Oxygen diffusion in a spherical cell subject to nonlinear Michaelis–Menten kinetics: Mathematical analysis by two exact methods

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A nonlinear model representing oxygen diffusion accompanied by the Michaelis–Menten consumption kinetics inside a spherical cell is solved analytically by the differential transform method (DTM) and the modified Adomian decomposition method (MADM). A perfect agreement between the literature data and the results from the proposed

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solutions is found. The advantages and drawbacks of the two approaches are discussed and their efficiencies are compared through a CPU-time analysis.

Keywords: Cellular oxygen transfer; mathematical modeling; differential transform method; modified Adomian decomposition method.

Mathematics Subject Classification 2010: 92Bxx, 34Bxx, 34L30, 34K07

1. Introduction

An original mathematical modeling on oxygen transfer inside a single spherical cell involving nonlinear uptake is due to Lin [1]. The work was later revisited by McElwain to reveal its erroneous approach [2]. Subsequent to these two initial conflicting studies, a number of efforts have been made to pursue the solution to the mentioned model approximately [3–7]. Of recent related works, reference can be made of Rashidinia et al. who have exploited non-polynomial spline to numerically tackle a similar problem [8]. Additionally, Lima and Morgado have reported on application of shooting method algorithms to pursue numerical solutions to the problem [9]. Most recently, Simpson and Ellery have managed to implement McLaurin series expansion to build analytical approximates to the model successfully [10].

Here, we present the mentioned model by establishing a steady-state mass balance over a spherical shell of infinitesimal thickness while assuming a generalized Michaelis–Menten uptake mechanism as:

\[
\frac{d^2 c}{dr^2} + \frac{2}{r} \frac{dc}{dr} = \frac{\alpha c}{\beta + c},
\]

(1.1)

\[
BC1 : \frac{dc(0)}{dr} = 0,
\]

(1.2)

\[
BC2 : \frac{dc(1)}{dr} = H(1 - c(1)),
\]

(1.3)

where the first boundary condition stems from symmetric oxygen distribution at the center of the cell and the second one implies a linear driving force (LDF) rate expression at cell’s external surface. The known parameter \(H\) accounts for membrane permeability.

It is the goal of this paper to explore the analytical solution of the discussed mathematical model via the differential transform method (DTM) and the modified Adomian decomposition method (MADM) through a comparative study. In case for utilization of the DTM, we have devised a helpful theorem and some corollaries to construct a recurrence, which ultimately affords the sought-after solution. However, the analysis by the MADM seems to be more routine. As it will be shown in the sequel, the both methods lead to an identical solution which matches the numerical data from the literature in an excellent way. It should be noted that Eq. (1.1) can possibly be solved by the homotopy perturbation method and the variational iteration method as well, however it is beyond the scope of this paper and the interested reader is referred to [11–17] to learn about these two methods.
2. Basics of the DTM

The DTM was initially conceptualized independently by Zhou [18] and Pukhov [19] in 1986. From then on, it has been enhanced and applied to yield series solutions of a broad class of functional equations or systems of equations in various fields of science and engineering [20–38]. We restrict our focus on one-dimensional differential transform and omit its generalizations to the n-dimensional space as it is beyond the scope of this paper. The interested reader is referred to [28] for further details in this regard.

The one-dimensional differential transform and its relevant inverse transform are defined by the following formulae, respectively:

\[ D_T \{ c(r) \} = \hat{c}(k) = \frac{d^k c(r)}{dr^k} \bigg|_{r=0}, \]  
\[ D_T^{-1} \{ \hat{c}(k) \} = c(r) = \sum_{k=0}^{\infty} \hat{c}(k)r^k, \]

where \( c(r) \) is an analytic function and \( \hat{c}(k) \) denotes its transformed function. For the sake of brevity, we let \( D_T(\cdot) \) and \( D_T^{-1}(\cdot) \) represent direct and inverse differential transform operators, orderly.

