Static Strength of V-Notches With End Holes Under Combined Tension-Shear Loading: Experimental Measurement by the Disk Test and Theoretical Prediction by the Local Energy
A. R. Torabi,1 A. Campagnolo,2 and F. Berto2

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Reference

ABSTRACT
The aim of the present research work was twofold. Firstly, to provide some new experimental results regarding brittle fracture in V-notches with end holes (VO-notches) under mixed mode I/II loading; secondly, to check the suitability of the well-established averaged strain energy density criterion (SED) in predicting the load-carrying capacity of notched specimens. A new test specimen, called Brazilian disk containing central VO-notches (VO-BD), made of PMMA was proposed and utilized to perform fracture tests at room temperature for different mode mixities. It was found that the average SED criterion seems to successfully evaluate the static strength of VO-BD specimens for different notch angles and various notch radii.

Keywords
V-notch with end hole (VO-notch), brittle fracture, average strain energy density (SED), Brazilian disk, mixed mode I/II loading

Introduction
About two decades ago, a new branch was born inside the classic fracture mechanics (FM), called the notch fracture mechanics (NFM), which investigates failure of notched domains made of brittle and ductile materials. Most of the investigations in the NFM concentrate on the static and monotonic failure of notched members. The modes of loading for a notched member are
three-point bending [9]. The conventional rectangular specimen containing an edge blunt V-notch (RV-TPB specimen) and the RV-SCB specimen have also been employed in Ref. [10] for conducting Mode I brittle fracture tests on coarse-grain polycrystalline graphite aiming the measurement of the NFT. The four-point bend (FPB) specimen weakened by blunt V-notches of very small tip radii and made of PMMA and MACOR has also been employed in Ref. [40] to measure Mode I and mixed mode I/II NFTs. A new version of the well-known compact-tension (CT) specimen containing an edge key-hole notch (instead of a sharp crack) was proposed in Ref. [41] to measure the mixed mode I/II NFT. A simple rectangular specimen containing a central slit with two U-shaped ends made of fine-grain isotropic graphite and subjected to remote tension has recently been suggested by Berto et al. [28] in order to measure the Mode I and mixed mode I/II NFTs for U-notches. The same graphite specimen with central key-hole notches was also proposed and utilized in Ref. [42] to measure the NFT for key-hole notches under pure Mode I and mixed mode I/II loading conditions.

Regarding out-of-plane loading conditions (e.g., torsion), the round specimens containing circumferential V-, U- and semicircular notches have been suggested in the literature to measure the Mode III and mixed mode I/III NFTs [8,43–45]. In Ref. [8], many Mode III brittle fracture tests have been carried out on fine-grain isotropic graphite by means of round bars weakened by circumferential V-, U- and semicircular notches. The load-carrying capacity of the graphite round bars have successfully been evaluated by means of the local strain energy density (SED) criterion [8]. The same experiments have been reported in Ref. [43] on PMMA at room temperature and the experimentally obtained Mode III NFT has successfully been estimated by means of the SED criterion. In order to avoid possible influences of the plastic deformations of PMMA at room temperature on the fracture test results, the same experiments reported in Ref. [43] have recently been repeated by Berto et al. [44] at –60 °C, for which PMMA exhibits linear elastic behavior till the final breakage. Again, SED has been shown to have successful predictions to the test results [44]. Susmel and Taylor [45] were probably the first ones who investigated brittle fracture in notched components under mixed mode I/III loading. They tested several V-notched PMMA specimens subjected to combined tension-torsion and measured the load-carrying capacity of the specimens. The experimental results were well predicted by using the theory of critical distances (TCD) [45].

Undoubtedly, the Brazilian disk (BD) specimen is one of the most well-known and widely used specimens for conducting fracture tests on brittle and quasi-brittle materials. The centrally cracked BD (CBD) specimen has frequently been employed by many investigators to measure experimentally the fracture toughness of various brittle materials under different loading conditions, e.g., Mode I, Mode II and mixed mode I/II loadings...
For indirectly measuring the tensile strength of brittle materials, such as rock materials, the BD specimen without crack was also utilized [52].

