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IMMISCIBLE DISPLACEMENT OF VISCOPLASTIC WAXY CRUDE OILS: A NUMERICAL STUDY

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The displacement of waxy crude oils obeying the Bingham model was numerically investigated in a water-flooding operation. A more accurate bundle-of-tubes model was developed based on the Rabinowitsch correction method to represent a porous medium. The classic Darcy’s law was used as the governing equation using the concept of effective viscosity. Owing to the nondifferentiability of the Bingham model, regularized versions of this rheological model were used to calculate the effective viscosity for the Bingham model. Use was made of the implicit-pressure/explicit-saturation numerical method for solving the governing equations for the saturation profiles. Based on the results obtained in the present work, it is concluded that regularized models are well capable of representing displacement flow of Bingham fluids. It is also concluded that the viscoplastic behavior of waxy crude oils has a negative effect on the water-flooding operation.

KEY WORDS: Bingham fluid, viscosity regularization, waxy crude oil, porous media, Buckley–Leverett problem, five-spot problem, water flooding

1. INTRODUCTION

Water flooding is a well-known operation in the oil industry. In this operation, water is used as the pusher fluid to drive the oil left in the porous rocks toward the production well (Poettmann, 1971). For the low–Reynolds numbers flows frequently encountered in most oil reservoirs, one can rely on the Darcy’s law as the governing equation (Getachew et al., 1998; Vafai, 2015). But, while ordinary crudes are Newtonian and the Darcy’s law is valid for them, waxy crude oils do not obey this empirical law. This is mainly because such oils exhibit a variety of non-Newtonian behaviors particularly at temperatures below their wax appearance temperature. The most striking non-Newtonian feature of waxy crude oils is their viscoplastic behavior. This means that these oils do not flow unless a certain stress threshold, the so-called yield stress, is exceeded (Wardhaugh et al., 1988; Chang et al., 1999; Davidson et al., 2004; Dong et al., 2013). The existence of yield stress makes a theoretical study of the water-flooding operation much more difficult for such fluids. This is mainly because the Darcy’s law cannot be applied to such fluids in its original form. Although it is possible to modify the Darcy’s law in such a way that it incorporates fluids’ rheology in a nontrivial way (often giving rise to nonlinear versions of the Darcy’s model), a simpler idea is to keep the (apparent) linear form of the Darcy’s model intact and just replace the Newtonian viscosity with an effective viscosity. (By so doing, it is the effective viscosity of the fluid that takes care of the nonlinear behavior of the fluid’s rheology.) Having decided on the Darcy’s model and the fluid’s rheological behavior, one has to decide on how to model the porous medium and determine its permeability to each phase. Most works carried out in the past have modeled the porous medium using the bundle-of-tubes approach (i.e., straight, identical, parallel tubes all oriented in the same direction). The Darcy’s law (whether linear or nonlinear) combined with the Poiseuille law then provides the permeability of the bundle in that direction.
The biggest advantage of the bundle-of-tubes model is its simplicity as compared with the pore-scale model (Sochi, 2010). This makes this approach highly suitable for analytical and/or semi-analytical studies. Having said this, it should be conceded that this simplistic model has its own limitations. For example, it disregards the morphology of the pore space and also the pore size distribution. Such drawbacks might have a significant influence on the pressure variation for non-Newtonian fluids. So, the use of this simplistic model should be justified for any given fluid. Al-Fariss and Pinder (1984) and Chevalier et al. (2013) have experimentally shown that for waxy crude oils obeying the Herschel–Bulkley model, the bundle-of-tubes approach works quite well. Similar conclusions have been reached for two other types of waxy crude oils: one obeying the Collins–Graves model and the other the Bingham model (Wu et al., 1992; Aadnøy and Ravnøy, 1994; Wu and Pruess, 1998).

Of particular importance in this area is the work carried out by Wu et al. (1992). They modified the Darcy’s law for Bingham fluids in such a way that it incorporates a minimum pressure gradient which is proportional to the fluid’s yield stress (Mendes et al., 2002; Zhang et al., 2014). Using the idea of minimum pressure gradient, they studied the displacement of Bingham fluids in the one-dimensional Buckley–Leverett problem and showed that there exists an ultimate displacement saturation which depends on the oil’s yield stress beyond which the sweep efficiency does not improve no matter how long the displacement operation is continued. In a more robust work, Wu and Pruess (1998) resorted to a potential-gradient approach to account for the non-Newtonian behavior of Bingham fluids. They relied on an “integral finite difference” scheme for numerically solving the equation governing the displacement of a Bingham fluid in the five-spot problem. Their results confirm the previous finding that the sweep efficiency of water-flooding operation is deteriorated whenever the oil exhibits a yield stress. Also, Yi (2004) developed an analytical model for the one-dimensional displacement of Herschel–Bulkley fluids in a porous medium using Al-Fariss and Pinder’s modified Darcy’s law and Wu’s methodology.