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(r) \pm v(r) )</td>
<td>( \hat{u}(k) \pm \hat{v}(k) )</td>
</tr>
<tr>
<td>( au(r) )</td>
<td>( a\hat{u}(k) )</td>
</tr>
<tr>
<td>( \frac{d^m u(r)}{dr^m} )</td>
<td>( \frac{d^m}{dr^m} \hat{u}(k+m) )</td>
</tr>
<tr>
<td>( u(r)v(r) )</td>
<td>( \sum_{l=0}^{k} \hat{u}(l)\hat{v}(k-l) )</td>
</tr>
<tr>
<td>( r^m )</td>
<td>( \delta(k-m) = \begin{cases} 1; &amp; k = m, \ 0; &amp; k \neq m \end{cases} )</td>
</tr>
<tr>
<td>( \exp(ar) )</td>
<td>( \frac{a^k}{k!} )</td>
</tr>
<tr>
<td>( \sin(ar) )</td>
<td>( \frac{a^k}{k!} \sin \left( \frac{k\pi}{2} \right) = \begin{cases} 0; &amp; k \text{ is even}, \ (-1)^{\frac{k-1}{2}} \frac{a^k}{k!}; &amp; k \text{ is odd} \end{cases} )</td>
</tr>
<tr>
<td>( \cos(ar) )</td>
<td>( \frac{a^k}{k!} \cos \left( \frac{k\pi}{2} \right) = \begin{cases} 0; &amp; k \text{ is even}, \ (-1)^{\frac{k-1}{2}} \frac{a^k}{k!}; &amp; k \text{ is odd} \end{cases} )</td>
</tr>
<tr>
<td>( r^n u(r); \ n \in \mathbb{Z} )</td>
<td>( \begin{cases} 0; &amp; k &lt; n, \ \hat{u}(k-n); &amp; k \geq n \end{cases} )</td>
</tr>
<tr>
<td>( N(u(r)) )</td>
<td>( \Theta_k(\hat{u}(0), \hat{u}(1), \hat{u}(2), \ldots, \hat{u}(k)) )</td>
</tr>
<tr>
<td>( f(r) = \int_{r_0}^{r} u(t)dt )</td>
<td>( \hat{f}(k) = \begin{cases} 0; &amp; k = 0, \ \frac{\hat{u}(k-1)}{k}; &amp; k \geq 0 \end{cases} )</td>
</tr>
<tr>
<td>( f(r) = \int_{r_0}^{r} g(t)u(t)dt )</td>
<td>( \hat{f}(k) = \begin{cases} 0; &amp; k = 0, \ \frac{1}{k} \sum_{l=0}^{k-1} { \hat{g}(l)\hat{u}(k-l-1) }; &amp; k \geq 0 \end{cases} )</td>
</tr>
</tbody>
</table>

\(^1\)The \( \Theta_k \) are the Adomian polynomials decomposing the nonlinear operator \( N \). In other words, \( \sum_{i=0}^{\infty} = N(u(r)) \).
Regarding Eqs. (2.1) and (2.2), special mention can be made of their inherent simplicity, especially for the inverse transform. A number of fundamental operations of the one-dimensional differential transform are listed in Table 1. Note that \( \delta \) symbolizes the Kronecker delta function and \( m \) is a non-negative integer. One can easily derive most of the proofs from Eqs. (2.1) and (2.2) or by consulting [29] and the references therein.

3. Mathematical Analysis by the DTM

First, let us establish a theorem which will come in handy during the treatment of Eq. (1.1) by the DTM.

**Theorem 3.1.** Given \( DT\{ f(r) \} = \hat{f}(k) \), it holds for any non-negative integer \( m \) that

\[
DT\{ r^m f(r) \} = \left\{ \begin{array}{ll}
\hat{f}(k-m); & k \geq m, \\
0; & 0 \leq k < m.
\end{array} \right.
\]  

(3.1)

**Proof.** Set \( u(r) = f(r) \) and \( v(r) = r^m \) in the fourth property and then utilize the fifth property mentioned in Table 1 to yield

\[
DT\{ r^m f(r) \} = \sum_{i=0}^{k} \{ \hat{f}(i) \delta(k-i-m) \}.
\]  

(3.2)

According to the definition of the Kronecker delta function, all components of the preceding summation become zero except for the one with \( i = k - m \). Therefore,

\[
DT\{ r^m f(r) \} = \hat{f}(k-m).
\]  

(3.3)

Since \( i \geq 0 \), Eq. (3.3) is valid for \( k-m \geq 0 \) or \( k \geq m \). To find differential transform of \( r^m f(r) \) for \( 0 \leq k < m \), we have to refer to Eq. (2.1):

\[
DT\{ r^m f(r) \} = \frac{1}{k!} \left[ \frac{d^k(r^m f(r))}{dr^k} \right]_{r=0}.
\]  

(3.4)

It follows from the general Leibniz rule that

\[
DT\{ r^m f(r) \} = \frac{1}{k!} \left[ \sum_{i=0}^{k} \binom{k}{i} \left( \frac{d^i(r^m)}{dr^i} \right) \frac{d^{k-i}f(r)}{dr^{k-i}} \right]_{r=0},
\]  

(3.5)

where \( \binom{k}{i} \) is the binomial coefficient.

