Experimental verification of two stress-based criteria for mixed mode I/III brittle fracture assessment of U-notched components

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ABSTRACT

In the present study, a series of fracture experiments are conducted on PMMA U-notched samples using an improved loading configuration, which is redesign of the apparatus recently presented for doing fracture tests under combination of tensile and out-of-plane shear loading. In order to assess the effect of notch tip radius, the tested specimens are fabricated with three different radii. Two brittle fracture criteria, namely the point stress (PS) and mean stress (MS), are employed to estimate the fracture resistance and the out-of-plane angle of fracture onset of the experiments. These criteria have previously been extended to general loading type of mixed mode I/II/III for studying fracture in engineering components containing blunt V-notches. Consequence of comparing the theoretical and experimental results for different notch tip radii reveals that both criteria are accurate enough to predict the fracture behavior of the U-notched engineering members subjected to mixed mode I/III loading. It is not found any considerable difference between the curves of fracture initiation angle and fracture resistance resulted from the PS and MS criteria. The criteria also show that the out-of-plane fracture angle due to pure mode III loading is constant and independent of the notch tip radius.

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1. Introduction

Similar to crack problems, various deformation modes of opening (mode I), in-plane sliding (mode II) and out-of-plane tearing (mode III) or any combination of the three basic modes, so-called mixed-mode loading conditions, may be imposed to notched components in engineering structures. Fracture is a catastrophic failure by which the notched member may suddenly break and result in severe damages. Theoretical failure criteria have a vital role in safety evaluation of engineering components weakened by various notch types such as O-, U- and V-shaped notches under different loading conditions.

As a basic failure problem in notched domains, the brittle fracture behavior of sharp and round-tip notches subjected to pure mode I loading has been employed in the past by numerous researchers. Generally speaking, few main failure models exist in the literature for mode I brittle fracture assessment of notched components, namely the strain energy density (SED) [1–5], the cohesive zone model (CZM) [6,7], the point-stress (PS) and mean-stress (MS) [8–14], the finite fracture mechanics (FFM) [15,16] and the generalized J-integral [17–20] criteria. Relatively fewer failure models have been presented for...
notched domains subjected to mixed mode I/II loading. The most active brittle fracture criteria in the context of notched members under mixed mode I/II and pure mode II loadings are based on the strain energy density (SED) [21–32], the point-stress (PS) and mean-stress (MS) criteria [10,33–44]. However, some limited papers have also been published on mixed mode I/II brittle fracture investigation of notches using the FFM [45] and the CZM [46] criteria.

In comparison with the in-plane loading conditions, there are very few papers dealing with brittle fracture for spatial (out-of-plane) loading cases including mode III loading. In this field, a fracture criterion based on the SED averaged over a well-defined material dependent control volume which embraces the notched edge has been applied to the torsion loading case of U- and V-notches [47–49]. This local SED criterion has also been implemented recently for V-notches subjected to tension–torsion loading conditions [50] as it has been utilized for dealing with the mixed mode I/III fracture of cracked specimens [51]. In another study, Zheng et al. [52] used the fracture data of ceramic notched specimens under combined tension/torsion to assess the applicability of a failure criterion based on the critical normal stress. In addition, a fracture model has been suggested for notched elements made of brittle materials under pure mode III loading [53], which is based on a combined normal stress/Griffith energy fracture criterion.

Lately, both the PS and MS criteria were extended to the loading case of general mixed mode I/II/III to provide two theoretical criteria for investigating brittle fracture of round-tip V-notches [54]. The estimations of the extended failure models in the loading case of mixed-mode I/III were assessed by comparing with the experimental results of V-notched graphite round bars.