Ayatollahi and Torabi [9] suggested a new version of the CBD specimen weakened by a central rhombic hole (instead of crack), namely the rounded-tip V-notched BD (RV-BD) specimen to perform the NFT tests on PMMA at room temperature under Mode I, Mode II, and mixed mode I/II loadings. They successfully predicted the NFTs and the fracture initiation angles (FIAs) by using the rounded-tip V-notched maximum tangential stress (RV-MTS) criterion [9]. The same fracture tests and theoretical estimations were performed in Ref. [12] on sharp V-shaped notches. To check the repeatability of the NFT tests by the RV-BD specimen and to see if the RV-MTS criterion works also well on various brittle materials, Ayatollahi and Torabi carried out the same analyses for polycrystalline graphite [10] and soda-lime glass [11], and achieved positive responses for both the RV-BD specimen and the RV-MTS criterion. Regarding NFT measurement for U-notches, the U-notched BD (UNBD) specimen has been proposed in Ref. [7] for performing Mode I and Mode II brittle fracture experiments on PMMA and soda-lime glass. The experimental results (i.e., the NFT and the FIA) have been well estimated by means of the U-notched MTS (UMTS) criterion [7]. The same analyses have recently been performed on the UNBD specimen made of polycrystalline graphite under Mode I [6], Mode II [8], and mixed mode I/II loading conditions. The most recent versions of the notched BD specimen are certainly those containing central V-notches with end holes (VO-notch) and key-hole notches, so-called as the VO-BD [53] and the Key-BD [54] specimens in the literature. Theses specimens have been shown to be useful for evaluating experimentally the Mode I NFT for VO- and key-hole notches [53,54]. In Refs. [53,54], fracture tests have been carried out on PMMA at room temperature under pure Mode I loading, and the two well-known brittle fracture criteria, namely the point stress (PS) and the mean stress (MS), were successfully employed to predict the experimental NFTs.

As mentioned earlier, various failure concepts have been proposed in the literature to predict brittle fracture in notched domains. The strain energy density (SED) criterion, proposed originally by Sih [55] for sharp cracks, is undoubtedly one of the most well-known criteria in the field of brittle fracture. According to the Sih’s [55] criterion, brittle fracture takes place in a cracked brittle member when the strain energy density factor (SEDF) attains its critical value [55]. The crack growth direction could be determined by setting a minimum condition on the SEDF [55]. During the past two decades, other researchers extended the SED concept to sharp and blunt notches by averaging SED over a specified control volume which embraces the notch edge, in order to predict brittle fracture in notched components under different loading conditions [23–29] as well as the fatigue strength of notched components [56–58]. The SED predictions have frequently been verified by using the experimental results obtained from testing different materials, specimens, and notches under various loading conditions, e.g., Mode I, Mode II, and mixed mode I/II etc [23–29]. Some recent results on the fracture analysis of blunt notches by means of SED are those published in Refs. [32–34]. The fracture load of specimens weakened by rounded V-notches and made of PMMA has been predicted under pure Mode I [32] and pure Mode II [33,34] loadings by means of the SED criterion. In two other papers, it has been demonstrated that the SED criterion works well also under torsion [30] and compression (31) loadings.

Consider a V-notched structural component damaged by a crack emanating from the notch border (see Fig. 1). The most common repairing method for the component is to remove the crack by drilling a hole with radius usually equal to the crack length. Such a repairing method changes the original notch feature (i.e., V-shaped notch) to a V-notch with end hole.

**FIG. 1**
V-notch with end hole (VO-notch) resulted from removing a small crack from a V-notch border.

![Cracked V-notch](image1)
![Hole-drilling](image2)
![V-notch with end hole (VO-notch)](image3)
Due to the difference between the geometries of V- and VO-notches, the stress distribution and the stress gradient at the vicinity of the VO-notch are different from those of the V-notch, and a new fracture investigation is needed.

In the present study, the Brazilian disk specimen weakened by V-notches with end holes, called VO-BD specimen, made of PMMA, was utilized to experimentally measure the static strength of VO-notches under mixed mode I/II loading for various mode mixities. The load-carrying capacity of the VO-BD specimens is theoretically evaluated by means of the average strain energy density (SED) criterion. A good agreement is shown to exist between experimental and theoretical results for various notch angles and different values of notch radius.