The present work can be regarded as an extension of the work carried out by Wu et al. (1992) and Wu and Pruess (1998) in the sense that, instead of relying on the “minimum pressure gradient” approach (which requires the Darcy’s model to be modified nonlinearly) or the “potential-gradient approach” (which is only asymptotically accurate), we have decided to rely on the concept of an effective viscosity in the classic Darcy’s law even for the Bingham fluids. This approach is suitable in that well-established numerical schemes developed in the past for Newtonian fluids can easily be extended to Bingham fluids. The main obstacle of implementing this idea is the notion that, in its original form, the Bingham model is nondifferentiable. In fact, this drawback has made it virtually impossible to capture the true shape of the displacement front in works dealing with viscoplastic oils. In the present work, it will be shown that by resorting to the regularized form(s) of the Bingham model, it is possible to implement this idea for the displacement flows of immiscible fluids as long as they are inelastic (Bercovier and Engelman, 1980; Papanastasiou, 1987; Allouche et al., 2000). Although this idea has already been shown to be applicable for Bingham fluids in a variety of complex geometries (Frigaard and Nouar, 2005), to the best of our knowledge, it has not previously been used for studying the displacement flow of Bingham fluids. With this in mind, the main objective of the present work is to investigate the role played by a fluid’s yield stress on the sweep efficiency and breakthrough time when it is being displaced by a Newtonian fluid. For ease of analysis, we have decided to focus mostly on the homogeneous case. Having said this, it should be stressed that the methodology developed in this work can be extended to nonhomogeneous situations as well. In fact, we have verified our code with a composite case for the Newtonian fluids. Limited numerical results have also been obtained for the Bingham fluid in a typical composite reservoir.

To reach its objective, the manuscript is organized as follows. In the next section, the mathematical formulation of the problem is developed and the viscosity regularization methods are introduced. The numerical method is then described, followed by a presentation of the numerical results obtained for the viscoplastic fluids in two classic benchmark problems: (1) the one-dimensional Buckley–Leverett problem (Buckley and Leverett, 1942) and (2) the two-dimensional quarter, five-spot problem (Willhite, 1986). The work is concluded by highlighting its major findings.

2. MATHEMATICAL FORMULATION

We consider the displacement of a viscoplastic waxy crude oil by a Newtonian fluid in a porous medium in the water-flooding operation. To make the analysis more tractable, we have relied on several simplifying assumptions, chief
among them the following: (1) the porous medium is isotropic, (2) both liquids are incompressible, (3) the liquids are immiscible, (4) each phase contains one component only, and (5) capillary and gravity forces are negligible. In a two-dimensional, finite control volume of the porous reservoir, the conservation of mass law for the water and waxy oil can be written as (Ertekin et al., 2001)

\[ -\frac{\partial}{\partial x} (u_{wx} A_x) \Delta x - \frac{\partial}{\partial y} (u_{wy} A_y) \Delta y = V_b \frac{\partial}{\partial t} (\varphi S_w) - q_{wsc} \]  

\[ -\frac{\partial}{\partial x} (u_{ox} A_x) \Delta x - \frac{\partial}{\partial y} (u_{oy} A_y) \Delta y = V_b \frac{\partial}{\partial t} (\varphi S_o) - q_{osc} \]

where \( \Delta x \) and \( \Delta y \) are the dimensions of the control volume, \( u = (u_x, u_y) \) is the phase velocity vector, \( A_x \) is the cross-sectional area of the control volume normal to the \( x \)-direction, \( A_y \) is the cross-sectional area of the control volume normal to the \( y \)-direction, \( V_b \) is the total volume of the control volume, \( \varphi \) is the reservoir’s porosity, \( t \) is the time, \( S \) is the phase saturation, and \( q_{osc} \) is the volumetric flow rate of a fluid to/from the control volume. Note that the subscripts \( o \) and \( w \) stand for the oil and the water phase, respectively. For Newtonian/Newtonian (N/N) displacement in a typical porous medium, the phase velocities can be related to the pressure gradient, fluid viscosity, and porosity of the medium through the classic Darcy’s law. For Newtonian/Bingham (N/B) displacement, we need to extend this well-known law to yield stress fluids. To that end, we write the water and oil phase velocities as

\[ u_w = -k \frac{k_{rw}}{\mu_w} \nabla p \]  

\[ u_o = -k \frac{k_{ro}}{\mu_o} \nabla p \]