**Nomenclature**

- $E$: elastic modulus
- $d_c$: critical distance from the notch tip
- $d_{c,V}$ or $d_{c,U}$: critical distance from the origin of the notch polar coordinate system
- $K_{I,U}^{I}$: mode I stress intensity factor of U-notch
- $K_{Ic}$: mode I notch fracture toughness
- $K_{II,U}^{II}$: mode III stress intensity factor of U-notch
- $K_{III}$: plane-strain mode I fracture toughness
- $K_{IIIc}$: mode III fracture toughness
- $K_{eff,U}^{II}$: normalized effective notch stress intensity factor
- $Me_U$: mode mixity parameter
- $r_0$: distance of the coordinate system origin from notch tip on the notch bisector line
- $r_c$: critical radial distance from the notch tip
- $r_{c,V}$ or $r_{c,U}$: critical radial distance from the origin of the notch polar coordinate system
- $(r, \theta, z)$: cylindrical coordinates
- $\alpha$: half of notch opening angle
- $\beta$: loading angle
- $\lambda_i$ (i = 1, 2, 3): eigen values
- $\theta_f$: in-plane angle of fracture initiation
- $\phi_f$: out-of-plane angle of fracture initiation
- $\rho$: notch tip radius
- $\sigma_c$: critical value of the tangential stress
- $\sigma_{r,1}(=r, \theta, z)$: stress components
- $\sigma_r$: ultimate tensile strength
- $\tau_{max}$: maximum elastic shear stress at the notch tip
- $\nu$: Poisson’s ratio
- CZM: cohesive zone model
- DOF: degree of freedom
- EMC: equivalent material concept
- FE: finite element
- FFM: finite fracture mechanics
- LEFM: linear elastic fracture mechanics
- NSIF: notch stress intensity factor
- PMMA: polymethyl-metacrylate
- PS: point stress
- MS: mean stress
- MTS: maximum tangential stress
- SED: strain energy density
In the present paper, a series of fracture experiments conducted by means of a new mixed mode I/III test apparatus on the U-notched specimens made of PMMA with various notch tip radii are first elaborated. Then, the extended PS and MS criteria are evaluated in predicting the experimental values of fracture resistance and fracture initiation angles of the U-notched test specimens fractured under combined tension/out-of-plane shear. Finally, the fracture behavior of brittle materials in the presence of U-notches are investigated using the curves of fracture resistance and fracture angles resulted from the PS and MS criteria in the whole range of mixed mode I/III loading from pure mode I to pure mode III.

2. Mixed mode I/III fracture experiments

In order to evaluate the validity of the theoretical predictions of the PS and MS criteria under mixed mode I/III loading conditions, a series of fracture experiments are conducted using the test configuration illustrated in Fig. 1. This configuration consists of a two-segment loading fixture and a rectangular test sample weakened by a U-shaped notch on one side. Each segment of the fixture comprises five loading holes. The uniaxial tensile load applied in suitable holes imposes several combinations of modes I and III from pure mode I to pure mode III to the test specimens. The positions of the mixed mode loading holes are determined with the aim of attaining some well-distributed combinations of tensile and out-of-plane loadings (mixed mode I/III ratios) between pure mode I and pure mode III. Depending on the angle between the specimen longitudinal direction and the loading direction (the loading angle, $\beta$), pure mode III ($\beta = 90^\circ$), pure mode I ($\beta = 0^\circ$) and three middle mixed mode I/III loading cases ($\beta = 40^\circ, 65^\circ, 72^\circ$) can be applied to the specimen by the fixture. The dimensions of the test specimen are shown in Fig. 2. The test sample is installed in the fixture via the two holes by bolt and nut.

Being a relatively homogeneous and isotropic material, Polymethyl-methacrylate (PMMA) with the commercial name of Plexiglas is utilized for the considered experiments as a proper choice for brittle fracture studies. The optical transparency of this glassy thermoplastic polymer allows direct observation of fracture surface and comparatively easy measurement of fracture angle. These advantages have encouraged many researchers to use the PMMA material for conducting the fracture experiments with the purpose of brittle fracture investigation (for instance, see [7,47,48,55–57]). Some of material properties of the tested PMMA are presented in Table 1.