Since \( 0 \leq k < m \), it is resulted that \( \frac{d^i(r^m)}{dr^i} \bigg|_{r=0} = 0 \), therefore \( DT\{ r^m f(r) \} = 0 \) for \( 0 \leq k < m \). Hence, we conclude the proof.

In addition, the following corollaries are easily resulted from the preceding theorem along with the operations summarized in Table 1:

\[
DT \left\{ r \frac{d^2 c}{dr^2} \right\} = k(k+1)\hat{c}(k+1),
\]  

(3.6)
or equally

\[ D_T \left\{ \frac{d^2 c}{dr^2} \right\} = \sum_{l=0}^{k} \{ l(l+1) \hat{c}(l+1) \hat{c}(k-l) \}. \quad (3.7) \]

We get the following equation by cross-multiplication of Eq. (1.1):

\[ D_T \{ rc \} = \hat{c}(k-1); \quad k \geq 1 \quad (3.8) \]

and

\[ D_T \left\{ \frac{dc}{dr} \right\} = \sum_{l=0}^{k} \{ (l+1) \hat{c}(l+1) \hat{c}(k-l) \}. \quad (3.9) \]

Back to our problem under investigation, one may get frustrated by the severity of the nonlinear expression included in Eq. (1.1) and fail to derive its differential transform. This obstacle can be overcome by a simple action: cross-multiplication. We get the following equation by cross-multiplication of Eq. (1.1):

\[ \beta r \frac{d^2 c}{dr^2} + r c \frac{d^2 c}{dr^2} + 2 \beta \frac{dc}{dr} + 2c \frac{dc}{dr} = \alpha r c. \quad (3.10) \]

In view of Eqs. (3.6)–(3.9), we take the differential transform from both sides of Eq. (3.10) to obtain

\[
\beta k(k + 1) \hat{c}(k + 1) + \sum_{l=0}^{k} \{ l(l+1) \hat{c}(k-l) \} + 2\beta(k+1) \hat{c}(k+1)
\]

\[
+ 2 \sum_{l=0}^{k} \{ (l+1) \hat{c}(l+1) \hat{c}(k-l) \} = \alpha \hat{c}(k-1); \quad k \geq 1, \quad (3.11)
\]

or equally

\[
\hat{c}(k+1) = \frac{\alpha \hat{c}(k-1) - \sum_{l=0}^{k-1} \{ (2l+1)(l+1) \hat{c}(l+1) \hat{c}(k-l) \}}{(\hat{c}(0) + \beta)(k^2 + 3k + 2)}; \quad k \geq 1. \quad (3.12)
\]

The transformed analogue of Eq. (3.2) suggests that \( \hat{c}(1) = 0 \). Recursively, from Eq. (3.12) it follows that:

\[
\hat{c}(2) = \frac{1}{6} \frac{\alpha \hat{c}(0)}{\hat{c}(0) + \beta}, \quad \hat{c}(3) = 0,
\]

\[
\hat{c}(4) = \frac{1}{20} \frac{\alpha \hat{c}(2) - 6[\hat{c}(2)]^2}{\hat{c}(0) + \beta} = \frac{1}{120} \frac{\alpha^2 \hat{c}(0)}{[\hat{c}(0) + \beta]^3}, \quad \hat{c}(5) = 0,
\]

\[
\hat{c}(6) = \frac{1}{42} \frac{\alpha \hat{c}(4) - 26 \hat{c}(2) \hat{c}(4)}{\hat{c}(0) + \beta} = \frac{1}{15120} \frac{\alpha^3 \beta \hat{c}(0)[3\beta - 10 \hat{c}(0)]}{[\hat{c}(0) + \beta]^5}, \quad (3.13)
\]