Experiments
A large bulk of new experimental results is provided in this study on a recently proposed VO-notched disk-type specimen. Details of the experiments are presented hereafter.

MATERIAL
The material is the polymethyl-methacrylate (PMMA) with the mechanical properties presented in Table 1 at room temperature. The PMMA tested is completely the same as that previously examined in Refs. [7,9,12,53,54].

TEST SPECIMEN
A new version of the well-known Brazilian disk specimen weakened by central V-notches with end holes, called VO-BD specimen, which has recently been proposed for performing Mode I fracture tests on VO-shaped notches [53], is utilized in this study to conduct mixed mode I/II fracture experiments on VO-notches. Fig. 2 shows that various in-plane loading conditions can be achieved for the VO-notch by changing the loading angle \( \beta \) (i.e., the angle between the loading direction and the slit bisector line) from zero (for pure Mode I loading) to larger values (for mixed mode I/II loading). In other words, different \( \beta \) values mean different mode mixities, i.e., various contributions of tensile and shear stresses at the notch tip vicinity.

The disk diameter (D), the overall slit length (d), and the disk thickness are equal to 80 mm, 40 mm and 10 mm, respectively. For producing the VO-BD specimens, a PMMA plate that was 10 mm thick was first provided. Then, the drawing of each specimen was given to a high-precision 2-D CNC water-jet cutting machine for fabrication. Three notch angles of 30, 60, and 90° and four notch radii of 0.5, 1, 2, and 4 mm were tested, meaning twelve different VO-notch geometries.

Two loading angles \( \beta \), different from zero, have been examined for each notch geometry providing mixed mode I/II loading conditions. In order to select appropriate \( \beta \) angles for mixed mode I/II experiments, the \( \beta \) value for which pure Mode II loading occurs (\( \beta = \beta_{II} \)) is needed. It was found by the finite element (FE) stress analysis that \( \beta_{II} \) considerably depends upon the notch angle and slightly on the notch radius (\( \beta_{II} \) values were obtained to be approximately equal to 25, 28, and 33° for the notch angles 30, 60, and 90°, respectively). For notch angles 30, 60, and 90°, the loading angles \( \beta \) were considered to be (10,15), (10,20), and (10,20), respectively. Each test was repeated three times in order to check the repeatability of the tests. All in all, 72 fracture tests were performed in this study. The method of determining \( \beta_{II} \) is briefly described in next paragraph.

For \( \beta = 0° \), the VO-ends of the central slit encounter opening; thus they experience pure Mode I loading. In this state, the notch bisector line at the neighborhood of the notch experiences only the tensile stresses. As \( \beta \) increases gradually from zero, the value of tensile stress decreases and, conversely, the value of the shear stress increases. For a specific \( \beta \) value, called \( \beta_{II} \), the tensile stress on the notch bisector line becomes zero meaning that the notch does not experience opening. This trivially means pure Mode II loading. In other words, to achieve \( \beta_{II} \), one should increase \( \beta \) from zero till the tensile stress at the notch bisector line becomes equal to zero.

Fig. 3 represents a VO-BD specimen subjected to mixed mode I/II loading inside the test machine. Some VO-BD

TABLE 1 Some of the mechanical properties of the PMMA tested at room temperature [7,9,12,53,54].

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus, E (GPa)</td>
<td>2.96</td>
</tr>
<tr>
<td>Poisson’s ratio, ( \nu )</td>
<td>0.38</td>
</tr>
<tr>
<td>Ultimate tensile strength (MPa)</td>
<td>70.5</td>
</tr>
<tr>
<td>Plane-strain fracture toughness (MPa m^{0.5})</td>
<td>1.96</td>
</tr>
</tbody>
</table>
specimens after fracture are also shown in Fig. 4. The test speed was set to be 0.5 mm/min providing monotonic loading conditions.

