The asymmetric elastic wavefields in a model comprising a liquid layer overlying an anisotropic solid seabed due to an arbitrary source within the solid

Amirhossein Bagheri¹, Ali Khojasteh b, Mohammad Rahimian a,⁎, Stewart Greenhalgh c, Reza Attarnejad a  

a Department of Civil Engineering, University of Tehran, P.O. Box 11155–4563, Tehran, Iran  
b School of Engineering Science, College of Engineering, University of Tehran, P.O. Box 11155–4563, Tehran, Iran  
c Institute of Geophysics, ETH Zurich, Sonneggstrasse 5, Zurich 8092, Switzerland

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The asymmetric three-dimensional radiation pattern and resultant elastodynamic response of stress waves in a model comprising a compressible water column overlying a transversely isotropic seabed in which a time-harmonic source acts is theoretically investigated. The use of potential functions, the Hankel transform, and a Fourier series expansion are adopted to deal with the equations of motion for both media. Closed-form integral expressions are developed for the potentials and the stress/displacement components. The expressions and introduced procedure are sufficiently flexible to incorporate various types of source loads. To evaluate the field quantities, the residue method and a robust integration scheme are utilized to handle the poles and branch points within the integrand. Any possible number of dispersive propagation modes are taken into account in the integral evaluation. The deduced velocity dispersion curves depict the characteristics of the various modes. They also indicate the existing singular points (poles) for a specific dimensionless frequency and the surface wave type associated with each pole. Numerical results are presented for the hydrodynamic pressure and displacement in the liquid layer and stress and displacement components in the solid seabed due to distributed and concentrated source excitations. The formulation and the numerical scheme are valid for calculating the wavefield anywhere within the model including both far- and near-field effects. The sensitivity of the results to different parameters is also analyzed. Both analytical and numerical comparisons with existing solutions for simpler cases are made to confirm the validity of the results. The results are especially useful in seismic hazard assessment of submarine earthquakes, landslides, and tsunamis. They can also be extended to deal with the fluid-solid-structure interaction problems.

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1. Introduction

Modelling the radiation pattern and resultant elastic wave response of a liquid layer overlying a solid medium due to an external excitation within the solid is encountered in various fields. In particular, one can refer to undersea seismic exploration, earthquake-hazard assessment, structural engineering, marine dynamics, tsunami generation and offshore hydrocarbon production. Stoneley [1] performed pioneering work on the effect of the ocean on the seismic response of a contacting solid seabed. Other notable contributors in the early development of wave propagation in liquid-solid systems include Biot [2], Haskell [3], Tolstoy [4], Ewing et al. [5], and Bennett and Hermann [6]. Until now, numerous scientists have investigated wave motion and dispersion in such models. Roberts [7] considered the dynamic elastic response of an isotropic solid half-space in contact with a liquid half-space due to a three-dimensional point load acting in either the solid or in the fluid. Novikova et al. [8] studied analytically a water column overlying a layered, isotropic solid in connection with
the tsunamis and the Rayleigh wave propagation process. Zhu et al. [9] and Rodriguez-Castellanos et al. [10,11] also utilized the integral transform method and boundary element techniques, respectively, to produce curves of pressure variation in the water in contact with an isotropic solid half-space. Mohapatra and Sahoo [12] studied the interaction of the surface waves with an elastic bed utilizing small amplitude water wave theory and the plate deflection assumption for the solid. Das et al. [13] investigated the flexural gravity wave motion over a poroelastic bed using Biot’s consolidation theory and incorporating the effect of the ocean stratification.

Anisotropy of the solid seabed, in which the wavespeed is directionally dependent, is a common assumption made in many oceanic crustal models. Normally, the specialized form of transverse isotropy in which the axis of symmetry is vertical (a VTI medium) is assumed. It may be the result of sediment layering, existence of horizontally aligned microcracks or preferred mineral orientation. The elastodynamics of a transversely isotropic solid material has been studied by numerous authors. Stoneley [14] described the role of anisotropy in the propagation of elastic waves in a transversely isotropic solid body. Further accounts are given, for example, by Payton [15], and Rajapakse and Wang [16]. Eksandari-Ghadi [17] introduced a special potential function method to solve the equations of motion for a transversely isotropic solid. This method was utilized by Khojasteh et al. [18] to analyze the asymmetric Green’s functions for a dynamic load (source excitation) acting within a transversely isotropic body. They applied the Residue Theorem and an adaptive integration scheme for numerical evaluation of the infinite integrals. They also computed the dynamic Green’s functions for layered half- and full-spaces, comprising dissimilar transversely isotropic materials [19,20]. Pan et al. [21] also developed a complete set of exact closed-form solutions to present elastic displacements and strains due to general polygonal dislocations in a transversely isotropic half-space.

Following on from the early work of Abubakar and Hudson [22], Sharma et al. [23] tackled the dispersion of surface waves in a half-space composed of a water column overlying a layered solid media. In a subsequent paper [24], Sharma presented the wave dispersion characteristics in an ocean crust model incorporating the effect of cracks. In both papers, only the dispersive characteristics of the models are investigated, which implicitly assumes plane waves (or the far-field situation) and ignores the source term. All these studies show a significant change in wave propagation characteristics due to the presence of the liquid layer.

The main objective of this research is to find stresses and displacements induced in a coupled ocean-anisotropic seabed model due to a harmonic excitation within the seabed and to analyze the characteristics of the associated elastic waves. The solid bed is transversely isotropic and the source can be of arbitrary shape, including both finite sized and point sources. To deal with the equations of motion, the method of potential functions along with an integral (Hankel) transform and Fourier series expansion in azimuth, introduced by Khojasteh et al. [18], are extended and utilized. By appropriately expressing the source term and applying the boundary conditions, closed-form solutions for the stress and displacement components are derived in the frequency-wavenumber domain. To find the response in the actual physical (frequency-space) domain, the resulting integrals are evaluated utilizing a specific integration scheme. According to the oscillatory nature of the integrands and the existence of singularities, special attention should be paid to poles, branch points, and how to choose the path of integration.

As a considerable improvement compared to other similar works, the method employed here allows for wavefield computation at any point in the media, i.e., far from the source or near to it. Furthermore, the method takes into account the reverberations and higher modes in the liquid layer. The time-harmonic excitation can act as either a force or a displacement over a finite area. The wavefield functions associated with each case are derived by suitably adjusting the source expression on the right side of the governing equations. The source can be at an arbitrary depth within the solid bed, either shallow or deep.

By involving the liquid layer and an asymmetric source excitation, this research provides the means to deal with a wide range of problems arising in marine seismology and offshore earthquake engineering. The formulated expressions and results of this study will be useful in applications that interpret seismic amplitudes from processes such as submarine earthquakes, landslides, tsunamis, explosions, and volcanic tremors. In particular, the effect of source excitation acting within the seabed is an area of active research for tsunami-hazard assessment (see for example [25]).

Furthermore, the derived Green’s functions can be conveniently used to solve the interaction of structures located in the described model. They can be also applied in underwater seismic seabed characterization when modeling accelerograms. The elastic wavefield expressions developed in this paper are also relevant in investigations of earthquake physics. Determination of the earthquake source characteristics is of considerable importance in seismology. In this regard, Green’s functions and associated field quantities are commonly employed in inversion methods used for this purpose (see, for example [26–28]). The basic formulation given in this paper can also be conveniently used for a model of a liquid layer overlying a two layered solid seabed made up of different materials.
2. Statement of the problem and governing equations

The model under consideration, shown in Fig. 1, comprises a compressible liquid layer (ocean water) of finite thickness \( h \), overlying a homogeneous transversely isotropic solid seabed. The solid is semi-infinite in the vertical direction where both the solid and water column extend infinitely in the horizontal directions. The cylindrical coordinate system \( (r, \theta, z) \) is used with the origin located at the free surface and \( z \) axis downward. Hence, the liquid layer lies in the region defined by \( 0 < z < h \), \( r > 0 \), and \( 0 < \theta < 2\pi \), where the solid seabed lies in the region \( h < z, r > 0 \), and \( 0 < \theta < 2\pi \). As shown in Fig. 1, a harmonic excitation source of temporal angular frequency \( \omega \) acts within the solid bed at depth \( z=s > h \) over a finite region \( \Omega \) where \( s \) is measure from the free surface of the water column. In the figure, the excitation is schematically through the three components of stress or displacement acting within the excitation region. The excitation source can be natural (earthquake or landslide) or artificial (seismic source used for geophysical purposes) as explained in the previous section.