where \( \mu \) is the effective viscosity (to be defined shortly), \( \nabla p \) is the pressure gradient vector, \( k \) is the reservoir’s permeability, and \( k_r \) is the relative permeability. It needs to be mentioned that, for simplicity, we have ignored the effect of salinity on the relative permeability of oil and water (Shaddel et al., 2014; Hassani et al., 2016). Now, because \( S_o + S_w = 1 \), by substituting Eqs. (3) and (4) into Eqs. (1) and (2), we obtain

\[ \frac{\partial}{\partial x} (k A_x \frac{k_{rw} \partial p}{\mu_w} \partial x) \Delta x + \frac{\partial}{\partial y} (k A_y \frac{k_{rw} \partial p}{\mu_w} \partial y) \Delta y = V_b \frac{\partial}{\partial t} (\varphi S_w) - q_{wsc} \]

\[ \frac{\partial}{\partial x} (k A_x \frac{k_{ro} \partial p}{\mu_o} \partial x) \Delta x + \frac{\partial}{\partial y} (k A_y \frac{k_{ro} \partial p}{\mu_o} \partial y) \Delta y = V_b \frac{\partial}{\partial t} [\varphi (1 - S_w)] - q_{osc} \]

Equations (5) and (6) are the governing equations for the displacement of a viscoplastic fluid by a Newtonian fluid in a porous medium. To account for the viscoplastic nature of the oil phase, we need to determine the oil’s viscosity, \( \mu_o \), in Eq. (6) through invoking an appropriate rheological model.

### 2.1 Rheological Model

There are many field data which suggest that reservoirs of paraffin-rich oils exhibit a yield stress. This means that a finite pressure gradient, called the threshold pressure gradient (Lu, 2012), must be exceeded before the flow can start (Barry, 1971). In the present work, we assume that the displaced liquid obeys the Bingham model. In simple shear flow, the Bingham model reads as (Bird et al., 1977)

\[
\begin{align*}
\tau & = \tau_y + \mu_B \dot{\gamma} ; \quad \tau > \tau_y \\
\dot{\gamma} & = 0 ; \quad \tau \leq \tau_y
\end{align*}
\]

where \( \tau \) is the shear stress, \( \tau_y \) is the yield stress, \( \mu_B \) is the plastic viscosity, and \( \dot{\gamma} \) is the shear rate. Based on Eq. (7), we can develop a pressure drop/flow rate equation for a Bingham fluid in pipe flow. To that end, we start from the
Rabinowitsch formula for the Poiseuille flow of generalized non-Newtonian fluids in a circular pipe (Bird et al., 1977):

\[ \frac{Q}{\pi R^3} = \frac{\tau}{R^3} \int_{0}^{\tau} \dot{\gamma} d\tau \]  

(8)

where \( Q \) is the volumetric flow rate, \( R \) is the radius of the pipe, and \( \tau_w \) is the shear rate at the pipe wall which is obtained from a simple force balance as

\[ \tau_w = \frac{R^2}{2} \left( -\frac{\partial p}{\partial x} \right) \]  

(9)

On the other hand, from Eq. (7), we obtain

\[ \begin{cases} \dot{\gamma} = \frac{\tau - \tau_y}{\mu_B} ; & \tau > \tau_y \\ \dot{\gamma} = 0 ; & \tau \leq \tau_y \end{cases} \]  

(10)

Inserting Eq. (10) into Eq. (8) yields

\[ \begin{cases} \frac{Q}{\pi R^3} = \frac{\tau}{R^3} \int_{0}^{\tau} \left( \frac{\tau - \tau_y}{\mu_B} \right) d\tau ; & \tau_w > \tau_y \\ \frac{Q}{\pi R^3} = 0 ; & \tau_w \leq \tau_y \end{cases} \]  

(11)

Using Eq. (11), the average velocity of the Bingham fluid in the pipe is obtained as

\[ \begin{cases} \bar{v} = \frac{R^2}{8\mu_B} \left( -\frac{\partial p}{\partial x} \right) \left[ 1 - \frac{4}{3} \left( \frac{\tau_y}{\tau_w} \right) + \frac{1}{3} \left( \frac{\tau_y}{\tau_w} \right)^4 \right] ; & \tau_w > \tau_y \\ \bar{v} = 0 ; & \tau_w \leq \tau_y \end{cases} \]  

(12)

Having obtained the average velocity in a pipe for a Bingham fluid, we now proceed to model the porous media as a bundle of separate capillaries all having the same radius, \( R \) (Teeuw and Hesselink, 1980). The superficial velocity in the porous medium can then be obtained from Eq. (12) as

\[ \begin{cases} u = \frac{k}{\mu_B} \left( -\frac{\partial p}{\partial x} \right) \left[ 1 - \frac{4}{3} \left( \frac{\tau_y}{\tau_w} \right) + \frac{1}{3} \left( \frac{\tau_y}{\tau_w} \right)^4 \right] ; & \tau_w > \tau_y \\ u = 0 ; & \tau_w \leq \tau_y \end{cases} \]  