\[
\hat{c}(7) = 0, \quad \hat{c}(8) = \frac{1}{72} \frac{\alpha \hat{c}(6) - 48 \hat{c}(2) \hat{c}(6) - 20[\hat{c}(4)]^2}{\hat{c}(0) + \beta}
\]

\[
= \frac{1}{1088640} \frac{\alpha^4 \beta \hat{c}(0)[70 \hat{c}(0)]^2 - 52 \beta \hat{c}(0) + 3[\hat{c}(0)]^2}{[\hat{c}(0) + \beta]^7}, \quad \hat{c}(9) = 0,
\]

\[
\vdots
\]
Table 2. Oxygen concentration in the cell at $r = 0$ for various values of $\alpha$, $\beta$, and $H$: Results by the DTM (or equally the MADM) and data from literature.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$H$</th>
<th>$c(0)$ {from [9]}</th>
<th>Solution by the DTM (or equally the MADM)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$c(0), O(r^2)$</td>
</tr>
<tr>
<td>#1</td>
<td>0.38065</td>
<td>0.03119</td>
<td>5.0</td>
<td>0.91404</td>
<td>1</td>
</tr>
<tr>
<td>#2</td>
<td>0.38065</td>
<td>0.03119</td>
<td>0.5</td>
<td>0.69583</td>
<td>1</td>
</tr>
<tr>
<td>#3</td>
<td>0.76129</td>
<td>0.03119</td>
<td>5.0</td>
<td>0.82848</td>
<td>1</td>
</tr>
<tr>
<td>#4</td>
<td>0.38065</td>
<td>0.31187</td>
<td>5.0</td>
<td>0.93311</td>
<td>1</td>
</tr>
</tbody>
</table>
Therefore,
\[ c(r) = \hat{c}(0) + \hat{c}(2)r^2 + \hat{c}(4)r^4 + \hat{c}(6)r^6 + \hat{c}(8)r^8 + O(r^{10}). \quad (3.14) \]

To complete the solution it is only needed to determine the free parameter namely \( \hat{c}(0) \), which equates the oxygen concentration in the center of the cell, in Eq. (3.14). This end is achieved by invoking the second boundary condition of the governing equation, i.e. Eq. (1.3). In order to study the convergence behavior of the aforementioned free parameter, the polynomial solution (Eq. (3.14)) with truncation orders ranging from \( O(r^2) \) to \( O(r^8) \) is substituted into Eq. (1.3) and numerically solved, e.g. by the Newton–Raphson method, for \( \hat{c}(0) \). From Eq. (3.14) it is obvious that \( \hat{c}(0) = c(0) \). The results for different values of constants \( \alpha, \beta \), and \( H \), i.e. four different case studies, are presented in Table 2. As it is observed \( c(0) \) is rapidly convergent for all the four cases and the results by the DTM perfectly fit the approximates available in the literature; see the fourth column of Table 2.

4. Fundamentals of the MADM

As the MADM is actually a refinement of the ADM, we start with a quick review over the latter technique to explain the way to construct the MADM solution of Eq. (1.1).

The Adomian decomposition method was first developed and promoted by the renowned applied mathematician, Professor George Adomian (1922–1996) in mid-1980s [39]. Being so powerful in handling various types of functional equations, the ADM has enjoyed a great deal of fame and popularity among the users of mathematics, either scientists or engineers. The literature abounds with studies concerning applications of the ADM in applied mathematics, physics, mechanical engineering, aerodynamics, heat transfer and chemical engineering, just to mention a few areas [40–67].

In what immediately follows, we present a brief review over the methodology of the ADM for the ease of the reader.

Think of a general functional equation as follows:
\[ L(u) + N(u) + R = g, \quad (4.1) \]

where \( L(\cdot) \) is chosen as an easily invertible linear operator, \( N(\cdot) \) is a nonlinear operator, which maps a Hilbert space \( H \) to \( H \), and \( R(\cdot) \) denotes the remainder linear operator, and obviously \( u \) represents the unknown function. By defining the inverse operator of \( L(\cdot) \) as \( L^{-1}(\cdot) \), we conclude that
\[ L^{-1}(L(u)) + L^{-1}(N(u)) + L^{-1}(R) = L^{-1}(g). \quad (4.2) \]