To survey the effect of the notch tip radius $\rho$, three radii of 0.5, 1 and 2 mm are chosen for fabricating the test specimens. For each notch tip radius, pure mode I and some mixed mode I/III fracture tests are carried out. Since each fracture test is considered to be repeated at least three times, a total number of 33 test samples are made by a laser-cutting machine from a PMMA sheet of 8 mm thick according to the drawing of Fig. 2. As observed in this figure, the notch depth of all the specimens is fixed (30 mm). Fig. 3 displays some U-notched specimens fabricated from PMMA.

After fastening the specimens in the loading fixture, they are loaded up to their final fracture by using a uniaxial tension-compression test machine under displacement-control condition with a constant rate of 1 mm/min. Fig. 4 depicts the test configuration under the mixed mode conditions due to the loading angle of $65^\circ$. Loads and displacements data of the tests are recorded by the test machine computer. All the load-displacement curves obtained from the experiments show that fracture takes place abruptly, which confirms brittleness of the tested PMMA (for instance, see Fig. 5). Table 2 summarizes the experimentally measured fracture loads of all the U-notched PMMA samples.

![Fig. 1. Loading configuration for mixed mode I/III fracture tests.](image)
It might be beneficial to remind that dissimilar to pure tensile loading which does not change the fracture surface of a notched component from the direction of initial notch, applying in-plane shear leads to kinking the fracture surface while out-of-plane shear loading causes twisting it (see Fig. 6). Therefore, the out-of-plane angles of fracture initiation ($\phi_f$) of the U-notched specimens broken under mixed mode I/III loading are also measured in the current study. The angles are determined by using a proper graphical software from the fracture surfaces images taken from a suitable angle of view.
and with an enough magnification. Magnitudes of the fracture angles of $\phi_f$ measured from the tested samples are also listed in Table 2. For instance, the specimens with $\rho = 1$ mm fractured under different loading cases have been depicted in Fig. 7.

3. FE analyses of the test apparatus

In order to compare the theoretical predictions of the PS and MS criteria with the experimental results, it is necessary to determine the notch stress intensity factors (NSIFs) of the tested U-notched specimens. This section elaborates the finite element (FE) modeling carried out for calculating the mode I and mode III NSIFs of the fractured specimens from the normal and out-of-plane shear stresses resulted from the fracture load.

Fig. 8 illustrates a FE model created by using ABAQUS v6.12 commercial software for the whole of loading configuration employed for the experiments. This FE model includes the meshed models of two segments of the fixture, the test specimen and the bolts installing the specimen within the fixture. The connecting bolts and the loading fixture are modeled as rigid
bodies because they have been fabricated from a high strength alloy steel and accordingly, they are much stiffer than the test sample. For meshing the test specimen model displayed in Fig. 9(a), the quadratic brick elements with 20 nodes are utilized. Because of high stress gradient, very fine elements are employed near the notch tip. Minimum element size at the notch tip

<table>
<thead>
<tr>
<th>Notch radius, $\rho$ (mm)</th>
<th>Loading mode, $\beta$ (°)</th>
<th>Fracture load (N)</th>
<th>Magnitude of out-of-plane fracture angle (°)</th>
</tr>
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<tbody>
<tr>
<td>0.5</td>
<td>0 (Mode I)</td>
<td>364.2</td>
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<td>809.4</td>
<td>32.1</td>
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Fig. 6. Fracture onset direction based on the loading type.
in both circumferential and thickness direction is 0.2 mm. Fig. 9(b) has focused on the portion of the mesh at the notch vicinity. Performing a mesh convergence study is the basis of deciding on the appropriate size of elements, which is sufficient for accurate modeling the specimen. Since the surfaces of the fixture and the specimen are smooth, the properties of the contact conditions established between the specimen and the connecting bolts and also between the specimen and the fixture are considered to be hard contact for normal behavior and frictionless for tangential behavior. Fig. 10 schematically shows the load and boundary conditions defined in the described FE model to simulate the real loading conditions of the experiments.