(13)

where the porous media’s permeability, \( k \), is related to its porosity and capillary radius by

\[ k = \frac{\varphi R^2}{8} \]  

(14)

We now define a new parameter, \( G \), as

\[ G = \frac{8\tau_y}{3R} \]  

(15)

By substituting Eq. (15) into Eq. (13), the pressure drop–velocity relationship for a Bingham fluid can be expressed as

\[ \begin{cases} u = \frac{k}{\mu_B} \left( -\frac{\partial p}{\partial x} \right) \left[ 1 - \frac{G}{4\left( -\partial p/\partial x \right)} \left( \frac{3}{4} - \frac{G}{4\left( -\partial p/\partial x \right)} \right)^3 \right] ; & \left| \frac{\partial p}{\partial x} \right| > G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} - \frac{G}{4\left( -\partial p/\partial x \right)} \right)^3 \right] \\ u = 0 ; & \left| \frac{\partial p}{\partial x} \right| \leq G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} - \frac{G}{4\left( -\partial p/\partial x \right)} \right)^3 \right] \end{cases} \]  

(16)
For Eq. (16), to be used for modeling the displacement operation in a porous medium (which is a two-phase flow in its true sense), the relative permeability should be introduced in this equation. For the oil phase, we then obtain

\[ u_o = \frac{kkr_o}{\mu_B} \left( -\frac{\partial p}{\partial x} \right) \left\{ 1 - G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left( -\frac{\partial p}{\partial x} \right) \right)^3 \right] \right\} ; \quad \left| \frac{\partial p}{\partial x} \right| > G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left( -\frac{\partial p}{\partial x} \right) \right)^3 \right] \]

\[ u_o = 0; \quad \left| \frac{\partial p}{\partial x} \right| \leq G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left( -\frac{\partial p}{\partial x} \right) \right)^3 \right] \]

Equation (17) is one of the main contributions of the present work, as it has not been reported previously in any published work. In the limit of \( \left( -\frac{\partial p}{\partial x} \right) \to \infty \), Eq. (17) can be approximated as

\[ u_o = \frac{kkr_o}{\mu_B} \left( -\frac{\partial p}{\partial x} \right) \left[ 1 - G \left( -\frac{\partial p}{\partial x} \right) \right] ; \quad \left| \frac{\partial p}{\partial x} \right| > G \]

which is the equation widely used in modeling the flow of Bingham fluids in porous media. In the present work, however, we rely on Eq. (17) for modeling a typical porous medium because it is more accurate than Eq. (18). It is worth mentioning that the threshold pressure gradient (i.e., the pressure gradient below which no flow can materialize for a Bingham fluid in a porous medium) is equal to 0.75\( G \) in Eq. (17), whereas it is equal to \( G \) in Eq. (18). Now, after some simple mathematical manipulations, and in analogy with the Darcy’s law for Newtonian fluids, Eq. (17) can be reformulated as

\[ u_o = \frac{kkr_o}{\mu_{o,\text{eff}}} \left( -\frac{\partial p}{\partial x} \right) \]

where \( \mu_{o,\text{eff}} \) is the effective viscosity of the Bingham fluid in a porous medium, defined by

\[ \mu_{o,\text{eff}} = \mu_B + kkr_o \frac{G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left| \frac{\partial p}{\partial x} \right| \right)^3 \right]}{\left| u_o \right| + 1/m} \]

The effective viscosity is seen to be discontinuous at the limit of \( u_o \to 0 \). Thus numerical computation of the effective viscosity is not straightforward using this equation for the classic Bingham model. To overcome this drawback, it is convenient to employ a regularization technique to smoothen the effective viscosity function. The three most commonly used regularization techniques are as follows:

1. The linear regularization method, for which we have (Allouche et al., 2000)

\[ \mu_o = \mu_B + kkr_o \frac{G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left( \left| \frac{\partial p}{\partial x} \right| \right) \right)^3 \right]}{\left| u_o \right| + 1/m} \]

2. The regularization method proposed by Bercovier and Engelman (1980), for which we have

\[ \mu_o = \mu_B + kkr_o \frac{G \left[ 1 - \frac{1}{4} \left( \frac{3}{4} G \left( \left| \frac{\partial p}{\partial x} \right| \right) \right)^3 \right]}{\sqrt{\left| u_o \right|^2 + (1/m)^2}} \]
3. The regularization method proposed by Papanastasiou (1987), for which we have

$$\mu_o = \mu_B + kkro \left( 1 - e^{-m|u_o|} \right) \frac{G \left[ 1 - \frac{3}{4} \left( \frac{G}{4(|\partial p/\partial x|)} \right)^3 \right]}{|u_o|}$$