Now, choosing \( L(\cdot) \) as an \( n \)th-order derivative operator, \( L^{-1}(\cdot) \) becomes an \( n \)-fold integration operator. Thus, it is followed that \( L^{-1}(L(u)) = u - a \), where \( a \) comprises the constants of integrations. The ADM proposes the final solution in the form of \( u = \sum_{n=0}^{\infty} u_n \); that is why it is called the decomposition method. Letting
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\[ u_0 = L^{-1}(g) + a, \text{ Eq. (4.2) yields} \]

\[ u = u_0 - L^{-1}(N(u)) - L^{-1}(R). \quad (4.3) \]

To proceed, the nonlinearity \( N(u) \) shall be decomposed into an especial infinite series expansion known as the Adomian polynomials as

\[ N(u) = \sum_{n=0}^{\infty} A_n, \quad (4.4) \]

where the \( A_n \) are defined as [68]:

\[ A_n = A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}. \quad (4.5) \]

A number of newer alternative schemes for derivation of the Adomian polynomials are worthy of [69–73].

Thus, a recurrence can be established to calculate the remnant solution components as

\[ u_{i+1} = -L^{-1}(A_i) - L^{-1}(R(u_i)); \quad i \geq 0. \quad (4.6) \]

It is worthwhile to mention that in certain cases such as singular boundary value problems, the classic ADM fails due to its choice of forward and inverse linear operators, that is when the MADM proposes a suitable candidate for operators \( L(\cdot) \) and \( L^{-1}(\cdot) \). According to the MADM [74, 75], the suitable forward and inverse linear operators for an ordinary differential equation in the form of

\[ \frac{d^2y}{dx^2} + \frac{\alpha}{x} \frac{dy}{dx} + f(x, y) = 0, \quad (4.7) \]

subject to

\[ \frac{dy(0)}{dx} = 0, \quad Ay(1) + B \frac{dy(1)}{dx} = C, \quad (4.8) \]

correspond respectively to

\[ L(\cdot) = x^{-\alpha} \frac{d(x^\alpha \frac{d(\cdot)}{dx})}{dx}, \quad (4.9) \]

and

\[ L^{-1}(\cdot) = \int_0^x x^{-\alpha} \int_0^x x^\alpha (\cdot) dx dx. \quad (4.10) \]

5. Mathematical Analysis by the MADM

Based on what is discussed above, we nominate the following operators for Eq. (1.1):

\[ L(\cdot) = r^{-2} \frac{d(r^2 \frac{d(\cdot)}{dr})}{dr}, \quad (5.1) \]

\[ L^{-1}(\cdot) = \int_0^r r^{-2} \int_0^r r^2 (\cdot) dr dr. \quad (5.2) \]
Consequently,

\[ c(r) = c(0) + \int_0^r r^{-2} \int_0^r r'^2 \frac{\alpha c}{\beta + c} drdr. \]  

(5.3)

According to the ADM, we let

\[ c(r) = \sum_{i=0}^{+\infty} c_i \]  

(5.4)

and obtain the solution via the following recurrence

\[
\begin{cases}
  c_0 = c(0), \\
  c_{k+1} = \int_0^r r^{-2} \int_0^r r'^2 A_k drdr; \quad k \geq 0,
\end{cases}
\]

(5.5)

where the \( A_k \) denote the Adomian polynomials relating to the nonlinearity \( \frac{\alpha c}{\beta + c} \).

We can list them as:

\[
A_0 = \frac{\alpha c_0}{\beta + c_0}, \quad A_1 = \frac{\alpha c_1}{\beta + c_0} - \frac{\alpha c_0 c_1}{(\beta + c_0)^2},
\]

\[
A_2 = \frac{\alpha c_2}{\beta + c_0} - \frac{\alpha c_1^2}{(\beta + c_0)^2} + \frac{\alpha c_0 c_2}{(\beta + c_0)^3},
\]

\[
A_3 = \frac{\alpha c_3}{\beta + c_0} - \frac{2\alpha c_1 c_2}{(\beta + c_0)^3} + \frac{\alpha c_0 c_3}{(\beta + c_0)^4} + \frac{2\alpha c_2 c_1 c_2}{(\beta + c_0)^5} - \frac{\alpha c_0 c_1}{(\beta + c_0)^5}.
\]

(5.6)