In the next section, two failure criteria recently developed to mixed mode I/III brittle fracture are introduced in brief. These criteria are utilized in Section 5 to predict the fracture resistance and fracture initiation angle of the tested U-notched specimens fabricated from PMMA.
4. Brittle fracture criteria

In the past, two stress-based fracture criteria entitled as PS (Point Stress) and MS (Mean Stress) criteria have been presented to study brittle fracture of the components containing a round-tip V-notch under mixed mode I/II loading [34]. Fig. 11 portrays a typical round-tip V-notch together with its polar and Cartesian coordinate systems. The PS and MS criteria were founded based on the two well-known failure concepts of the maximum tangential stress (MTS) and the mean-stress.
(MS), respectively. Recently, the authors have extended the two failure criteria to general mixed mode I/II/III loading and as a reduced case, they formulated the criteria to mixed mode I/III loading conditions [54]. As it is evident, when the notch opening angle, $2\alpha$, is zero, the round-tip V-notch becomes a U-notch. Therefore, the formulations of the mentioned criteria can be used here for studying brittle fracture of U-notched members by substituting $2\alpha$ with zero.

The main hypotheses of the PS criterion state that fracture onset position is a point on the notch edge at which the tangential stress is a maximum. Also, the crack grows along a direction perpendicular to the maximum tangential stress. In addition, the criterion assumes that brittle fracture initiates when the tangential stress at a critical radial distance from the origin of the polar coordinate system, $r_{c,V}$, reaches a critical value $\sigma_c$. Based on the MS criterion, fracture takes place when the mean value of the tangential stress over a specified critical distance, $d_{c,V}$, attains a critical value $\sigma_c$. The critical value of the tangential stress $\sigma_c$ is a material property and is usually considered equal to the ultimate tensile strength $\sigma_u$ for quasi-brittle and brittle materials. For the sake of brevity, details and formulations of the PS and MS criteria are not repeated here and they can be found in [54].

The PS and MS criteria can provide a set of fracture curves for V-notched members made of a brittle or quasi-brittle material with an arbitrary notch tip radius and notch angle. Therefore, these curves can be used to predict the fracture resistance and fracture initiation angle (out-of-plane fracture angle) in U-notched PMMA components subjected to any combination of tension and out-of-plane shear (mixed mode I/III) loading. The fracture curves are similar to those of the conventional MTS criterion frequently used for analyzing the cracked components [58]. The main parameters required for using the fracture curves are the mode I and mode III NSIFs $K_{II}^{\rho}$ and $K_{III}^{\rho}$, the mode I notch fracture toughness $K_{IC}^{\rho}$, the out-of-plane fracture initiation angle $\phi_I$, and the mode mixity parameter $M_U^{\rho}$. $K_{IC}^{\rho}$ can be determined experimentally and it is not a fixed material property because it depends on the notch tip radius. This parameter can also be predicted theoretically by some appropriate fracture criteria such as the PS and MS ones. The other mentioned parameters are defined and explained in detail in [54] and $M_U^{\rho}$ is defined as follows

$$M_U^{\rho} = \frac{2}{\pi} \tan^{-1} \left( \frac{K_{II}^{\rho}}{K_{III}^{\rho}} \right)$$

$M_U^{\rho}$ varies from zero for pure mode III loading to one for pure mode I loading. By calculating the fracture initiation angle $\phi_I$ in the range between modes I and III using the PS or MS criteria, $\phi_I$ can be plotted in terms of $M_U^{\rho}$ to achieve the fracture initiation angle curve for the case of mixed mode I/III loading. Using the PS and MS criteria, another useful curve, which represents the fracture resistance of the material, can be plotted as the dimensionless normalized NSIF of $K_{II}^{\rho}/K_{IC}^{\rho}$ versus $K_{I}^{\rho}/K_{IC}^{\rho}$.