Following Aki and Richards [29], the source can be defined as a discontinuity of stress or displacement at \( z=s \) by one of the following equations:

\[
\begin{align*}
\sigma_{rr} (r, \theta, s^{-}) - \sigma_{rr} (r, \theta, s^{+}) &= \begin{cases} P (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
\sigma_{\theta\theta} (r, \theta, s^{-}) - \sigma_{\theta\theta} (r, \theta, s^{+}) &= \begin{cases} Q (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
\sigma_{zz} (r, \theta, s^{-}) - \sigma_{zz} (r, \theta, s^{+}) &= \begin{cases} R (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
\end{align*}
\]

\[(1a)\]

\[
\begin{align*}
u_r (r, \theta, s^{-}) - u_r (r, \theta, s^{+}) &= \begin{cases} P' (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
u_\theta (r, \theta, s^{-}) - u_\theta (r, \theta, s^{+}) &= \begin{cases} Q' (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
u_z (r, \theta, s^{-}) - u_z (r, \theta, s^{+}) &= \begin{cases} R' (r, \theta) & (r, \theta) \in \Omega \\ 0 & (r, \theta) \notin \Omega \end{cases} \\
\end{align*}
\]

\[(1b)\]

In the above expressions, \( \sigma_{zz} \) is the normal stress and \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) are the shear or tangential stress components, and \( u_r, u_\theta \) and \( u_z \) represent the displacement components in the radial, vertical and azimuthal directions, respectively. The functions \( P, Q, \) and \( R \) are arbitrary functions describing the stress source, where \( P', Q', \) and \( R' \) are those used to describe a known displacement source. Here, \( \Omega \) is the finite region over which the source is acting.

To find the elastodynamic response of the coupled liquid-seabed model due to the defined excitation, we utilize the equations of motion for each subdomain as given in sections 2.1 and 2.2. We then make use of the method of potential functions and the Hankel transform to deal with the equations.

2.1. Solid domain

In this section, the governing equations of motion for the homogeneous transversely isotropic solid half-space subjected to a source excitation are presented. These equations are utilized to find the stress and displacement components at any point within such a solid. In
the cylindrical co-ordinates \((r,\theta, z)\) described before, the time-harmonic equations of motion for the displacement components \(u_r\), \(u_\theta\) and \(u_z\) are given by Aki and Richards [29]:
\[
\begin{align*}
\c_{11} \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) + \c_{66} \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - 2\c_{11} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} + \\
\c_{44} \frac{\partial^2 u_r}{\partial z^2} + \frac{\c_{11} + \c_{12}}{2} \left( \frac{1}{r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) + \left( \c_{13} + \c_{44} \right) \frac{\partial^2 u_z}{\partial r \partial z} + \rho_s \omega^2 u_r = 0, \\
\c_{66} \left( \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right) + \c_{11} \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \c_{44} \frac{\partial^2 u_\theta}{\partial z^2} + \\
2\c_{11} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{\c_{11} + \c_{12}}{2} \left( \frac{1}{r} \frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) + \left( \c_{13} + \c_{44} \right) \frac{1}{r} \frac{\partial u_z}{\partial \theta} + \rho_s \omega^2 u_\theta = 0, \\
\c_{44} \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} \right) + \c_{33} \frac{\partial^2 u_z}{\partial z^2} + \left( \c_{13} + \c_{44} \right) \times \\
\left( \frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial \theta \partial z} \right) + \rho_s \omega^2 u_z = 0.
\end{align*}
\]
(2)

Here, \(c_i\) are the five elastic constants which characterize the anisotropic solid seabed and \(\rho_s\) is the mass density of the solid. The time factor \(\exp(i \omega t)\) is implicit and therefore suppressed. (Alternatively, we Fourier transform the original equation with respect to time, noting that \(\frac{\partial}{\partial t} \rightarrow i \omega\).) A pair of complete potential functions \(F\) and \(\chi\) are utilized to effectively deal with the coupled equations of motion in the solid [17]. The potential functions are related to the displacement components as follows (Khojasteh et al. [18]):
\[
\begin{align*}
u_r (r, \theta, z) &= -\alpha_3 \frac{\partial F (r, \theta, z)}{\partial z} - \frac{1}{r} \frac{\partial \chi (r, \theta, z)}{\partial \theta}, \\
u_\theta (r, \theta, z) &= -\alpha_3 \frac{1}{r} \frac{\partial^2 F (r, \theta, z)}{\partial \theta \partial z} + \frac{\partial \chi (r, \theta, z)}{\partial r}, \\
u_z (r, \theta, z) &= \left(1 + \alpha_1\right) \nabla^2 \nabla^2 \chi + \alpha_2 \frac{\partial^2 \chi}{\partial z^2} + \rho_s \omega^2 \chi \right) F (r, \theta, z),
\end{align*}
\]
(3)

where the dimensionless \(\alpha\) terms are defined in terms of the elastic moduli through the ratios:
\[
\alpha_1 = \frac{c_{12} + c_{66}}{c_{66}}, \quad \alpha_2 = \frac{c_{44}}{c_{66}}, \quad \alpha_3 = \frac{c_{13} + c_{44}}{c_{66}}.
\]
(4)

The expressions for the displacement components (Eq. (3)) are substituted into Eq. (2) to decouple the equations of motion:
\[
\left( \nabla^2 \nabla^2 + \delta \omega^2 \frac{\partial^2}{\partial z^2} \right) F = 0,
\]
\[
\nabla^2 \chi = 0.
\]
(5)

where,
\[
\nabla^2 = \nabla^2_r + \frac{1}{s_1^2} \frac{\partial^2}{\partial z^2} + \frac{\rho_s \omega^2}{\mu_0} \\
\mu_0 = 1,
\]
\[
\mu_1 = \alpha_2 = \frac{c_{44}}{c_{66}}, \quad \mu_2 = 1 + \alpha_1 = \frac{c_{11}}{c_{66}}, \\
\delta = \left[-\frac{1}{c_{44}s_2^2} \frac{1}{c_{11}s_1^2} + \frac{1}{c_{11}} \left(1 + \frac{c_{13}}{c_{44}} \right) \right].
\]
(6)

where \(s_0 = 1/\sqrt{\alpha_2}\) and quantities \(s_1\) and \(s_2\) are the roots of the strain energy function:
\[
c_{33}c_{44}s_1^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{22}c_{22})s_1^2 + c_{11}c_{44} = 0.
\]
(7)

Here, we apply the Hankel transform of \(m\)th order in the radial \((r)\) direction and use a Fourier series expansion in the azimuthal \((\theta)\) direction. This simplifies solving the PDEs and later applying the boundary conditions. The discontinuities in equations (1a) and (1b) will also be
removed by using the integral transform. Hence, the partial differential equations for the transformed Fourier series components of the potential functions, $F_m$ and $\tilde{\chi}_m$ can be expressed as:

$$\left( \hat{\psi}_{1m}^2 \hat{\psi}_{2m}^2 + \delta \omega^2 \frac{d^2}{dz^2} \right) \hat{F}_m(z) = 0,$$

$$\hat{\psi}_{0m}^2 \hat{\chi}_m^2 = 0. \quad (8)$$

The superscript on the potentials indicates the order of the Hankel transform and the subscript denotes the component order in the Fourier expansion series. In the above expressions, the terms $\delta$ and $\hat{\psi}^2$ are defined by:

$$\hat{\psi}_{im}^2 = \frac{\rho_0 \omega^2}{\mu_{equ}} - \xi^2 + \frac{1}{\xi^2} \frac{d^2}{dz^2}, \quad i = 0, 1, 2. \quad (9)$$

where $\xi$ is the Hankel transform variable (or radial wavenumber). General solutions for the transformed Fourier components of the potential functions $\hat{F}_m(\xi, z)$ and $\hat{\chi}_m(\xi, z)$ are given by:

$$\hat{F}_m(\xi, z) = A_m(\xi) e^{\lambda_1 z} + B_m(\xi) e^{-\lambda_1 z} + C_m(\xi) e^{\lambda_2 z} + D_m(\xi) e^{-\lambda_2 z},$$

$$\hat{\chi}_m(\xi, z) = E_m(\xi) e^{\lambda_3 z} + F_m(\xi) e^{-\lambda_3 z}, \quad (10)$$

where

$$\lambda_1 = \sqrt{a \xi^2 + b + \frac{1}{2} c \xi^4 + d \xi^2 + e},$$

$$\lambda_2 = \sqrt{a \xi^2 + b - \frac{1}{2} c \xi^4 + d \xi^2 + e},$$

$$\lambda_3 = \sqrt{\xi^2 - \frac{\rho_0 \omega^2}{\mu_{equ}}}. \quad (11)$$

In the above equations:

$$a = \frac{1}{2} (s_1^2 + s_2^2), \quad b = -\frac{1}{2} \rho_0 \omega^2 \left( \frac{1}{c_{33}} + \frac{1}{c_{44}} \right), \quad c = (s_1^2 - s_2^2)^2,$$

$$d = -2 \rho_0 \omega^2 \left[ \left( \frac{1}{c_{13}} + \frac{1}{c_{44}} \right) (s_1^2 + s_2^2) - 2 \frac{c_{11}}{c_{33}} \left( \frac{1}{c_{11}} + \frac{1}{c_{44}} \right) \right], \quad e = \rho_0^2 \omega^4 \left( \frac{1}{c_{33}} - \frac{1}{c_{44}} \right)^2. \quad (12)$$

The as yet unspecified coefficients $A_m, B_m, C_m, D_m, E_m,$ and $F_m$ are determined from the boundary conditions. In the above expressions, $\lambda_1, \lambda_2, \text{ and } \lambda_3$ are multi-valued functions and are made single-valued by specifying the branch cuts emanating from the branch points $\xi_{\lambda_1} = \pm \omega \sqrt{\rho_0 / c_{11}}, \xi_{\lambda_2} = \pm \omega \sqrt{\rho_0 / c_{44}}$ and $\xi_{\lambda_3} = \pm \omega \sqrt{\rho_0 / \mu_{equ}}$ in the complex horizontal wavenumber ($\xi$)-plane, such that the real parts of $\lambda_1, \lambda_2, \text{ and } \lambda_3$ are non-negative for all values of $\xi$. The positive sign of these branch points denotes the wavenumbers of the body waves, $P, SV, \text{ and } SH,$ respectively.

### 2.2. Liquid domain

In this section, the governing equation of motion for the overlying liquid layer is analyzed when subjected to a forced excitation. For a homogeneous and compressible liquid medium, the governing equation of motion in the cylindrical coordinate system is [5]:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{K}{\rho_l} \nabla^2 \psi (r, \theta, z), \quad (13)$$

where $\psi$, $K$, and $\rho_l$ are the displacement potential, bulk modulus and mass density, respectively. Assuming harmonic time variation $\exp(i \omega t)$, the Fourier transformed normal component of displacement and excess pressure can be expressed in terms of $\hat{\psi}_m^0$ as:

$$\hat{p}_{zm}^m = \frac{\partial \hat{\psi}_m^0}{\partial z},$$

$$\hat{F}_m^m = -K \nabla^2 \hat{\psi}_m^0 = \rho_l \omega^2 \hat{\psi}_m^0. \quad (14)$$

In a similar fashion to what we did for the solid medium, a Fourier series expansion is made with respect to the azimuthal coordinate. The $m$th order Hankel transform is also taken with respect to the radial coordinate to yield the following equation for the liquid layer:

$$\left( \frac{\rho_0 \omega^2}{K} - \xi^2 + \frac{d^2}{dz^2} \right) \hat{\psi}_m^0 = 0. \quad (15)$$

where again $\xi$ is the Hankel transform variable. A general solution of this equation is:

$$\hat{\psi}_m^0 = S_m(\xi) e^{-\xi^2 z} + R_m(\xi) e^{\xi^2 z}, \quad (16)$$
where:

$$\lambda_4 = \sqrt{\xi^2 - \frac{\alpha^2}{c_l^2}}$$

(17)

Here, $\lambda_4$ is a multi-valued function which is made single-valued using the same procedure mentioned previously for $\lambda_1$, $\lambda_2$ and $\lambda_3$. Quantity $c_l = \sqrt{K/\rho_l}$ is the velocity of the sound wave in the liquid. The unknown coefficients $S_m$ and $K_m$ will be determined through the boundary conditions.

### 3. Boundary conditions and frequency equation

In order to allow for the interaction between the two media and to find the response of the coupled model, the boundary conditions should be appropriately applied. The entire two-layered model is subdivided into three depth intervals (see Fig. 1): the liquid layer ($0 < z < h$) (denoted by superscript I), the part of the seabed above the source depth and lying in the depth interval $h < z < s$ (denoted by superscript II), and the part of the seabed below the source depth, $z > s$, (denoted by superscript III). Separate potential functions should be defined for each subdomain and respective unknown variables in each domain are denoted with superscripts. Hence, the total number of unknown coefficients is fourteen (six associated with part III, six associated with part II and two associated with part I). However, according to the Sommerfeld radiation condition that the field is zero at infinity (taking $z$ as a large positive value in Eq. (10)), $A_m^{III}$, $C_m^{III}$, and $D_m^{III}$ vanish.

The free-surface condition of zero pressure on the top of the liquid requires $\theta_i^I (z = 0, \xi) = 0$ which yields $S_m = -K_m$. Hence, for the reduction of the number of unknown coefficients by one, the transformed potential function can be expressed in a simpler form as:

$$\psi_m^I = S_m^I \sinh(\lambda_4 z)$$

Note that $S_m^I$ now differs from the previous definition by a constant multiplying factor.

The boundary conditions at the solid-liquid interface include the continuity of the normal components of both stress and displacement. Since the liquid is assumed inviscid, the shear (tangential) stress components in the solid part vanish at the interface. The remaining six equations are obtained by applying boundary conditions at $z = s$. When the excitation is enforced as a discontinuity of some components of stress in the solid (Eq. (1a)), the continuity of the three displacement components is preserved. However, a spatial excitation described by Eq. (1b) imposes a continuity of the stress components and a discontinuity in the displacement components in the solid. Note that in the liquid, $P = -\sigma_{zz}$.

The boundary condition formulation using the above equations provides us with two separate systems of equations in terms of the unknown coefficient functions as follows:

$$M_1 \begin{bmatrix} S_m^I & A_m^I & B_m^I & C_m^I & D_m^I & A_m^{III} & B_m^{III} \end{bmatrix}^T = L_1,$$

(19a)

$$M_2 \begin{bmatrix} E_m^I & F_m^I \end{bmatrix}^T = L_2,$$

(19b)

where,

$$M_1 = \begin{bmatrix} \lambda_4 \cosh \lambda_4 h & -\theta_1 e^{k_1 h} & -\theta_1 e^{-k_1 h} & -\theta_2 e^{k_2 h} & -\theta_2 e^{-k_2 h} & 0 & 0 \\ -\rho_0 \alpha^2 \sinh \lambda_4 h & c_{33} e^{k_1 h} \eta_1 & -c_{33} e^{-k_1 h} \eta_1 & c_{33} e^{k_2 h} \eta_2 & -c_{33} e^{-k_2 h} \eta_2 & 0 & 0 \\ 0 & -c_{44} e^{k_1 h} \eta_1 & -c_{44} e^{-k_1 h} \eta_1 & c_{44} e^{k_2 h} \eta_2 & -c_{44} e^{-k_2 h} \eta_2 & 0 & 0 \\ 0 & c_{44} k_1 e^{k_1 h} \eta_1 & c_{44} k_1 e^{-k_1 h} \eta_1 & c_{44} k_2 e^{k_2 h} \eta_2 & c_{44} k_2 e^{-k_2 h} \eta_2 & -c_{44} k_1 e^{k_2 h} \eta_2 & c_{44} k_1 e^{-k_2 h} \eta_2 \\ 0 & c_{33} k_1 e^{k_1 h} \eta_1 & c_{33} k_1 e^{-k_1 h} \eta_1 & c_{33} k_2 e^{k_2 h} \eta_2 & c_{33} k_2 e^{-k_2 h} \eta_2 & -c_{33} k_1 e^{k_2 h} \eta_2 & c_{33} k_1 e^{-k_2 h} \eta_2 \\ -i\lambda_4 c_{44} e^{k_1 h} & i\lambda_4 c_{44} e^{-k_1 h} & e^{-k_3 s} & e^{-k_3 s} & -e^{-k_3 s} & e^{-k_3 s} & e^{-k_3 s} \end{bmatrix},$$