(22)

In all three regularization methods, $m$ is an adjustable parameter, which, if chosen to be sufficiently large, can make the model virtually the same as the original Bingham model. These equations, when inserted into Eq. (4), would allow us to extend the Darcy’s law to the flow of Bingham fluids in the two-dimensional five-spot problem by simply replacing $|\partial p/\partial x|$ with $|\nabla p|$ in Eqs. (20)–(22). It should be noted that in the derivation of Eqs. (20)–(22), the permeability, $k$, has been assumed to be a scalar, implying that the reservoir has tacitly been assumed to be isotropic. But we would like to stress that there is no restriction in extending our analysis to heterogeneous reservoirs in which the permeability is allowed to vary in the domain. As a matter of fact, for code-verification purposes, we intend to resort to known results for the heterogeneous case. It should also be emphasized that the oil’s viscosity in Eqs. (20)–(22) depends on the pressure gradient, the oil phase velocity, and the oil phase relative permeability. This means that the oil viscosity itself is a nonlinear function in Eq. (4), and so its computation may require an iterative procedure. In the next section, an appropriate linearization method is introduced that does not need iteration.

3. NUMERICAL METHOD

To obtain a numerical solution for the water saturation, the governing equations have to be discretized. To achieve this goal, use was made of the finite difference method through generating a block-centered grid. To discretize the second-order spatial derivatives in the left-hand side of Eqs. (5) and (6), we rely on the central-difference scheme. To discretize the temporal terms, we rely on the first-order backward approximation scheme. The discretized equations are nonlinear and have to be linearized. To linearize these equations, an explicit scheme was used, which means that the relative permeabilities and the Bingham fluid’s viscosity are evaluated first before computing the pressure. To solve the discretized equations, we employ the implicit-pressure/explicit-saturation (IMPES) method. In this well-established numerical method, the water-phase saturation is omitted from the flow equations by adding Eqs. (5) and (6) together. This enables the pressure to be solved implicitly. Knowing the pressure field, the water-phase saturation is calculated explicitly by solving the water-phase flow equation. Using the explicit linearization and the IMPES method significantly lowers the amount of computational effort needed at each time step, simply because this method does not need iteration to solve the equations. The price is that, to stabilize the solution scheme, the time step needs to be very small, which means more computational time (Ertekin et al., 2001). Still, thanks to the notion that we have neglected certain nonlinearities, which could have arisen through invoking the oil’s compressibility and/or the capillary forces at the interface, the numerical method employed in the present work still turned out to be good for our purposes.

To carry out the computations, a FORTRAN code was developed in this work. To verify the code developed in the present work, we have decided to use it for simulating the N/N displacement in certain benchmark problem(s). The code could easily reproduce analytical solutions long established in the literature for the classical one-dimensional Buckley–Leverett problem for the homogeneous N/N case (Buckley and Leverett, 1942). As a more demanding test, we have found it advisable to check the performance of the code in dealing with the nonhomogeneous N/N Buckley–Leverett problem studied previously by Wu et al. (1993) see also Wu et al., 2010). Figure 1 shows the composite domain used for such an analysis. As can be seen in this figure, the whole domain consists of two subdomains each with a different permeability. To carry out these simulations, parameter values shown in Table 1 were used.

It is worth mentioning that, like Wu et al. (1993), the relative permeabilities for the oil and water in subdomain 1 are computed by the correlation proposed by Willhite (1986):

$$\begin{align*}
    k_{ro} &= 0.75 (1 - S_D)^2 \\
    k_{rw} &= 0.75 (S_D)^2
\end{align*}$$

(23)
where \( S_D = S_w / 1 - S_{or} \). With the same reasoning, the relative permeabilities in subdomain 2 are calculated using the equations presented by Honarpour et al. (1986):

\[
\begin{align*}
\{ k_{ro} &= 0.75 \left( 1 - 1.25 S_w \right)^2 \left( 1 - 1.652 S_w^2 \right) \\
\{ k_{rw} &= 1.831 S_w^4 \\
\end{align*}
\]  

(24)

To decide on the appropriate number of grid points, a mesh-sensitivity study was undertaken first. Figure 2 shows the effect of the number of grid points (\( N_{\text{max}} \)) on the quality of the numerical results. (The right plot is just a zoomed-in version of the left plot.) As can be seen in this figure, more accurate results can indeed be obtained by increasing the number of grid points. In fact, the saturation discontinuity at the water front can be captured more accurately by mesh refinement. However, we have reached the conclusion that 3200 nodes can ensure grid-independent results and at the same time keep the computational time/cost at a reasonable level. So, in all results to be presented shortly, we have set \( N_{\text{max}} = 3200 \).