According to the PS criterion, the fracture angle of $\phi_I$ and the normalized NSIFs of $K_{III}^{\rho}/K_{IC}^{\rho}$ and $K_{I}^{\rho}/K_{IC}^{\rho}$ for each arbitrary mode mixity are obtained by solving the following equations simultaneously [54]:

![Fig. 11. Typical round-tip V-notch.](image-url)
\[
K_{Ic}^{U,\rho} \left[ v \left( L(A + R) + M \left( \frac{r_c u}{r_0} \right)^p (B + V) \right) - L(R + S\chi_{d1}) - M \left( \frac{r_c u}{r_0} \right)^p (\chi_{c1} + V\chi_{d1}) \right] \sin 2\phi_f - \frac{2K_{Ic}^{U,\rho}}{r_c^{U,\rho}} \left[ 1 + \left( \frac{r_c u}{r_3} \right)^p \right] \cos 2\phi_f = 0
\]

(2)

\[
K_{Ic}^{U,\rho} \left[ \frac{L(R + S\chi_{d1}) + M \left( \frac{r_c u}{r_0} \right)^p (\chi_{c1} + V\chi_{d1})}{\cos 2\phi_f} - \frac{K_{II}^{U,\rho}}{K_{Ic}^{U,\rho}} \left[ 1 + \left( \frac{r_c u}{r_3} \right)^p \right] \right] \sin 2\phi_f

+ v \frac{K_{Ic}^{U,\rho}}{K_{II}^{U,\rho}} \left[ \frac{L(A + R) + M \left( \frac{r_c u}{r_0} \right)^p (B + V)}{\sin 2\phi_f} = L(R + S\chi_{d1}) + M \left( \frac{r_c u}{r_0} \right)^p (\chi_{c1} + V\chi_{d1}) \right]
\]

(3)

All the parameters of Eqs. (2) and (3) except from \(K_{Ic}^{U,\rho}, K_{II}^{U,\rho}, K_{Ic}^{U,\rho}, r_0, r_c, v, \) and \(\phi_f\) are the constant coefficients which depend upon the notch opening angle (which this angle is zero for U-shaped notch) and they have been reported in Ref. [54]. \(r_{c,U}\) is the critical radial distance from the origin of the notch polar coordinate system and has the following relation with \(r_c\) which is the critical distance measured from the notch tip:

\[
r_{c,U} = r_0 + r_c
\]

(4)

For brittle and quasi-brittle materials under the loading cases containing mode III, the authors have recently derived a relation for \(r_c\) which is based on the mode III fracture toughness of material \((K_{Ic})\) [54]. This relation is

\[
r_c = \frac{1}{2\pi} \left( \frac{K_{Ic}}{\sigma_u} \right)^2
\]

(5)

where \(\sigma_u\) is ultimate tensile strength of material.

Similarly, to plot the fracture curves of \(K_{II}^{U,\rho}/K_{Ic}^{U,\rho}\) versus \(K_{Ic}^{U,\rho}/K_{Ic}^{U,\rho}\) and \(\phi_f\) versus \(M_{U}^{\rho}\) based on the MS criterion, the following equations should be simultaneously solved [54]:

\[
K_{Ic}^{U,\rho} (vC^* - A^*) \sin 2\phi_f + 2K_{II}^{U,\rho} B^* \cos 2\phi_f = 0
\]

(6)

\[
K_{Ic}^{U,\rho} \frac{K_{II}^{U,\rho}}{K_{Ic}^{U,\rho}} A^* \cos^2 \phi_f + K_{II}^{U,\rho} B^* \sin 2\phi_f + v \frac{K_{Ic}^{U,\rho}}{K_{Ic}^{U,\rho}} C^* \sin^2 \phi_f = A^*
\]

(7)

The relations determining the constant parameters \(A^*, B^*\) and \(C^*\) in terms of notch parameters have been presented in Ref. [54]. \(d_{c,U}\) which is the critical distance measured from the origin of the notch polar coordinate system is obtained as follows

\[
d_{c,U} = r_0 + d_c
\]

(8)

where \(d_c\) is the critical distance measured from the notch tip. To ascertain \(d_c\) for brittle and quasi-brittle materials subjected to the loading conditions comprising mode III, the following relation has previously been proposed by the authors [54]:

\[
d_c = \frac{2}{\pi} \left( \frac{K_{Ic}}{\sigma_u} \right)^2
\]

(9)

Based on the properties of the PMMA material tested in the current work, which has been listed in Table 1, \(r_c\) and \(d_c\) are calculated equal to 0.115 mm and 0.460 mm, respectively.