So,

\[
c_1 = \frac{1}{6} \frac{\alpha c_0}{\beta + c_0} r^2, \quad c_2 = \frac{1}{120} \frac{\alpha^2 c_0 c_0}{(\beta + c_0)^2} r^4,
\]

\[
c_3 = \frac{1}{15120} \frac{\alpha^3 c_0 (3\beta - 10c_0)}{(\beta + c_0)^5} r^6,
\]

\[
c_4 = \frac{1}{1088640} \frac{\alpha^4 c_0 (70c_0^2 - 52\beta c_0 + 3\beta^2)}{(\beta + c_0)^7} r^8,
\]

(5.7)

\[
\vdots
\]

Hence, the final solution reads:

\[
c(r) = c_0 + \frac{1}{6} \frac{\alpha c_0}{\beta + c_0} r^2 + \frac{1}{120} \frac{\alpha^2 c_0 c_0}{(\beta + c_0)^2} r^4 + \frac{1}{15120} \frac{\alpha^3 c_0 (3\beta - 10c_0)}{(\beta + c_0)^5} r^6
\]

\[
+ \frac{1}{1088640} \frac{\alpha^4 c_0 (70c_0^2 - 52\beta c_0 + 3\beta^2)}{(\beta + c_0)^7} r^8 + O(r^{10}),
\]

(5.8)

with \( c_0 = c(0) \).
Comparing Eq. (5.8) with Eq. (3.14), one clearly concludes that the DTM and the MADM have both provided an identically similar solution. Furthermore, it is remarkable that unlike the DTM, the MADM has automatically discarded its trivial solution components. For illustration, Fig. 1 plots the oxygen concentration profiles throughout the cell for the four case studies characterized in Table 2. To provide insights into the convergence behavior of the solution yielded from the DTM (or equally the MADM), Fig. 2 plots the oxygen concentration, typically at

![Fig. 1](image1)

**Fig. 1.** Oxygen concentration profiles obtained by the DTM (or equally the MADM) with truncation error order of $O(r^{10})$ for the four case studies characterized in Table 2.

![Fig. 2](image2)

**Fig. 2.** Convergence behavior of the four case studies (see Table 2): The oxygen concentration on a sphere of radius $r = 0.5$ inside the cell obtained from the truncated DTM (or equally the truncated MADM) series solution while keeping only its first $n$ nonzero terms.
Fig. 3. Comparison of the CPU times required for obtaining the first $n$ nonzero terms of the series solution to the model, Eq. (1.1), by the DTM and the MADM.

$r = 0.5$, obtained from the series solution truncated after $n$ terms. As shown, a rapid convergence is achieved for all the case studies; i.e. the concentration variations vanish sharply as $n$ exceeds four.

In order to compare the computational speed of the two applied solution strategies, a CPU time analysis was conducted on a personal computer with a 2.66 GHz processor and 2 GB of RAM using MATLAB 7.0. The results of this comparison are depicted in Fig. 3. It is observed from Fig. 3 that the DTM solves Eq. (1.1) faster than the MADM especially as the parameter $n$ increases. This can be justified by the fact that the computation of the solution components by the MADM is subject to a two-folded integration as well as evaluation of the Adomian polynomials while the DTM merely requires simple arithmetic operations in course of the solution; compare Eq. (5.5) with Eq. (3.12). It is also worthwhile to recall that the evaluation of the Adomian polynomials involves high-order differentiations.

6. Conclusion

In this study, we have managed to apply two powerful mathematical methods to provide convenient analytical solutions for a nonlinear differential equation arising in biology. To enable the DTM treat the nonlinearity entailed in the model, the governing ODE was cross-multiplied to generate its variant involving some new product expressions and nonlinearities. The differential transform of the terms in the latter equation was computed with the help of a presented theorem and a couple of corollaries. Finally, the desired solution was reverted from $k$-domain to that of its original. The analysis by the MADM did not require such manipulations and hence is evaluated to be more straightforward. However, this advantage comes with the penalty of computation of the Adomian polynomials which even though cannot be
considered as a demerit in the presence of present rapid computers. Amazingly, it was revealed that the both methods furnish identical solutions that perfectly match numerical approximates from the literature. According to our CPU time analysis, it was revealed that the MADM solves the model slower since it entails two-folded integrations and computation of the Adomian polynomials in course of the solution while the DTM merely involves a number of simple arithmetic operations.

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References
Oxygen diffusion in a spherical cell


Oxygen diffusion in a spherical cell