(20)

$$M_2 = \begin{bmatrix} -i\lambda_3 e^{k_3 s} & -i\lambda_3 e^{-k_3 s} & 0 \\ e^{k_3 s} & e^{-k_3 s} & -e^{-k_3 s} \\ -i\lambda_3 e^{k_3 s} & -i\lambda_3 e^{-k_3 s} \\ c_{44} e^{k_3 s} & c_{44} e^{-k_3 s} \\ -c_{33} e^{k_3 s} & -c_{33} e^{-k_3 s} \end{bmatrix}.$$  

(21)

In the above terms, $L_1$ and $L_2$ are the load vectors and $i' = -1$ and,

$$\eta_i = (\alpha_3 - \alpha_2) \lambda_i^2 + \xi^2 (1 + \alpha_1) - \frac{\rho_0 \alpha^2}{c_{66}}, \quad \theta_i = \alpha_3 \lambda_i^2 - \eta_i,$$

(22)

$$v_i = \left( \eta_i - \alpha_3 \frac{c_{13}}{c_{33}} \xi^2 - \alpha_3 \lambda_i^2 \right) \lambda_i, \quad i = 1, 2.$$

For setting up the equations (20) and (21), the following equations which connect the transformed stress and displacement components to the transformed potential functions are used:

$$\psi_m^I = \left[ \frac{\alpha_2}{\rho_0} \frac{d^2}{dz^2} + \frac{\rho_0 \alpha^2}{c_{66}} - \xi^2 (1 + \alpha_1) \right] \psi_m^I.$$


\[ \mathbf{p}^{m+1}_{m} - i \mathbf{q}^{m+1}_{m} = \alpha_3 \xi \frac{d\mathbf{P}^{m}_{m}}{dz} - i \xi \mathbf{\hat{X}}^{m}_{m} \],

\[ \mathbf{p}^{m+1}_{m} - i \mathbf{q}^{m+1}_{m} = -\alpha_3 \xi \frac{d\mathbf{P}^{m}_{m}}{dz} - i \xi \mathbf{\hat{X}}^{m}_{m} \],

\[ \mathbf{\hat{a}}_{zm}^{m} = \frac{d}{dz} \left[ \alpha_3 c_{13} \xi^2 + c_{33} \left( \frac{\rho_0 \alpha^2}{c_{66}^2} - \xi^2 (1 + \alpha_1) \right) \right] \mathbf{F}^{m}_{m} - c_{44} \xi^2 \frac{d\mathbf{\hat{X}}^{m}_{m}}{dz} \],

\[ \mathbf{\hat{a}}_{zm}^{m+1} + i \mathbf{\hat{a}}_{zm}^{m+1} = c_{44} \xi \left[ (\alpha_3 - \alpha_2) \frac{d^2}{dz^2} + \xi^2 (1 + \alpha_1) - \frac{\rho_0 \alpha^2}{c_{66}^2} \right] \mathbf{F}^{m}_{m} - c_{44} \xi \frac{d\mathbf{\hat{X}}^{m}_{m}}{dz} \].

The load vectors \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) can be defined in various forms, depending on the type and location of the excitation. By properly defining the load vector, one may define excitations at either the seafloor interface or within the seabed at \( z = s \). Any assumed excitation may be in the form of either a known displacement or stress component. Furthermore, a pure compressional source can be assumed to act at the free surface of the ocean layer. Eqs. (16a) and (16b) can be solved to develop the response for such an excitation with some slight changes in the formulation.

The right side of Eqs. (16a) and (16b) can be written as follows for an asymmetric stress source at \( z = s \):

\[ \mathbf{L}_1 (\xi) = \left[ 0 \ 0 \ 0 \ 0 \ \frac{(X_m - Y_m)}{2\xi} \right] \mathbf{Z}_m^T, \]

\[ \mathbf{L}_2 (\xi) = \left[ 0 \ 0 \ \frac{-(X_m + Y_m)}{2\xi} \right] \mathbf{Z}_m^T. \]

If one is dealing with a displacement source at \( z = s \), \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) are written as:

\[ \mathbf{L}_1 (\xi) = \left[ 0 \ 0 \ \frac{(X_m' - Y_m')}{2\xi} \ 0 \ 0 \right] \mathbf{Z}_m', \]

\[ \mathbf{L}_2 (\xi) = \left[ 0 \ \frac{-(X_m' + Y_m')}{2\xi} \ 0 \right] \mathbf{Z}_m'. \]

In the above terms,

\[ X_m = \mathbf{\hat{p}}^{m+1}_{m} (\xi) - i \mathbf{\hat{q}}^{m+1}_{m} (\xi), \ Y_m = \mathbf{\hat{p}}^{m+1}_{m} (\xi) + i \mathbf{\hat{q}}^{m+1}_{m} (\xi), \ Z_m = \mathbf{\hat{R}}^{m}_{m} (\xi), \]

\[ X_m' = \mathbf{\hat{p}}^{m+1}_{m} (\xi) - i \mathbf{\hat{q}}^{m+1}_{m} (\xi), \ Y_m' = \mathbf{\hat{p}}^{m+1}_{m} (\xi) + i \mathbf{\hat{q}}^{m+1}_{m} (\xi), \ Z_m' = \mathbf{\hat{R}}^{m}_{m} (\xi). \]

Note that putting \( |\mathbf{M}_1| = 0 \) (\( |\mathbf{M}_1| \) represents the determinant of \( \mathbf{M}_1 \)), the frequency (or dispersion) equation for the surface waves in such a model is obtained. The liquid layer has an appreciable effect on the solution of this equation. It is identical to that presented by Abubakar and Hudson [22] who derived the equation without consideration of the source; they investigated only the velocity dispersion characteristics. According to the impressed frequency, the material properties of the two subdomains, and the thickness of the water column, the frequency equation gives the wavenumber \( \xi \) associated with either Rayleigh or Scholte waves. In addition, roots associated with the wavenumbers of the higher acoustic modes of propagation may arise. The roots of the frequency equation are the singular points of the integrands discussed in the section 5 and should be carefully handled while evaluating the inverse Hankel transform integrals. The phase velocity of the surface waves can be calculated as a function of dimensionless frequency to obtain the dispersive characteristic of surface waves in the model under consideration.

It should be mentioned that there is no appearance of the oceanic liquid layer in Eq. (21). This implies that the inviscid liquid layer has no effect on this part (Love wave response), which is purely associated with shear stress. Notwithstanding this, it does have some effect even on the tangential components of stress and displacement in the solid medium. On the other hand, even pure shear stress (or horizontal displacement) acting within the seabed may induce pressure and displacements within the liquid layer. Note also that since \( \mathbf{M}_2 \) is associated with the horizontally polarized shear waves, the frequency equation, \( |\mathbf{M}_2| = 0 \), does not give a root pertinent to a surface wave and this implies no Love wave arises.

Solving Eqs. (19a) and (19b) for the unknown coefficients, the potential functions and the resultant stress and displacement components are obtained for the assumed load vectors \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \). By means of Eqs. (14) and (23), the temporal Fourier components of the stress and displacement fields can be determined in the Hankel domain. The physical (space) domain quantities can be determined with the aid of inverse transform integration.

**4. Potential functions and elastic wavefields**

In this section, expressions are presented for the stress and displacement wavefields in the water column and the seabed when the model is subjected to some specific excitations. A point source and an equally distributed patch excitation are studied in more details. For the point source (delta function in space), the wavefields become the frequency-domain Green’s functions for the system. The expressions for
Fig. 3. Comparison of the real and imaginary parts of the normal stress $\sigma_{zz}$ versus depth within a transversely isotropic half-space subjected to time-harmonic patch load at depth $s = A$ (excitation source radius) for a normalized frequency $A \omega \sqrt{\rho_0/c_0} = 1$.