5. Results and discussion

As mentioned before, to plot the experimental results in the form of the curves of fracture resistance and fracture initiation angle, it is necessary to convert the fracture loads listed in Table 2 to the corresponding NSIFs values at fracture, so-called the critical NSIFs. The FE model described in Section 3 is utilized for the conversion. Using the normal and out-of-plane shear stresses obtained from the FE analyses, the NSIFs \(K_{Ic}^{U,\rho}\) and \(K_{II}^{U,\rho}\) are calculated from the following equations [54]:

\[
K_{Ic}^{U,\rho} = \sqrt{2\pi} \frac{\sigma_{\text{out}}(r_0, 0) r_0^{1-\lambda_1}}{1 + \omega_1}
\]

(10)

\[
K_{II}^{U,\rho} = \sqrt{2\pi} \frac{\tau_{\text{max}} r_0^{1-\lambda_3}}{\omega_3}
\]

(11)

The parameters \(\omega_1\) and \(\omega_3\) and the eigen values \(\lambda_1\) and \(\lambda_3\) have been presented in [54]. \(\sigma_{\text{out}}(r_0, 0)\) in Eq. (10) is the tangential stress at the U-notch tip and \(\tau_{\text{max}}\) is the maximum elastic shear stress at the notch tip. \(\tau_{\text{max}}\) can be determined from the following relation [54]:

\[
\tau_{\text{max}} = \frac{\omega_3 (\sigma_{\text{out}})_{r=r_0, \theta=0}}{1 + (r_0/r_3)^{\mu_3 - \lambda_3}}
\]

(12)
The parameters $l_3$ and $r_3$ have also been presented in [54].

Table 3 lists the normal and shear stresses $\sigma_{ww}(r_0, 0)$ and $\sigma_{ws}(r_0, 0)$ due to the fracture loads of Table 2 which have been computed by means of the aforementioned FE model. Also presented in Table 3 are the values of mode mixity ratio, $M^p_\varepsilon$, of all the fractured samples based on the NSIFs $K^{\varepsilon I}_\varepsilon$ and $K^{\varepsilon III}_\varepsilon$ calculated by using Eqs. (10)–(12). In addition, based on the results of mode I experiments, the values of mode I notch fracture toughness for the tested PMMA, $K^{\varepsilon Ic}_\varepsilon$, are obtained equal to 2.17, 2.65 and 4.00 MPa$\sqrt{\text{m}}$ for notch tip radii of 0.5, 1 and 2 mm, respectively.

Figs. 12 and 13 exhibit respectively the fracture curves and the curves of fracture angles predicted by the PS and MS criteria together with the experimental results of the U-notched PMMA specimens with three notch tip radii of 0.5, 1 and 2 mm.

The fracture resistance curves are drawn in terms of the dimensionless normalized NSIFs as $K^{\varepsilon I}_{\varepsilon} / K^{\varepsilon Ic}_\varepsilon$ versus $K^{\varepsilon III}_{\varepsilon} / K^{\varepsilon Ic}_\varepsilon$. While the authors believe based on their experience that existence of scatter in the test results is natural, the experimental data scatter observed in Figs. 12 and 13 might be questionable. The scatter seems to be related to the utilized multi-part test configuration with even slight inevitable clearances between its parts and small misalignment of the loading fixture elements and the specimen with respect to each other. Nevertheless, the significant advantages of the test configuration in comparison with the test specimens and apparatuses previously developed for mixed mode I/III fracture experiments (explained in [56]) encouraged the authors to use it.