Figure 3 shows a comparison between our numerical results and the analytical results reported by Wu et al. (1993) for the distribution of water-phase saturation after 10 and 25 hours of water injection. As can be seen in this figure, our code is doing a nice job in reproducing analytical results and recovering the saturation discontinuity at the interface of the subdomains.

4. RESULTS AND DISCUSSION

Having verified the code with known analytical N/N results, we are now ready to present our new results for the displacement of Bingham fluid obtained using the concept of effective viscosity for two benchmark water-flooding problems: (1) the one-dimensional Buckley–Leverett problem and (2) the two-dimensional five-spot problem.
FIG. 2: Effect of grid size on the quality of the numerical results obtained for the Buckley—Leverett problem of the N/N pair in a composite porous medium after $t = 10$ hours. The right plot highlights the situation in the vicinity of the water front.

FIG. 3: Comparison between our numerical results (symbol) and the analytical solution reported by Wu et al. (1993) for the Buckley–Leverett problem of the N/N pair in a composite porous medium obtained after (a) $t = 10$ hours and (b) $t = 25$ hours

4.1 The Buckley–Leverett Problem

To study the effect of yield stress on the Buckley–Leverett problem, we assume that the reservoir is homogeneous, having the same properties as given for subdomain 1 in Table 1 (see also Fig. 1). We have tried all three regularization methods (i.e., Eqs. (20)–(22)) and reached the conclusion that they render virtually the same results for this particular fluid mechanics problem. Figure 4 shows a typical comparison between numerical results obtained in the present work using the (exact) equation (17) with the results reported by Wu et al. (1992) obtained using the (approximate)
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FIG. 4: A comparison between our numerical results and the analytical solution reported by Wu et al. (1992) for the displacement of a Bingham fluid by a Newtonian fluid in the homogenous Buckley–Leverett problem obtained after $t = 10$ hours for $G = 10^4 \text{Pa/m}$.

equation (18). As can be seen in this figure, the regularization model is doing a nice job simulating the displacement of the Bingham fluids, with the maximum deviation being in the vicinity of the inlet section, where the maximum water saturation ($S_{\text{max},w}$) is concerned. The maximum saturation corresponds to the minimum pressure gradient required for a Bingham fluid to flow in the porous medium. As mentioned earlier, based on Eq. (17), the minimum pressure gradient needed by a Bingham fluid to flow is equal to $0.75G$, whereas, based on the approximate analysis in common use [see Eq. (17)], it is equal to $G$ instead. Therefore the results shown in Fig. 4 are as expected.

At the limit of $(-\partial p/\partial x) \rightarrow \infty$, the numerical solution obtained by the three viscosity regularization methods for the displacement of Bingham fluid should tend to the asymptotic analytical solution obtained for Bingham fluids by Wu et al. (1992). This notion provides us with a convenient framework to evaluate the accuracy of the three viscosity regularization methods proposed in this work. To that end, we define the error measure as

$$\text{Error} = \sqrt{\frac{\sum_{i=1}^{N_{\text{max}}} (S_{w,i}^{\text{numeric}} - S_{w,i}^{\text{analytic}})^2}{N_{\text{max}}}}$$  \hspace{1cm} (25)

Having calculated the errors this way, we have reached the conclusion that all three regularization methods are good for our purposes. Still, the regularization method proposed by Papanastasiou (1987) is marginally better than the other two methods in terms of accuracy. In terms of the CPU time, however, Papanastasiou’s model needs (roughly 50%) more time to render converged results, whereas the other two methods need roughly the same amount of time. Considering the two key factors of accuracy and computational cost, Bercovier and Engleman’s regularization method appears to be the most efficient technique for simulating the flow of yield stress fluids through porous media.

Having realized that the regularization method is well capable of simulating the Buckley–Leverett problem for Bingham fluids, the methodology was used to investigate the effect of the yield stress (or, equivalently, $G$) and the plastic viscosity, $\mu_B$, on the results. Figure 5(a) shows the effect of the threshold pressure gradient $G$ on the water saturation distribution. As can be seen in this figure, the ultimate saturation at the inlet decreases by an increase in the threshold pressure gradient (or, equivalently, yield stress). This is not surprising, given that the resistance of Bingham fluids to flow increases when $G$ is increased, and this boosts the propagation of the water front in the flow domain. This mechanism is expected to decrease the breakthrough time of the displaced fluid, as can be seen in Fig. 5(b).
FIG. 5: Effect of the displaced fluid’s yield stress (represented by the threshold pressure gradient, \(G\)) on the homogeneous Buckley–Leverett problem: (a) water saturation distribution after \(t = 10\) hours and (b) breakthrough time

That is to say, by an increase in the yield stress of the waxy crude oil, less time is needed by the water to reach the wellbore. So yield stress of waxy crude oils has a negative impact on the water-flooding operation, at least as far as homogeneous reservoirs are concerned.