The potential functions can be obtained for either a known displacement or a known load. As previously mentioned, the elastic wavefields for any desired excitation can be developed by substituting the appropriate function into the load vector.

4.1. Point source

Let us consider a time-harmonic point displacement of known magnitude at $(0, 0, h \leq z = s)$. For such a source, the displacement is decomposed into $U_v$ and $U_h$ components, being the horizontal (tangent to the plane $z = s$ in the optional direction $\theta_0$) and vertical directions, respectively. The displacement field is defined using the Dirac-delta function, $\delta$ as

$$U_v(r, z) = U_v \frac{\delta(r)}{2\pi r} \delta(z - s) e^{i\omega t} e_z,$$

$$U_h(r, z) = U_h \frac{\delta(r)}{2\pi r} \delta(z - s) e^{i\omega t} e_h.$$  \hfill (28-a)  

Here, $U_v$ and $U_h$ are the magnitudes of the displacement in the vertical and horizontal directions, $e_z$ is the unit vector in the vertical direction, and $e_h$ is the unit vector in the horizontal direction defined by $\theta = \theta_0$. The latter is given by:

$$e_h = e_r \cos(\theta - \theta_0) - e_\theta \sin(\theta - \theta_0).$$  \hfill (29)

where, $e_r$ and $e_\theta$ are the unit vectors in the radial and azimuthal directions, respectively. By virtue of the azimuthal expansions of the displacement discontinuities across the plane $z = s$ and invoking the orthogonality of the angular eigen functions $e^{i m \theta}$, the transformed expressions for the source terms $X_m, Y_m$ and $Z_m$ mentioned in Eq. (27) are:

$$X_1 = \frac{U_h}{2\pi} e^{-i \theta_0}, X'_m = 0, m \neq 1,$$

$$Y_{-1} = \frac{U_h}{2\pi} e^{i \theta_0}, Y'_m = 0, m \neq -1,$$

$$Z'_0 = \frac{U_v}{2\pi}, Z'_m = 0, m \neq 0.$$  \hfill (30)

The harmonic time factor $e^{i \omega t}$ has been omitted for brevity. Expressions for $X_m, Y_m$ and $Z_m$ due to a point force can be determined similarly.
4.2. Uniform patch source or disc excitation

A time-harmonic uniform circular stress source of radius $A$ acting at $(0, 0, h \leq z = s)$ can be defined with the aid of Heaviside and delta functions:

$$f_v(r, z) = F_v \frac{H(A - r)}{\pi A^2} \delta(z - s) e^{i \omega t} \mathbf{e}_z,$$

$$f_h(r, z) = F_h \frac{H(A - r)}{\pi A^2} \delta(z - s) e^{i \omega t} \mathbf{e}_h,$$

where $F_v$ and $F_h$ are the magnitudes of the vertical and tangential loads applied across the circular area, respectively. For this kind of excitation, one can obtain expressions for the source terms (see Eq. (24)) as:

$$X_1 = \frac{F_h}{\pi A s} J_1(\alpha s), \quad X_m = 0, \ m \neq 1,$$

$$Y_{-1} = \frac{F_h}{\pi A s} J_1(\alpha s), \quad Y_m = 0, \ m \neq -1,$$

$$Z_0 = \frac{F_v}{\pi A s} J_1(\alpha s), \quad Z_m = 0, \ m \neq 0.$$

The harmonic time factor $e^{i \omega t}$ is again implied but omitted for simplicity. $X_m, Y_m$ and $Z_m$ associated with a circular displacement can be determined in a similar manner.

4.3. Potential functions

The potential functions can now be derived by solving the system of Eqs. (16a) and (16b) and then substituting the proper expressions for $(X_m, Y_m$ and $Z_m$ or $X_m', Y_m'$ and $Z_m'$) into the load vectors $L_1$ and $L_2$. For a clearer interpretation, the potential function $F$ is decomposed.
into the parts, \( F_1 \) and \( F_2 \), related to the vertical \( (Z_m, Z_m) \) and horizontal \( (X_m, Y_m, \dot{X}_m, \dot{Y}_m) \) excitations, respectively. For a known stress acting at \( z = s > h \) one obtains the transformed Fourier components of potential functions as follows:

For the liquid layer, \( 0 < z < h \),

\[
\tilde{\phi}^m_m = \frac{2\alpha_3 c_3 c_{44} (\eta_2 \theta_1 - \eta_1 \theta_2) (\eta_2 \varphi_1 - \eta_1 \varphi_2) (e^{-\lambda_1 z} - e^{-\lambda_2 z}) + (e^{-\lambda_1 h} - e^{-\lambda_2 h})}{|M_1|} \sinh (\lambda_4 z) Z_m, \tag{33-a}
\]

\[
\tilde{\rho}^m_m = \frac{2\alpha_3 c_3 c_{44} (\lambda_2 \varphi_1 - \lambda_1 \varphi_2) (e^{-\lambda_1 h} - e^{-\lambda_2 h})}{|M_1|} \sinh (\lambda_4 z) (X_m - Y_m), \tag{33-b}
\]

for the solid seabed in the region \( h < z < s \),

\[
\tilde{\rho}^m_{1m} = \frac{2\alpha_3 c_3 c_{44} (\eta_2 \varphi_1 - \eta_1 \varphi_2) (X_m - Y_m)}{|M_1|} \times \left( c_3 \lambda_4 \times \left[ -2\eta_1 \lambda_2 \varphi_1 e^{-\lambda_1 z} - 2\eta_2 \lambda_1 \varphi_2 e^{-\lambda_2 z} + \lambda_2 (\eta_2 \varphi_1 - \eta_1 \varphi_2) e^{\lambda_1 h} + \lambda_1 (\eta_2 \varphi_1 - \eta_1 \varphi_2) e^{\lambda_2 h} \right] \sinh (\lambda_4 h) \right), \tag{34a}
\]

\[
\tilde{\rho}^m_{2m} = \frac{2\alpha_3 c_3 c_{44} (\lambda_2 \varphi_1 - \lambda_1 \varphi_2) (X_m - Y_m)}{|M_1|} \times \left( c_3 \lambda_4 \times \left[ -2\eta_1 \lambda_2 \varphi_1 e^{-\lambda_1 z} - 2\eta_2 \lambda_1 \varphi_2 e^{-\lambda_2 z} + \lambda_2 (\eta_2 \varphi_1 + \eta_1 \varphi_2) e^{\lambda_1 h} + \lambda_1 (\eta_2 \varphi_1 + \eta_1 \varphi_2) e^{\lambda_2 h} \right] \sinh (\lambda_4 h) \right), \tag{34b}
\]

\[
\tilde{\chi}^m_m = \frac{\rho_0 e^{-\lambda_3 z} \left( e^{2k_3 h} + e^{2k_3 z} \right) (X_m + Y_m)}{4c_{44} \xi \lambda_3}. \tag{35}
\]
and for the solid seabed in the region \( z > s \),

\[
\begin{align*}
\tilde{P}_{1m} &= \frac{2\alpha_3 c_4^2 (\eta_2 \vartheta_1 - \eta_1 \vartheta_2) Z_m}{|M_1|} \\
&\times\left\{c_3 \lambda_4 \left[ -2 \eta_1 \lambda_2 v_1 e^{-\lambda_1 z} - 2 \eta_2 \lambda_1 v_2 e^{-\lambda_1 z} + \lambda_2 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \lambda_1 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_1 (h-s-z)} e^{-\lambda_2 h} \right] \cosh (\lambda_4 h) - \\
&\left[ (-\lambda_1 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \lambda_1 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \lambda_2 e^{\lambda_1 (h-s-z)} e^{-\lambda_2 h} - \lambda_2 e^{\lambda_1 (h-s-z)} e^{-\lambda_2 h}) \rho \omega_0^2 (\eta_2 \vartheta_1 - \eta_1 \vartheta_2) \right] \sinh (\lambda_4 h) \right\},
\end{align*}
\]