The theoretical and experimental results depicted in Fig. 12 can be compared quantitatively using a dimensionless parameter called the normalized effective NSIF (NENSIF), $K^{\varepsilon I}_{\varepsilon eff}$, which is defined as

$$
K^{\varepsilon I}_{\varepsilon eff} = \sqrt{\left(\frac{K^{\varepsilon I}_{\varepsilon}}{K^{\varepsilon Ic}_\varepsilon}\right)^2 + \left(\frac{K^{\varepsilon III}_{\varepsilon}}{K^{\varepsilon Ic}_\varepsilon}\right)^2}
$$

Indeed, the NENSIF value is the length of the cord drawn from the coordinate origin in Fig. 12(a)–(c) to the intercept on the PS and MS fracture curves (theoretical values) and to the point on each experimental data. The theoretical values of the NENSIF belonging to the PS and MS criteria as well as the mean values of the experimental NENSIFs are listed in Table 4 containing the mean discrepancies. Likewise, Table 5 presents the values of $\phi_\varepsilon$ predicted by the PS and MS criteria together with the mean values of the experimental $\phi_\varepsilon$ including the average discrepancies.

**Table 3**

Normal and out-of-plane shear stresses at the notch tip of the test specimens and the related mode mixity ratios.

<table>
<thead>
<tr>
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<th>$\sigma_{ws}(r_0, 0)$ (MPa)</th>
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Tables 4 and 5 imply that the mean accuracies of both the PS and MS criteria in estimating the fracture angle and resistance are better for the notch tip radius of 0.5 mm than two larger ones. This is because the PS and MS criteria curves are obtained by using the critical distances of $r_{c,V}$ and $d_{c,V}$, respectively, which have been calculated based on the critical distance measured from the notch tip, $r_c$, as explained in [54]. This parameter is the critical distance defined and used for the sharp

Fig. 12. The fracture resistance curves of the PS and MS criteria together with the test results of the U-notched PMMA samples for different notch tip radii.
crack problems and consequently when the notch tip radius increases, the theoretical calculations deviate more from the experimental results. Moreover, Tables 4 and 5 show that the overall average discrepancies of the PS and MS criteria are in the range of 9.4%–11.4%. These acceptable discrepancies represent that both stress-based criteria are accurate enough.
to be used in predicting the mixed mode I/III fracture resistance and fracture angles of the U-notched PMMA specimens. One main cause for the discrepancies between the theoretical and experimental results is that the stress field utilized in the PS and MS criteria is an approximate expression only satisfying the boundary conditions in limited points on the notch edge and not on the whole edge [34]. Another reason for deviation of the theoretical predictions from the experimental results could be the development of a plastic region around the notch tip. It has previously been shown that as the mode III loading contribution increases, the plastic zone around the tip of the stress raisers such as cracks broadens [56,59,60]. Larger size of the plastic region near the notch tip is equivalent to more energy dissipation, which leads to increased fracture resistance. Therefore, the classical failure criteria such as PS and MS, which are based on the linear elastic fracture mechanics (LEFM) assumptions, cannot accurately predict the fracture resistance of material when the out-of-plane shear load enhances. Recently, a new concept, which is able to equate a ductile or quasi-brittle material with a virtual brittle material, namely the equivalent material concept (EMC), has been suggested in the literature [61,62]. By means of the EMC, the effects of the plastic deformations on fracture are directly considered in the standard tensile tests and there is no need to consider the plastic zone around the stress raiser. Therefore, utilizing the combination of the PS and MS criteria with the EMC may improve the theoretical predictions for mixed mode I/III fracture of U-notched members and reduce the discrepancies. This idea can be examined in future works.