**TEST RESULTS**

Table 2 summarizes the experimental fracture loads of the VO-BD PMMA specimens for various notch angles and different notch radii. Each specimen is identified by an index (see the first column of Table 2) consisting of three numbers like x-y-z in which x, y, and z are related to the notch angle, the notch radius, and the loading angle, respectively. For example, the index 60-1-20 belongs to a VO-notch with 60° angle and 1 mm radius, subjected to a compressive load with $\beta = 20°$. Note that $P_1$, $P_2$, and $P_3$ denote the fracture loads in the repeated tests. The average fracture loads are shown in the last column of Table 2.

<table>
<thead>
<tr>
<th>Specimen Index</th>
<th>$P_1$ (N)</th>
<th>$P_2$ (N)</th>
<th>$P_3$ (N)</th>
<th>$P_{av}$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30-0.5-10</td>
<td>4754</td>
<td>4332</td>
<td>4707</td>
<td>4597.6</td>
</tr>
<tr>
<td>30-0.5-15</td>
<td>4395</td>
<td>4650</td>
<td>5203</td>
<td>4749.3</td>
</tr>
<tr>
<td>30-1-10</td>
<td>4395</td>
<td>5505</td>
<td>4651</td>
<td>4850.3</td>
</tr>
<tr>
<td>30-1-15</td>
<td>4127</td>
<td>4533</td>
<td>4949</td>
<td>4536.3</td>
</tr>
<tr>
<td>30-2-10</td>
<td>6699</td>
<td>6082</td>
<td>6427</td>
<td>6402.6</td>
</tr>
<tr>
<td>30-2-15</td>
<td>5203</td>
<td>4815</td>
<td>5308</td>
<td>5108.6</td>
</tr>
<tr>
<td>30-4-10</td>
<td>6261</td>
<td>6156</td>
<td>6208</td>
<td>6208.3</td>
</tr>
<tr>
<td>30-4-15</td>
<td>5742</td>
<td>5456</td>
<td>6156</td>
<td>5784.6</td>
</tr>
<tr>
<td>60-0.5-10</td>
<td>3857</td>
<td>3834</td>
<td>3620</td>
<td>3770.3</td>
</tr>
<tr>
<td>60-0.5-20</td>
<td>3734</td>
<td>3935</td>
<td>3834</td>
<td>3834.3</td>
</tr>
<tr>
<td>60-1-10</td>
<td>4395</td>
<td>3834</td>
<td>4280</td>
<td>4169.6</td>
</tr>
<tr>
<td>60-1-20</td>
<td>4357</td>
<td>4318</td>
<td>4280</td>
<td>4318.3</td>
</tr>
<tr>
<td>60-2-10</td>
<td>4874</td>
<td>4993</td>
<td>4844</td>
<td>4903.6</td>
</tr>
<tr>
<td>60-2-20</td>
<td>4228</td>
<td>4561</td>
<td>4693</td>
<td>4494</td>
</tr>
<tr>
<td>60-4-10</td>
<td>5758</td>
<td>5305</td>
<td>5686</td>
<td>5583</td>
</tr>
<tr>
<td>60-4-20</td>
<td>5399</td>
<td>5686</td>
<td>5488</td>
<td>5524.3</td>
</tr>
<tr>
<td>90-0.5-10</td>
<td>2371</td>
<td>2353</td>
<td>2609</td>
<td>2444.3</td>
</tr>
<tr>
<td>90-0.5-20</td>
<td>2738</td>
<td>3013</td>
<td>2886</td>
<td>2879</td>
</tr>
<tr>
<td>90-1-10</td>
<td>2722</td>
<td>2915</td>
<td>2907</td>
<td>2848</td>
</tr>
<tr>
<td>90-1-20</td>
<td>3279</td>
<td>3279</td>
<td>2969</td>
<td>3175.6</td>
</tr>
<tr>
<td>90-2-10</td>
<td>3359</td>
<td>2862</td>
<td>2916</td>
<td>3045.6</td>
</tr>
<tr>
<td>90-2-20</td>
<td>3319</td>
<td>3439</td>
<td>3445</td>
<td>3401</td>
</tr>
<tr>
<td>90-4-10</td>
<td>3700</td>
<td>3911</td>
<td>3947</td>
<td>3852.6</td>
</tr>
<tr>
<td>90-4-20</td>
<td>3745</td>
<td>4071</td>
<td>3900</td>
<td>3905.3</td>
</tr>
</tbody>
</table>

It is found from the experiments that the load-displacement curves of the VO-BD specimens are linear up to final breakage (see an instance in Fig. 5). The recorded curves are in a good consistency with the experimental observations from which no considerable plastic deformations are obtained at the notch tip vicinity. Sudden fall of the load from a maximum (the fracture load) to zero in Fig. 5 suggests sudden fracture of the VO-BD specimen. Experimental observations confirm that fracture takes place suddenly in VO-BD specimens without any visible failure evidences such as excess deformations around the notch or crack initiation from the notch.

