem is ill posed and that effective regularization methods should be used to obtain reasonable solutions (Li and Oldenburg, 1998; Portniaguine and Zhdanov, 1999; Boulander and Chouteau, 2001; Silva and Barbosa, 2006). In geophysical inverse modeling, it is often assumed that the sources of interest are localized and separated by distinct interfaces. Thus, the inversion methodology should be able to provide sharp and focused images of the subsurface. Many different approaches have been used, including the compactness constraint (Last and Kubik, 1983), minimum gradient support (Portniaguine and Zhdanov, 1999; Zhdanov, 2002), total variation regularization (Bertet-Aguirre et al., 2002), applying the Cauchy norm on the model parameters (Pilkington, 2009), and using the L1-norm stabilizer (Loke et al., 2003; Farquharson, 2008; Vatankhah et al., 2017). In all these cases, the process is iterative and the model-space iteratively reweighted least-squares (IRLS) algorithm may be used. Here, we suppose a focused image of the subsurface is preferred and adopt the L1 inversion methodology presented in Vatankhah et al. (2017) for determining the solution of the under-determined inversion problem with \( m \) data measurements for recovery of a volume with \( n \) cells, \( m \ll n \).

For a forward-modeling operator \( G \in \mathbb{R}^{m \times n} \) with \( m \) and \( n \) relatively small, a physically acceptable numerical solution is obtained using singular-value decomposition (SVD), or generalized singular-value decomposition (GSVD), as appropriate. For large-scale inverse problems, it is no longer feasible, whether with respect to memory requirements or computational time, to rely on a direct solver (Oldenburg and Li, 1994; Li and Oldenburg, 2003). Rather, nowadays the LSQR algorithm based on Golub-Kahan bidiagonalization (GKB) is frequently used (Paige and Saunders, 1982a, 1982b; Kilmer and O’Leary, 2001; Chung et al., 2008; Voronin et al., 2015; Renaut et al., 2017; Vatankhah et al., 2017). For the LSQR algorithm using \( t \ll m \) steps of the GKB process, a Krylov subspace with dimension \( t \) is generated, and the solution is obtained on this subspace at negligible computational cost using the SVD of the subspace system matrix. On the other hand, randomized algorithms can be used to efficiently and directly approximate the SVD of \( G \) (Halko et al., 2011; Xiang and Zou, 2013; Voronin et al., 2015) yielding a rank \( q \) approximation of \( G \) in which \( q \ll m \). Here, we use the randomized singular-value decomposition (RSVD) algorithm.
tions obtained using the RSVD and the FSVD is very small. Furthermore, we demonstrated that the UPRE parameter choice rule can be used for the projected space without requiring truncation of small singular values. This is a significant difference, besides the CPU time, with the inversion methodology based on the LSQR algorithm. We showed the efficiency of the presented inversion methodology using different synthetic tests and real case data from the ME complex in the GAP in the center of Brazil.

ACKNOWLEDGMENTS

The authors would like to acknowledge the financial support of the University of Tehran for this research under grant number 30674/1/01. R.A. Renaut acknowledges the support of NSF grant DMS 1418377: “Novel Regularization for Joint Inversion of Nonlinear Problems.” The constructive comments of the three reviewers, Zhdanov, Shalivahan, and Hidalgo-Gato, guided improvements in the clarity of the manuscript.

APPENDIX A

OBTAINING SINGULAR VALUES AND VECTORS FROM EIGENVALUES AND VECTORS

The reduced SVD of matrix $B \in \mathbb{R}^{m \times n}$, step 4 of Algorithm 1, is given by $B = U \Sigma V^T$ (Trefethen and Bau, 1997). Here, $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are the left and the right singular vectors of $B$ with columns denoted by $u_i$ and $v_j$, respectively. The diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ contains singular values of the $B$ ordered as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0$. Furthermore, $\Sigma$ and $U$ are the singular values and left singular vectors of the low-rank approximation of $\hat{G}$, i.e., $\hat{G}_l$ (see Xiang and Zou, 2013; Voronin et al., 2015). The right singular vectors of the $\hat{G}_l$ are obtained via $v_j = Q v_j \in \mathbb{R}^{n \times l}$ (Xiang and Zou, 2013). Now for $B^*B = \hat{V} \Sigma^2 \hat{V}^T$ with $U^T U = I$ (Voronin et al., 2015), it is immediate that $B^*B = D \Sigma^2 D^T$, where $D = \hat{V} \Sigma$. This indicates that the eigen decomposition of $B^*B$ gives the singular values, $\sigma_i = \sqrt{\lambda_i}$, and the first $l$ right singular vectors of the $B$, $v_j$. Finally, to compute the left singular vectors, we note that $BV = UV \Sigma V^T V = U \Sigma V$, then $U = BV \Sigma V^{-1}$. In this way, we can avoid computing the SVD of matrix $B$ directly, instead we use the eigendecomposition of the smaller matrix $B^*B$. We also note that in the generation of the rank $q$ approximation, it is only the dominant $q$ terms of the spectral decomposition that are required, and hence all terms with $l$ columns can be replaced by those with just $q$ columns.

REFERENCES

Sifredi, M. S., and E. Tolstaya, 2004, Minimum support nonlinear parameterization using different synthetic tests and real case data from the ME complex in the GAP in the center of Brazil.