(36a)

\[
\begin{align*}
\tilde{P}_{2m} &= \alpha_3 c_3 c_4 (\lambda_2 \vartheta_1 - \lambda_1 \vartheta_2) (X_m - Y_m) \times \\
&\left\{c_3 \lambda_4 \left[ -2 \eta_2 \vartheta_1 v_2 e^{-\lambda_1 z} + \vartheta_1 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \vartheta_1 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_1 (h-s-z)} e^{-\lambda_2 h} - 2 \vartheta_2 \eta_1 v_1 e^{-\lambda_1 z} + \\
&\vartheta_2 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \vartheta_2 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h}) \cosh (\lambda_4 h) + \\
&\left[ (\vartheta_1 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} + \vartheta_1 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} - \vartheta_2 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h} - \vartheta_2 e^{\lambda_2 (h-s-z)} e^{-\lambda_1 h}) \rho \omega_0^2 (\eta_2 \vartheta_1 - \eta_1 \vartheta_2) \right] \sinh (\lambda_4 h) \right\},
\end{align*}
\]

(36b)

\[
\chi_m = -\frac{i e^{-\lambda_1 (s+z)} (e^{2\lambda_3 h} + e^{2\lambda_3 z}) (X_m + Y_m)}{4\alpha_3^2 \lambda_3^2 \xi}.
\]

(37)

If we are dealing with a source of known displacement components, then the Fourier components of transformed potential functions are derived as follows:

For the liquid layer, \( 0 < z < h \),

\[
\begin{align*}
\tilde{\phi}_{1m} &= \frac{-4\alpha_3^2 c_4^2 (e^{-\lambda_2 z} - e^{-\lambda_1 z}) \eta_2 v_1 - e^{-\lambda_1 h} (\eta_2 \vartheta_1 - \eta_1 \vartheta_2) Z_m \sinh (\lambda_4 z)}{|M_1|}, \\
\tilde{\phi}_{2m} &= \frac{2\alpha_3^2 c_4^2 (e^{-\lambda_2 z} - e^{-\lambda_1 z}) \eta_1 v_2 (\eta_2 \vartheta_1 - \eta_1 \vartheta_2)^2 (X_m - Y_m) \sinh (\lambda_4 z)}{\xi |M_1|}.
\end{align*}
\]

(38a)

(38b)
for the solid seabed in the region $h < z < s$,

$$\mathcal{P}_m = -2\alpha_4 c_3 c_4^2 (\lambda_2 \eta_1 - \lambda_1 \eta_2) Z_m$$

$$(39a)$$

$$\begin{align*}
(-c_3^2 \lambda_4 & \times [2e^{-\lambda_1 z} - 2e^{-\lambda_2 z} - 2e^{-\lambda_1 z} - 2e^{-\lambda_2 z}] - \eta_1 - \eta_2 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (-h - s)} e^{\lambda_1 h} + \eta_1 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (-h - s)} e^{\lambda_2 h} \\
& \quad - \eta_1 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_2 (h - s - z)} e^{\lambda_1 h} - \eta_2 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_2 (h - s - z)} e^{\lambda_2 h}) \cosh (\lambda_4 h) + \\
& \quad \left( \eta_1 e^{\lambda_2 (h - s - z)} e^{\lambda_1 h} - \eta_1 e^{\lambda_2 (h - s - z)} e^{\lambda_2 h} - \eta_2 e^{\lambda_2 (h - s - z)} e^{\lambda_2 h} + \eta_2 e^{\lambda_2 (h - s - z)} e^{\lambda_2 h} \right) \rho_0 a^2 (\eta_2 \partial_1 - \eta_1 \partial_2) \sinh (\lambda_4 h),
\end{align*}$$

$$(39b)$$

$$\mathcal{P}_m = c_3 c_4^2 (\eta_2 \partial_1 - \eta_1 \partial_2) (X'_m - Y'_m)$$

and for the solid seabed in the region $z > s$,

$$\mathcal{P}_m = -2\alpha_4 c_3 c_4^2 (\lambda_2 \eta_1 - \lambda_1 \eta_2) Z_m$$

$$(41a)$$

$$\begin{align*}
(-c_3^2 \lambda_4 & \times [2\eta_1 v_1 \eta_2 v_2 e^{\lambda_2 z} - 2\eta_1 v_1 \eta_2 v_2 e^{\lambda_2 z} - \eta_1 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_1 z} e^{\lambda_2 h} + \eta_1 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_1 z} e^{\lambda_2 h} \\
& \quad - \eta_1 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_1 z} e^{\lambda_2 h} - \eta_2 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_2 (h - s - z)} e^{\lambda_2 h} + \eta_2 e^{\lambda_2 (h - s - z)} e^{\lambda_2 h} + \eta_2 e^{\lambda_2 (h - s - z)} e^{\lambda_2 h}) \rho_0 a^2 (\eta_2 \partial_1 - \eta_1 \partial_2) \sinh (\lambda_4 h),
\end{align*}$$

$$(41b)$$
\[
\begin{align*}
\frac{F_m^{\delta_{2m}}}{2m} &= \frac{c_{33} \lambda_4^2 (\eta_2 v_1 - \eta_1 v_2) (X_m - Y_m)}{\xi |M_1|} \\
&= \frac{c_{33} \lambda_4^2 \left[ -2 \eta_2 v_1 v_2 e^{-\lambda_1 z} - 2 \eta_2 v_1 v_2 e^{-\lambda_2 z} - v_1 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_2 (z+z)} - v_2 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_1 (z+z)} + v_2 (\eta_2 v_1 - \eta_1 v_2) e^{\lambda_1 (z+z)} + v_1 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_1 (z+z)} + v_2 (\eta_2 v_1 + \eta_1 v_2) e^{\lambda_2 (z+z)} \right] \cosh (\lambda_4 h) +} \\
&\frac{\left[ (v_1 e^{\lambda_1 (z+z)} - v_1 e^{\lambda_2 (z+z)} - v_2 e^{\lambda_1 (z+z)} + v_2 e^{\lambda_2 (z+z)}) \rho_0 c_2^2 (\eta_2 v_1 - \eta_1 v_2) \right] \sinh (\lambda_4 h),}
\end{align*}
\]

Specifically, the potential functions due to a point source and a circular disc source can be obtained by substituting Eqs. (27) or (29) into Eqs. (30)-(39). Load vector components respectively to any other source shape should be similarly calculated and substituted into the expressions derived for the potential functions. Once the potential functions are determined, expressions for the transformed components of stress and displacement in the solid and liquid domains can be easily derived by means of Eqs. (11) and (20).

The results can then be transformed back into the space domain by means of the inverse Hankel transform and Fourier series expansion. The former is given by the infinite integral:

\[
f_m(r) = \int_0^\infty \tilde{f}_m^m(\xi) \rho_m(\xi) \, d\xi,
\]

where \(\tilde{f}_m^m(\xi)\) is the solution to the \(m\)th component of the Fourier series expansion of the variables in the wavenumber \(\xi\) domain, and \(f_m(r)\) is the solution in the space domain. The complete values of the variable can now be developed by substituting the terms in the Fourier series expansion; that is,

\[
f(r) = \sum_{-\infty}^{\infty} f_m(r, z) e^{im\theta}.
\]
5. Numerical evaluation scheme

The inverse Hankel transform integrals must be evaluated by means of an appropriate integration scheme. The integrands contain complicated terms as well as poles (see Eqs. (30)–(39)) in the integration path. Hence, they should be handled with the aid of a coupled Residue Theorem and an efficient numerical integration method.

For this model, for any combination of the parameters, there exists at least one pole associated with either the dispersive generalized Rayleigh or Scholte wave. In the absence of the liquid layer, there exists exactly one pole which is associated with the Rayleigh wave in the solid (not dispersive). When there is a liquid layer, however, the Rayleigh wave becomes dispersive. Moreover, if the liquid has a normal compressibility (e.g. water), an infinite number of dispersive higher modes are also possible. The wavenumbers associated with the fundamental or higher acoustic modes are obtained from the frequency equation. Hence, the frequency equation ($|\mathbf{M}_1| = 0$) might have several real roots, $\xi_n$, for a specific value of $\omega h$. The first mode which is associated with the pole at the highest wavenumber is separated into two branches at a critical phase velocity value $c = c_l$. The values of phase velocity greater than $c_l$ form the normal mode branch, and the lower values form the well-known Stoneley-or Scholte- branch [22]. The possibility of the higher acoustic modes depends on the frequency, material properties and the layer thickness. All of the higher modes exhibit a phase velocity higher than $c_l$. In other words, in the sequence of the real roots, $\xi_n$, there may exist only one root which exceeds $\xi_l = \omega / \sqrt{K}/\rho_l$. This root, if it exists, is associated with the Stoneley wavenumber $\xi_n$. The appearance of this type of wave depends on the specific model parameters.