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**FIG. 3** A VO-BD specimen subjected to mixed mode I/II loading inside the test machine.

**FIG. 4** Some VO-BD specimens after fracture.

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**TABLE 2** The experimental fracture loads of the VO-BD PMMA specimens subjected to mixed mode loading.
Fracture Criterion Based on the Local Strain Energy Density (SED)

With the aim of estimating the critical loads of PMMA components weakened by cracks or notches, designers need appropriate fracture criteria which take into account the local behavior of material around the stress concentrators. In the present section, a criterion based on the local strain energy density (SED) and useful for the theoretical estimation of the fracture loads of notched components is described. The SED criterion has been used in its original form (for cracks) to study three problems of structural failure [59], namely the problem of slow stable propagation of an inclined crack in a plate under uniaxial tension, the problem of fracture instability of a plate weakened by a central crack and two notches, and the problem of unstable crack propagation in a circular disc subjected to two equal and opposite forces. The stress analysis results have been combined with the SED theory to achieve the entire history of crack propagation from nucleation to instability [59]. A length parameter has been introduced to define the fracture instability of a mechanical system. Fracture trajectories have been obtained for fast unstable crack growth. The investigation of crack nucleation and growth is still an active research topic as is demonstrated by many recent works in the field [60–64]. This is especially true for mixed mode loading [65–67].

On the basis of the local strain energy density criterion (SED) [23,25,68], brittle fracture occurs when the SED averaged over a given control volume reaches the critical value, \( W_c \), which is a characteristic of the material. \( W_c \) depends only on the material and not on the notch geometry. With reference to materials characterized by linear elastic behavior up to fracture under static loading, the size of the control volume is thought of as a function of the tensile strength \( \sigma_t \) and the fracture toughness \( K_{IC} \).

This approach was first proposed in the literature for sharp V-notched components subjected to Mode I or mixed I/II loading conditions [23] and then also formulated for blunt notched components [24,68]. The latest developments and applications have been presented in Refs. [68–70] by highlighting the capacity to automatically take into account also the three-dimensional effects [71,72].

In the case of components weakened by cracks, the control volume becomes a circle centered at the tip and characterized by a radius \( R_c \) [23]. With reference to a state of plane strain, the size of the control volume, \( R_c \), can be estimated by means of the expression [23,68]:

\[
R_c = \left(1 + \nu\right)\left(5 - 8\nu\right)\frac{K_{IC}}{\sigma_t}^2
\]

where:
- \( K_{IC} \) = the fracture toughness of material,
- \( \nu \) = the Poisson’s ratio, and
- \( \sigma_t \) = the ultimate tensile strength of the smooth material.

In the case of components weakened by blunt notches and subjected to Mode I loading conditions, the control volume assumes a crescent shape [24], where \( R_c \) represents the size evaluated along the bisector line. The control volume (Fig. 6) is characterized by an outer radius that equals \( R_c + r_0 \). The parameter \( r_0 \) is a function of the opening angle (2\( \psi \)) and the notch tip radius (\( \rho \)) as is highlighted by the following relationship:

\[
r_0 = \frac{q - 1}{q} \rho
\]

where \( q \) is defined by:

\[
q = \frac{2\pi - 2\psi}{\pi}
\]

In the case of blunt notched components subjected to mixed mode loading conditions as the VO-BD specimens investigated in the present contribution, the control volume results to be centered on the point of the notch edge characterized by the maximum principal stress [26] rather than on the notch tip, as can be seen from Fig. 6. To learn more about the local stress field due to V-notches with end-holes under different loading conditions, the reader can refer to some recent works by Zappalorto and Lazzarin [73,74]. The basic idea behind the
The “equivalent local Mode I” approach, already used for U-notched and V-notched PMMA components \([26,68]\), is applied in this contribution to Brazilian disk specimens weakened by V-notches with end-holes and subjected to mixed mode loading. For this purpose, two different FE analyses have been performed with reference to each geometry: the first one aimed at determining the point characterized by the maximum principal stress; the second one aimed at evaluating the average value of the strain energy density, once the control volume has been defined properly in relation to the point determined in the first FE analysis.