Tables 4 and 5 also represent that the PS model gives slightly better estimates to the experimental fracture resistance than the MS criterion. Conversely, the MS criterion has been a bit more successful in predicting the out-of-plane angles of fracture initiation. Even so, the MS criteria has no preference for predicting the mixed mode I/III fracture of PMMA made U-notches over the PS one because of the observed negligible discrepancies.

Both the theoretical and experimental results displayed in Fig. 12(a)–(c) reveal that the notch fracture resistance enhances by increasing the notch tip radius because of lower stress concentration at the notch tip vicinity. This is similar
to the influence of the notch tip radius on the fracture resistance under mixed mode I/II loading [34,63]. Additionally, Fig. 12 indicates that the MS criterion is more conservative than the PS model in predicting the fracture resistance under the combined tension/out-of-plane shear.

Fig. 13 shows that according to the PS and MS criteria, the out-of-plane fracture initiation angle ($\phi_f$) is neither a function of the notch tip radius in pure mode I nor in pure mode III but it depends on the notch tip radius under each combination of tension and out-of-plane shear. According to this figure, as $M_{II}$ decreases from 1 (pure mode I) to 0 (pure mode III), the fracture angle of $\phi_f$, increased from zero up to the mode III fracture angle of 45° which is independent of the notch tip radius. This is in contrast to the in-plane fracture angle due to pure mode II which depends on the notch tip radius [34,63]. By a review of Fig. 13(a)–(c) one can realize that for a constant mode mixity parameter, the value of the $\phi_f$ angle caused by the exerted mixed mode I/III loading is somewhat larger for the U-notches with a smaller notch tip radius.

It is observed in Figs. 12 and 13 that there is no considerable difference between the PS and MS criteria estimations for the fracture resistance and fracture initiation angles of the U-notched PMMA samples subjected to mixed mode I/III loading conditions. Nonetheless, by increasing the mode III contribution in the applied mixed-mode loading, the discrepancy between the PS and MS criteria estimations slightly grows up to pure mode III. Moreover, the small deviation between the curves predicted by the two criteria decreases when the notch tip radius increases.

It is noteworthy that mode III or mode II deformation in cracked and notched components rarely exists independently along the thickness. It has been demonstrated in several researches that mode III deformation creates mode II (and vice versa) for the sake of the Poisson's ratio effect and boundary conditions [64–68]. This is a local mode coupling which for instance in the case of out-of-plane loading of a through-the-thickness sharp notch is concentrated close to the plate free surfaces. In other words, because of three-dimensional effects and the global deformation behavior, the nominal NSIFs are different from those of the local modes. Since the authors have been developed the PS and the MS criteria to general mixed mode I/III loading conditions [54], they are three-dimensional failure criteria. Accordingly, these criteria have the capability of surveying the fracture probability at each point along the specimen thickness and taking account of mode coupling effects mentioned above. This capability can be investigated by the authors in future works.

6. Conclusions

1. A series of new test fractures were conducted under combined tension/out-of-plane shear conditions by means of a new test apparatus on the U-notched specimens made of PMMA (Polymethyl-metacrylate) with different notch tip radii.
2. The mean stress (MS) and point stress (PS) fracture criteria newly formulated for mixed-mode I/III loading were utilized to predict the fracture resistance and the out-of-plane fracture initiation angles of the rectangular plates weakened by U-notches.
3. Good agreement found between the theoretical predictions and the experimental results verified the accuracy of the PS and MS criteria in estimating the fracture initiation angle and resistance of the U-notched PMMA specimens.
4. No meaningful difference was recognized between the curves of fracture initiation angle and fracture resistance predicted by the PS and MS criteria.
5. In contrast to the in-plane fracture initiation angle resulted from pure mode II loading, the out-of-plane fracture angle created by pure mode III loading is constant (45°) and independent of the notch tip radius.
6. The value of the fracture initiation angle caused by a given applied mixed mode I/III loading is slightly larger for the U-notches having smaller notch tip radii.
7. Both the PS and MS criteria were more accurate for U-notched components with smaller notch tip radii.

References