With this in mind, one may encounter several singular points, including poles and branch points, while evaluating the inverse Hankel transform integrals. Generally, the poles include at least one point, and may increase depending on the aforementioned parameters. The number of dispersion curves intersected by the vertical line $\omega_0 = \text{constant}$ specifies the number of poles arising during the integration. The intersection points also reveal the category to which the pole is associated, i.e. Rayleigh, Scholte or higher acoustic modes. The branch points include three points associated with the solid subdomain i.e. $\xi_{Gi}$, $\xi_{SV}$ and $\xi_{P}$ and one associated with the liquid, $\xi_C$.

The Residue Theorem is applied, and the procedure introduced by Khojasteh et al. [18] is modified and followed in order to evaluate the infinite integrals. Since the pole(s) at $\xi_n$ is (are) an interior singular point of the first order, the integrand may be written in the form of $q (\xi_n) / |\mathbf{M}_1|$, where $q (\xi_n)$ is analytic at $\xi_n$. Therefore, the integral over the limiting small semi-circle at the pole is equal to $-\pi i \text{Res} (\xi_n)$, where $\text{Res} (\xi_n) = \lim_{\xi \rightarrow \xi_n} \left[ q (\xi_n) / (d(|\mathbf{M}_1|)/d\xi) \right]$.

For the contour integration, a computer program in Mathematica software was developed. The program applies an adaptive quadrature approach for integration within the regions between the singular points and after the pole having the largest abscissa $\xi$ value out to
sufficiently large values. The integration path followed is depicted in Fig. 2. The poles with an asterisk superscript in the figure imply that the pole may or may not arise in this layout, according to the model parameters and frequency in question (Scholte and higher modes).

In contrast to the more common case discussed before, we now consider the case where the velocity of SV waves in the solid domain is less than the soundwave speed in the liquid ocean (i.e. the liquid is too incompressible). In this case, for any values of the other parameters, one single pole will arise belonging to the Scholte (Stoneley) wave class. Hence, the wave will be guided by the liquid-solid interface. In that case, the implementation followed in this contribution can also be well adopted without any remarkable change. The modification includes a conversion in the sequence of the branch points in the path of integration (Fig. 2) and removal of the poles related to higher modes. These lead to simpler calculations.

6. Special cases

Some special cases are now considered to present the results for simpler situations and to confirm the validity of the new formulation against some previously published solutions.

Case 1. If the liquid layer thickness, $h$, is set to zero then we are dealing with a uniform transversely isotropic half-space which was studied in [18]. In this case, the frequency (dispersion) equation reduces to the Rayleigh wave equation in a transversely isotropic half-space. Green's functions are also those of a transversely isotropic half-space which agree with the solution developed in the mentioned reference. This can be mathematically proved simply by substituting $h = 0$ in Eqs. (30)–(39), and is also confirmed numerically in the next section.
Fig. 16. Imaginary part of displacement $u_z$ at the solid-liquid interface due to a point load of unity acting in the $z$-direction.

Fig. 17. Real and Imaginary parts of $\sigma_{zz}$ at $r = 0$ due to a point load of unity acting in the $z$-direction at $s = 2h$.

**Case 2.** If the elastic constants are set as follows: $c_{11} = c_{33} = 2\mu + \lambda$, $c_{12} = c_{13} = \lambda$, and $c_{44} = c_{66} = \mu$, then the problem reduces to a liquid layer overlying an isotropic solid half-space with Lame constants $\lambda$ and $\mu$. This was investigated in detail by Ewing et al. [5]. By means of some mathematical manipulations, it can be shown that the potential functions due to a time-harmonic point displacement of unit magnitude, can be reduced to the identical form of those obtained by Ewing et al. [5]. In this case, $\lambda_1 = \sqrt{\frac{\mu^2 - \frac{\lambda^2}{2}}{\rho_s^2}}$, $\lambda_2 = \sqrt{\frac{\mu^2 - \frac{\lambda^2}{2}}{\rho_s^2}}$, $\beta = \sqrt{\frac{\mu}{\rho_s}}$ and $\alpha = \sqrt{(2\mu + \lambda)/\rho_s}$. Similarly, the frequency equation ($|M_{11}| = 0$) reduces to that of an isotropic seabed given in [5], if it is written in the following form:

$$-ho_0\omega^2\lambda_1\sin\lambda_1 h \frac{\beta^2}{\beta^2} + i\alpha\beta^2 \left[-4\beta^2\lambda_1\lambda_2 + \left(2\beta^2 - \beta^2\right)^2\right] \cos(\beta h) = 0.$$  \hspace{1cm} (45)

The well-known Rayleigh cubic equation for an isotropic half-space can also be obtained from this equation by putting $h = 0$, viz.,

$$-4\beta^2\lambda_1\lambda_2 + \left(2\beta^2 - \beta^2\right)^2 = 0.$$  \hspace{1cm} (45)

**Case 3.** When the excitation is applied at the seafloor, Green’s functions (or wavefields) can be derived by putting $s = h$ in the equations, or directly by applying the proper modification to the load vectors. In the latter case, the boundary conditions and respective equations at
z = s will be redundant. In this case, the entire number of necessary potential functions reduces by two and the number of the unknown coefficient functions reduces by six.

7. Numerical results

To confirm the validity of the results numerically, we first compare our results against previous work presented for the special case of no water column present. The solid seabed is taken to be the mineral Berly having the normalized elastic constants (relative to \(c_{44}/10\)) \(c_{11} = 41.3, c_{12} = 10.1, c_{44} = 10, c_{12} = 14.7, \) and \(c_{32} = 36.2\). Fig. 3 is a plot of the complex normal stress (real and imaginary parts) as a function of depth, the latter normalized against the source radius \(A\). It shows good agreement of the results with those in [18].

Fig. 4 is a plot of the fundamental mode group and phase velocity dispersion curves compared with the results obtained by Abubakar and Hudson [22]. The acoustic properties of the water layer and the elastic properties of the underlying VTI half-space are such that \(\rho_l/\rho_s = 2\), \(c_{SV}/c_l = \sqrt{2}\), and \(c_p/c_{SV} = 2.032\). In these ratios, quantities \(c_{SV}\) and \(c_l\) are the velocities of the SV and P waves in the solid, respectively. Agreement between our results and those of Abubakar and Hudson (op cit) are very good.

Next, we present results for a liquid layer (water) overlying Mesaverde sandstone. The sandstone is anisotropic and characterized by the five elastic constants \(c_{11} = 50.01\,\text{GPa}, c_{13} = -8.6\,\text{GPa}, c_{44} = 24.57\,\text{GPa}, c_{66} = 26.59\,\text{GPa}, \) and \(c_{33} = 45.05\,\text{GPa}\) and a density of 2870 kg/m\(^3\) [30]. The density and bulk modulus of the water are \(\rho_l = 1000\,\text{kg/m}^3\) and \(K = 2.2\,\text{GPa}\), respectively.

Fig. 5 shows the phase velocity dispersion curves for the first three modes. The horizontal axis is the dimensionless frequency \(\omega_0 = \omega h/\sqrt{\rho_s/c_{44}}\). The normalized group velocity dispersion curves are also depicted \((U/c_{SV})\). The normal mode velocity curve tends to the velocity of the Rayleigh wave in a VTI half-space when the water column thickness \(h\) tends to zero. The figure confirms that for any value of the dimensionless frequency, there will be at least one possible surface wave. The group velocity for the first mode has a minimum whereas the higher modes may have several local maxima and minima. The first mode tends to the limiting value of the Scholte wave in such a model where the higher acoustic modes have an asymptotic velocity equal to \(c_l\) at large values of \(\omega_0\). The upper bound on the velocity of the higher modes is the SV wavespeed of the solid seabed. For the current model, as indicated in Fig. 5, a single pole arises in the path of integration when \(\omega_0 = 1\). However, the figure indicates that velocity dispersion curves of three modes are intersected by the line \(\omega_0 = 4\), yielding three poles along the integration path. The figure shows that the largest pole falls on the Stoney branch. The exact values of the poles are given in Table 1. The table also shows, for comparison purposes, the Rayleigh wavenumber in the absence of the liquid layer (not dispersive). By determining the poles in the integration path, the results can now be developed by utilizing the procedure explained in section 5.