The main advantage of the strain energy density approach with respect to the local stress-based parameters is that it does not need very refined meshes in the close neighborhood of the stress concentration \([69]\). The averaged SED can be accurately evaluated also by means of coarse meshes \([69]\) because it directly depends on nodal displacements.

### SED Approach for the Fracture Assessment of the VO-BD PMMA Specimens

In this section, the failure criterion previously introduced is applied to PMMA specimens weakened by V-notches with end

### TABLE 3
Comparison between experimental and theoretical loads evaluated by means of SED. The SED in the Table has been evaluated applying in the numerical model \(P = P_{av}\).

| Specimen Index | \(\rho\) | \(\beta\) | \(P_1\) (N) | \(P_2\) (N) | \(P_3\) (N) | \(P_{av}\) (N) | SED (MJ/m\(^3\)) | \(P_{th}\) (N) | \(|\Delta|\%\) |
|----------------|--------|--------|-----------|-----------|-----------|-------------|---------------|-------------|------------|
| 30-0.5-10      | 0.5    | 10     | 4754      | 4332      | 4707      | 4597.7      | 0.74139       | 4893        | −15.2      | −7.9       | −17.0       |
| 30-0.5-15      | 0.5    | 15     | 4395      | 4650      | 5203      | 4749.3      | 0.87768       | 4645        | −2.3       | 6.7        | −1.3        |
| 30-1-10        | 1      | 10     | 4395      | 5505      | 4651      | 4850.3      | 0.65278       | 5501        | 5.4        | −2.0       | 6.8         |
| 30-1-15        | 1      | 15     | 4127      | 4533      | 4949      | 4536.3      | 0.63246       | 5227        | 15.9       | −5.3       | 11.0        |
| 30-2-10        | 2      | 10     | 6699      | 6082      | 6247      | 6402.7      | 0.85222       | 6335        | −2.2       | 4.3        | −0.5        |
| 30-2-15        | 2      | 15     | 5203      | 4815      | 5308      | 5108.7      | 0.59381       | 6075        | −10.3      | −0.1       | −5.8        |
| 30-4-10        | 4      | 10     | 6261      | 6156      | 6208      | 6208.3      | 0.65109       | 7050        | 0.8        | 7.4        | 4.5         |
| 30-4-15        | 4      | 15     | 5742      | 5456      | 6156      | 5784.7      | 0.63187       | 6668        | 6.1        | 7.7        | 6.9         |
| 60-0.5-10      | 0.5    | 10     | 3857      | 3834      | 3620      | 3770.3      | 0.89535       | 3651        | −5.6       | −3.8       | 2.3         |
| 60-0.5-20      | 0.5    | 20     | 3734      | 3935      | 3834      | 3834.3      | 0.83706       | 3840        | −0.4       | 0.2        | 5.7         |
| 60-1-10        | 1      | 10     | 4395      | 3834      | 4280      | 4169.7      | 0.81901       | 4222        | 3.5        | 3.1        | −7.4        |
| 60-1-20        | 1      | 20     | 4357      | 4318      | 4280      | 4318.3      | 0.88128       | 4215        | −4.3       | 9.0        | −1.5        |
| 60-2-10        | 2      | 10     | 4874      | 4993      | 4844      | 4903.7      | 0.80777       | 4999        | −6.5       | −8.8       | 4.9         |
| 60-2-20        | 2      | 20     | 4228      | 4561      | 4693      | 4494.0      | 0.69122       | 4953        | 1.6        | −0.8       | 2.2         |
| 60-4-10        | 4      | 10     | 5758      | 5305      | 5686      | 5583.0      | 0.73225       | 5978        | 3.1        | 1.9        | 0.7         |
| 60-4-20        | 4      | 20     | 5399      | 5686      | 5488      | 5524.3      | 0.67463       | 6163        | 6.6        | 13.9       | 7.7         |
| 90-0.5-10      | 0.5    | 10     | 2371      | 2353      | 2609      | 2444.3      | 0.90968       | 2348        | 2.5        | −5.7       | −0.5        |
| 90-0.5-20      | 0.5    | 20     | 2738      | 3013      | 2886      | 2879.0      | 0.77141       | 3003        | 21.1       | 21.7       | 13.1        |
| 90-1-10        | 1      | 10     | 2722      | 2915      | 2907      | 2848.0      | 0.89224       | 2763        | 7.0        | 10.5       | 13.7        |
| 90-1-20        | 1      | 20     | 3279      | 3279      | 2969      | 3175.7      | 0.82333       | 3207        | 15.1       | 9.1        | 9.3         |
| 90-2-10        | 2      | 10     | 3359      | 2862      | 2916      | 3045.7      | 0.61496       | 3559        | 20.6       | 15.3       | 15.3        |
| 90-2-20        | 2      | 20     | 3319      | 3439      | 3445      | 3401.0      | 0.66019       | 3835        | 12.4       | 25.4       | 24.0        |
| 90-4-10        | 4      | 10     | 3700      | 3911      | 3947      | 3852.7      | 0.51819       | 4904        | 22.9       | 19.5       | 19.0        |
| 90-4-20        | 4      | 20     | 3745      | 4071      | 3900      | 3905.3      | 0.51855       | 4969        | 25.5       | 21.3       | 20.6        |
holes and subjected to mixed mode loading conditions, with the aim of estimating the critical loads. The SED values have been evaluated by means of a FE model of each specimen. On the basis of the SED approach, the failure occurs when the strain energy density averaged over a control volume, $W$, equals the critical value $W_c$ of the material [23]. $W_c$ can be evaluated as a function of the ultimate tensile strength $\sigma_t$ by means of the Beltrami’s expression:

$$W_c = \frac{\sigma_t^2}{2E}$$  \hspace{1cm} (4)

According to Eq 1, in order to calculate the size of the control volume ($R_c$), it is necessary to know the mechanical properties of the material, in particular the fracture toughness $K_{IC}$ and the Poisson’s ratio $\nu$. The theoretical fracture loads of the PMMA specimens weakened by V-notches with end holes and subjected to mixed mode loading conditions can be estimated by imposing the equality between $W$ and the material critical energy $W_c$, which is supposed to be constant under Mode I, Mode II, and mixed mode I + II loading conditions as widely investigated in some recent contributions [25,68].

As is reported in the “Material” subsection, the PMMA examined in this work is characterized by the following mechanical properties: $\sigma_t = 70.5$ MPa, $K_{IC} = 1.96$ MPa$\sqrt{\text{m}}$, and finally a Poisson’s ratio $\nu = 0.38$. Accordingly, the critical value of the strain energy density for the considered material results to be $W_c = 0.839$ MJ/m$^3$ (see Eq 4), whereas the control radius is $R_c = 0.166$ mm (see Eq 1).

The strain energy density averaged over the control volume has been evaluated by means of finite element analyses performed in ANSYS, version 14.5. A FE analysis has been performed with reference to each geometry, defining a control volume (Fig. 6) in which the strain energy density has been averaged. Eight-node finite elements (PLANE 183) under a state of plane strain have been used for all the FE models.

The comparison between experimental values and theoretical estimations of the fracture loads for the PMMA notched components is reported in Table 3. The fracture loads obtained experimentally ($P_i$ with $i = 1, 2, 3$) and the theoretical estimations ($P_{th}$) determined according to the SED approach are provided in Table 3 with reference to each value of the notch tip radius, $\rho$. The values of the averaged SED have been evaluated from the numerical analyses by applying the mean value of the experimental loads $P = P_{av}$ to the FE model.