To examine the effect of \(G\) on the total volume of the oil recovered by water injection, in Fig. 6, we have shown the time variation of the recovered oil. As can be seen in this figure, by an increase in the threshold pressure gradient, less oil is retrieved from the porous media. As mentioned earlier, the magnitude of ultimate saturation at the inlet section depends on \(G\). With the progress of time, the water saturation gradually increases in the whole domain. But the water saturation cannot exceed the ultimate saturation at the inlet no matter how long the water injection process

FIG. 6: Effect of threshold pressure gradient, \(G\), on the volume of the recovered oil in the homogenous Buckley–Leverett problem
continues. So the total volume of the porous medium saturated by the water (i.e., the total volume of the recovered oil) is restricted by the ultimate saturation at the inlet and thus depends on $G$. Higher values of $G$ mean a lower inlet saturation (see Fig. 6). This means that a greater portion of the oil now behaves like a solid plug, and so it stays stationary in the reservoir. The correlation between the recovered oil and the ultimate saturation at the inlet, and also its dependence on $G$, is nothing new and has previously been reported by Wu et al. (1992).

Figure 7(a) shows the effect of the plastic viscosity on the water saturation profile, the breakthrough time, and the amount of recovered oil. As can be seen in this figure, the maximum saturation for the water phase does not change noticeably by an increase in the plastic viscosity. But any increase in the plastic viscosity lowers the local oil velocity and, at the same time, accelerates the water front advection so that the breakthrough time is decreased [see Fig. 7(b)]. Indeed, as can be seen in Fig. 8, the plastic viscosity does not affect the total volume of the recovered oil, although

**FIG. 7:** Effect of the plastic viscosity on the homogenous Buckley–Leverett problem: (a) water saturation distribution at $t = 10$ hours and (b) breakthrough time

**FIG. 8:** Effect of the plastic viscosity on the volume of the oil recovered in the homogenous Buckley–Leverett problem
it delays the recovery process. This was expected, because the total volume of recovered oil depends on the ultimate saturation (at the inlet section), which depends on yield stress, not the plastic viscosity of the displaced fluid.

The preceding results have been obtained for the homogeneous reservoirs. To see how nonhomogeneity of real reservoirs can affect the displacement of Bingham fluids, in Fig. 9, we have shown the effect of $G$ on the saturation distribution and also the volume of the recovered oil in a typical composite reservoir, as depicted schematically in Fig. 1. As can be seen in Fig. 9(a), there is a jump in the saturation profile at the dividing boundary of the two subdomains as caused by the discontinuity in their relative permeability. A comparison between Fig. 9 and Fig. 6 reveals that the volume of the recovered oil is increased for this specific composite reservoir. As mentioned earlier, for homogeneous reservoirs, the amount of oil recovered is restricted by the ultimate saturation at the inlet. But, for the composite porous medium depicted in Fig. 1, subdomain 2 has different absolute and relative permeabilities. This changes the ultimate saturation at the inlet section of subdomain 2. Because in this specific reservoir, the permeability of subdomain 2 is 1/10 that of subdomain 1, the water phase becomes less mobile, and its saturation increases significantly at the inlet section of subdomain 2. The increase in the ultimate saturation of subdomain 2 leads to an increase in the volume of recovered oil with respect to the homogeneous reservoir. That is, although the effect of yield stress is qualitatively the same for both types of wells, still more oil can be recovered if permeability decreases toward the outlet section.

### 4.2 The Five-Spot Problem

The simplicity of the viscosity regularization technique can enable us to study the flow of yield stress fluids through multidimensional porous domains. As an example, we tackle the well-known two-dimensional, homogeneous five-spot water-flooding problem (Albright et al., 1979; Geiger et al., 2004) in which a square domain initially saturated with Bingham oil is considered. As a result of water injection (at a constant flow rate) into a well located at the bottom left corner of the domain, the oil is eventually extracted from the upper right corner of the square domain (see Fig. 10).

We have tried different grid sizes and reached the conclusion that a $25 \times 25$ grid is sufficient for obtaining mesh-independent results for this particular fluid mechanics problem. For the five-spot problem, the pertinent parameters are set as shown in Table 2.