A matter of particular interest is to examine the influence of the liquid layer on the results. We show in Figs. 6 and 7 the computed vertical displacements \(u_r\) at the top surface of the solid, multiplied by \(\pi c_{44} A\), versus the radial distance \(r\) (normalized by twice the source radius) The figure shows the results for two different values of the normalized frequency \(A\omega_0/\sqrt{\rho_s/c_{44}} = 1\) and 4. The two sets of curves in each figure are the complex displacements (real and imaginary parts) when the liquid layer is present \((h = 2A)\) and when it is absent \(h = 0\) (solid half-space only). The two figures clearly show that the liquid layer affects the results significantly at both low and high frequencies. In Fig. 6, where \(A\omega_0/\sqrt{\rho_s/c_{44}} = 1\), the liquid layer damps 8.1% of the absolute value of the displacement at \(r = 0\) and this increases to 47.7% at \(r/(2A) = 3\). The corresponding values for the higher frequency case of \(A\omega_0/\sqrt{\rho_s/c_{44}} = 4\) are 24.4% and 94%, respectively.

Figs. 8–11 depict the real and imaginary parts of the normalized displacement parameters \((\pi c_{44} A, \pi c_{44} A_h)\) at the liquid-solid interface due to uniformly distributed loads acting within different distances from the free surface. Figs. 8 and 9 are for the real and imaginary parts of the horizontal displacement \(u_r\) due to a pure horizontal excitation whereas Figs. 10 and 11 are for the real and imaginary parts of the vertical displacement \(u_z\) due to a pure vertical excitation. To appreciate the influence of source depth on the results, the curves in each case are presented for two values of \(I(= h + 1)\) and \(I(= h/2)\), where \(I\) is distance of the source from the interface \((I = s - h)\). The figures are also given for two dimensionless frequencies, \(\omega_0 = 1\) and \(\omega_0 = 4\).

Shear stresses \(\sigma_{xz}\) versus depth, due to a uniform circular patch of horizontal load are provided in Fig. 12. Both real and imaginary parts of the shear stress, multiplied by the source area, are given. The depth is normalized by the liquid layer thickness \(h\). In all the cases, it is assumed that the radius \(A\) of the loading region is equal to \(h/2\). The curve is restricted to the range \(z > h\), since no shear stress can be carried by the liquid.

The depth dependence of the corresponding vertical displacement at \(u_r\) and the normal stress \(\sigma_{xz}\) due to a source of normal stress buried within the solid are presented in Figs. 13 and 14. Again, the plotted quantities have been normalized by the appropriate multiplying factors, and real and imaginary parts of the responses are given for two different normalized frequencies. A jump of unit magnitude at \(z = s\) where the load is applied, and a discontinuous gradient at \(z = h\) due to the change in material properties can be observed. The effect of source radiation pattern is clearly evident in the plots.

The real and imaginary parts of the displacements and stresses (Green’s functions) in the case of a point source are presented in Figs. 15–19. Figs. 15 and 16 show how the real and imaginary parts of the vertical displacement at the seafloor interface change as a function of normalized radial distance away from the source. Two cases are considered, involving source depths of 1.5\(h\) and 2\(h\), where \(h\) is the thickness of the water column. The curves indicate that the source location has a pronounced effect on the displacement received at the interface. This is especially true for the lower dimensionless frequency plot where the absolute value of the normal displacement is magnified by approximately 120% at \(r = 0\) at the interface when \(s = 1.5h\) compared to the case \(s = 2h\). The source is a vertically directed point stress. Again, results for two different normalized frequencies \(\omega_0 = 1, 4\) are given. In Fig. 17, we show how the normal stress (real and
imaginary parts) varies as a function of depth for a unit point load acting in the vertical direction along \( r = 0 \). The source depth (measured from the free surface) is twice the liquid layer thickness. Unlike the lower frequency case, the higher frequency case gives rise to noticeable pressure in the fluid domain. Fig. 18 shows the corresponding result for the vertical displacement. Finally, we show in Fig. 19 the depth dependence of the shear stress for the same model involving a point source, acting in the horizontal direction at a depth of twice the liquid layer thickness. The source lies vertically below the position \( r = 0 \). In all cases, the behavior due to a point source is more irregular than that for the patch load. This can be attributed to the higher density of stress components when concentrated at a single point. The plots for a point source excitation indicate very large values of stress and displacement exactly at the source point.

8. Summary and conclusions

In this study, an accurate mathematical formulation is developed for the elastodynamic response of a water column (ocean) overlying a transversely isotropic solid seabed. Time-harmonic wavefields are derived for asymmetric three-dimensional wave propagation due to an arbitrary source acting in the vertical and/or the horizontal directions at a specified depth within the solid. Potential functions and corresponding displacements and stresses are derived for each medium in the frequency-wavenumber domain.

Efficient computer code in Mathematica software was used to perform the numerical evaluation of the infinite integrals which arise in converting back from the wavenumber to the space domain. When utilizing the new procedure, it was necessary to pay careful attention to the singularities during the adaptive integration. The Residue Theorem and contour integration were applied to obtain accurate results for the induced stress and displacement values. By involving the liquid layer, the problem becomes more complicated compared to an isolated VTI solid half-space. The reason is that the Rayleigh wave becomes dispersive and the higher dispersive acoustic modes are also possible. Special care should be given to the number of the poles in the integrands which depends on the geometry, material properties and frequency in question. The singular points arising in the integration path along the real wavenumber axis are associated with the possible surface waves in the model which include the generalized Rayleigh wave, the Scholte (Stoneley) wave, and any possible higher acoustic modes.

Phase and group velocity dispersion curves are also presented for the model, by solving the frequency (dispersion) equation. These curves are used to identify the number of poles and the surface wave type associated with each of them. The higher the frequency and the thicker the ocean, the more probable is the existence of higher modes and more singular points. These conclusions pre-suppose that the velocity of the SV wave in the VTI solid seabed is greater than the compressional wave velocity of the liquid layer. The formulation can be suitably adapted for other kinds of excitation with appropriate minor modifications.
Stress and displacement wavefields are presented in numerous figures and plotted against dimensionless depth or radial distance for several dimensionless frequencies and source locations. A comparison is made between the displacements at the top surface of the VTI solid seabed with and without the effect of the overlying water column, with all other parameters identical. The curves show the significant role played by the interacting liquid layer.

Figures are also presented for stress/displacement Green’s functions for different depths of the source and frequencies of excitation. They reveal the notable role that dimensionless frequency has on the results. Higher frequencies undergo more substantial damping of the emitted disturbance. The results are also strongly sensitive to the distance of the source of excitation from the receiver point, compatible with the observations for deep and shallow focus offshore earthquakes. The figures reveal the considerable effect of the source location on the induced hydrodynamic pressure and displacement in the ocean. It is also found that considerable pressure and displacement occur in the liquid layer even due to a point source, if the frequency is high enough. The results also reveal that the maximum values of the stress and displacement are completely dependent on the various parameters in the model, and not just limited to the distance from the source. When the frequency is high enough, the maximum values of the field quantities within the seabed may occur at points which do not coincide with the source. However, when the frequency is low, i.e. closer to the static case, the maximum values generally occur exactly at the point in which the excitation is enforced. Absolute maximum values within the entire model may also occur in either liquid or solid subdomains, due to the significantly different radiation pattern in each subdomain caused by the considerable difference in the material properties. A considerable change in the gradient of the wavefield curves can be observed at the liquid-solid interface ($z/h = 1$) in the figures presented for the variation of the field quantities along the vertical direction. This is again a result of the significant difference in the stiffness of the two materials (although the curves remain continuous).

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