The table also shows the percentage deviation ($\Delta\%$) between the theoretical estimation of the fracture load and each of the three experimental values, with reference to all geometries considered. As has been highlighted in other contributions

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**Table 3**

<table>
<thead>
<tr>
<th>Geometries</th>
<th>$\beta$</th>
<th>$P_{th}$ (N)</th>
<th>$P_i$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\alpha = 30^\circ$</td>
<td>$10^\circ$</td>
<td>5.08</td>
<td>5.12</td>
</tr>
<tr>
<td>$2\alpha = 60^\circ$</td>
<td>$10^\circ$</td>
<td>6.00</td>
<td>6.05</td>
</tr>
<tr>
<td>$2\alpha = 90^\circ$</td>
<td>$10^\circ$</td>
<td>7.00</td>
<td>7.05</td>
</tr>
</tbody>
</table>

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**Fig. 7** Comparison between theoretical fracture loads obtained by SED and experimental data for $2\alpha = 30^\circ$. (a) $\beta = 10^\circ$ and (b) $\beta = 15^\circ$.

**Fig. 8** Comparison between theoretical fracture loads obtained by SED and experimental data for $2\alpha = 60^\circ$. (a) $\beta = 10^\circ$ and (b) $\beta = 20^\circ$. 
[25,68], acceptable values from the engineering point of view are between −20 % and +20 %. It can be easily observed from Table 3 that this range is satisfied for the great majority of the present data with only a few exceptions falling outside the range from −20 to +20 %.

The results reported in Table 3 are also represented in the diagrams of Figs. 7–9, in which the fracture loads experimentally determined (open dots) are compared with the theoretical estimations obtained according to the SED approach (solid line), for a notch opening angle $2\alpha$ equal to 30, 60, and 90°, respectively. In particular, the abscissa axis of each figure reports the value of the notch root radius $\rho$ of each specimen. The figures clearly show a good agreement between theoretical predictions and experimental data, for all the different notch opening angles ($2\alpha$) and loading angles ($\beta$) taken into consideration in this contribution.

Fig. 10 shows the square root value of the ratio between the SED averaged over the control volume with radius $R_c$ and the constant value of the critical energy of the material (0.839 MJ/m$^3$) as a function of the notch radius $\rho$. Being the strain energy density proportional to the square of the applied load, the plotted parameter results to be proportional to the ratio between the experimental and the theoretical loads. The purpose is to evaluate the influence of the notch root radius and that of the opening angle on the estimation of the critical loads according to the SED approach. From the diagram, it results that the scatter of the new data is very limited and almost independent of the notch geometries. In fact, the great majority of the experimental values obtained from the PMMA notched specimens fall inside a scatter band ranging from 0.8 to 1.2 with only a few values outside this range. Note that the majority of the results are inside a scatter ranging from 0.9 to 1.1.

These considerations underline the very good accuracy of the SED approach for the fracture assessment of notched components made of PMMA under mixed mode I + II loading conditions, once the control volume has been properly modeled.

Conclusions

In the present contribution, some new experimental results regarding brittle fracture in V-notches with end holes (VO-notches) under mixed mode I + II loading are provided.

A new test specimen, called Brazilian disk containing central VO-notches (VO-BD), made of PMMA has been proposed and utilized to perform the fracture tests at room temperature for different mode mixities. Static tests aimed at obtaining the fracture loads have been conducted on specimens characterized by different notch geometries, such as the notch opening angle $2\alpha$, the notch root radius $\rho$ and the loading angle $\beta$.

Then, the suitability of the well-established averaged strain energy density criterion (SED) in predicting the load-carrying capacity of PMMA notched specimens has been checked. It is found that the averaged SED criterion, used in combination with the equivalent local Mode I concept, can successfully predict the static strength of VO-BD specimens subjected to mixed mode loading and characterized by different values of notch opening angle, loading angle and notch root radius. The scatter of the great majority of the data, that represents the deviation between the experimental loads and the theoretical ones, is found to be very limited (within ±20 %) as a demonstration of the effectiveness of the SED approach.
From the synthesis based on SED criterion, it can be inferred that the choice of the crescent shape for the control volume is appropriate to characterize the PMMA behavior under mixed mode I/II loading, and also that the critical SED $W_c$ and the control radius $R_c$ are both constant material properties.

References