Figure 11 shows a sequence of the water saturation contours at four different time instances. As can be seen in this figure, the IMPES numerical method equipped with the viscosity regularization scheme is well capable of

![FIG. 9: Effect of threshold pressure gradient, $G$, on the heterogeneous Buckley–Leverett problem: (a) water saturation distribution after $t = 15$ hours and (b) volume of recovered oil](image-url)
resolving the diagonal displacement of the heavy crude oil as soon as water is injected. The water front is seen to travel through the domain until it reaches the extraction well at the breakthrough time of roughly 7.9 hours. As can be seen in Fig. 11, during the initial stages of the injection process, the saturation front is circularly shaped, which is in qualitative agreement with the five-spot problem for Newtonian fluids (Albright et al., 1979; Geiger et al., 2004). As time progresses, the saturation front stretches and aligns itself in the diagonal direction before reaching the low-pressure extraction well at the upper right of the domain. This notion can best be seen in Fig. 12, which shows pressure contours at \( t = 10 \) hours.

As a final note, it is of interest to examine the effect of the fluid’s yield stress (or \( G \)) on the distribution of the water-phase saturation in the five-spot water-flooding problem. Figure 13 shows the contours of the water saturation for three values of \( G = 5, 10, \) and 20 kPa/m. As can be seen in this figure, by an increase in the yield, the injected water flows closer to the diagonal of the fluid domain (i.e., where the pressure gradient is maximum) to counteract the intensified flow resistance of the heavy oil. (It can be shown that, similar to the case of the one-dimensional Buckley–Leverett problem, due to a larger diffusion length for the water front, the maximum of water saturation is decreased by an increase in \( G \).)

5. CONCLUDING REMARKS

In this work, the flow and displacement of a viscoplastic waxy crude oil by water was numerically investigated in a homogeneous and nonhomogeneous porous medium. The oil was assumed to obey the Bingham model as its constitutive equation. We have relied on the Darcy’s law in its original form for the analysis, with the only difference being

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**TABLE 2:** Input data for the homogenous five-spot problem for the N/B case

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the square-shaped domain</td>
<td>( L )</td>
<td>1.0 m</td>
</tr>
<tr>
<td>Porosity</td>
<td>( \phi )</td>
<td>0.2</td>
</tr>
<tr>
<td>Permeability</td>
<td>( k )</td>
<td>1.0 Darcy</td>
</tr>
<tr>
<td>Oil threshold pressure gradient</td>
<td>( G )</td>
<td>( 10.0 \times 10^{-3} ) Pa.m(^{-1})</td>
</tr>
<tr>
<td>Oil plastic viscosity</td>
<td>( \mu_B )</td>
<td>0.004 Pa.s</td>
</tr>
<tr>
<td>Volumetric flow rate of injection</td>
<td>( Q_{in} )</td>
<td>( 2.0 \times 10^{-6} ) m(^3).s(^{-1})</td>
</tr>
<tr>
<td>Residual water saturation</td>
<td>( S_{wr} )</td>
<td>0.0</td>
</tr>
<tr>
<td>Residual oil saturation</td>
<td>( S_{or} )</td>
<td>0.2</td>
</tr>
</tbody>
</table>
FIG. 11: Contours of water saturation for the five-spot problem obtained at (a) $t = 1$ hour, (b) $t = 2$ hours, (c) $t = 5$ hours, and (d) $t = 10$ hours

FIG. 12: Pressure contours for the five-spot problem obtained at $t = 10$ hours
that the Newtonian viscosity in this law was replaced with an effective viscosity to take care of the non-Newtonian nature of the oil. Owing to the nondifferentiability of the Bingham model, we have resorted to the regularized versions of this model to determine its effective viscosity. Three different regularized models were tried, and among them, Papastasiou’s regularization method was realized to be the most accurate. To further increase the accuracy level, we have modified the bundle-of-tubes model using the Rabinowitsch correction factor. Numerical results obtained using the IMPES method were found to be in good agreement with an asymptotic analytical solution found for the Bingham fluids in the classic homogenous Buckley–Leverett problem, except close to the inlet section, where the Rabinowitsch correction factor utilized in the present work renders our results more accurate. All in all, the oil’s yield stress is predicted to have a negative effect on the water-flooding operation as it lowers the volume of the recovered oil; it also reduces the breakthrough time. The negative effect of the yield stress on the displacement operation remains qualitatively the same for nonhomogeneous wells, although for such wells, the sweep efficiency is larger if the permeability decreases toward the production well. The methodology was then extended to the two-dimensional five-spot water-flooding problem with relative ease. It was found that the negative effect of the yield stress on the oil recovery is qualitatively the same as that for the Buckley–Leverett problem. In general, the work shows that the

FIG. 13: Effect of $G$ on the water saturation distribution for the homogenous five-spot problem after $t = 5$ hours from the start of water imbibition for (a) $G = 5$ kPa/m, (b) $G = 10$ kPa/m, and (c) $G = 20$ kPa/m
IMPES numerical method combined with Papanastasiou’s regularized model is well capable of simulating the displacement flows for the Bingham fluids. An important aspect of the method is that it can also handle heterogeneous wells.

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REFERENCES


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