Tsunami generation and associated waves in the water column and seabed due to an asymmetric earthquake motion within an anisotropic substratum

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Abstract In this paper, closed-form integral expressions are derived to describe how surface gravity waves (tsunamis) are generated when general asymmetric ground displacement (due to earthquake rupturing), involving both horizontal and vertical components of motion, occurs at arbitrary depth within the interior of an anisotropic subsea solid beneath the ocean. In addition, we compute the resultant hydrodynamic pressure within the seawater and the elastic wavefield within the seabed at any position. The method of potential functions and an integral transform approach, accompanied by a special contour integration scheme, are adopted to handle the equations of motion and produce the numerical results. The formulation accounts for any number of possible acoustic-gravity modes and is valid for both shallow and deep water situations as well as for any focal depth of the earthquake source. Phase and group velocity dispersion curves are developed for surface gravity (tsunami mode), acoustic-gravity, Rayleigh, and Scholte waves. Several asymptotic cases which arise from the general analysis are discussed and compared to existing solutions. The role of effective parameters such as hypocenter location and frequency of excitation is examined and illustrated through several figures which show the propagation pattern in the vertical and horizontal directions. Attention is directed to the unexpected contribution from the horizontal ground motion. The results have important application in several fields such as tsunami hazard prediction, marine seismology, and offshore and coastal engineering. In a companion paper, we examine the effect of ocean stratification on the appearance and character of internal and surface gravity waves.

1. Introduction

Phenomena such as submarine earthquakes, landslides, slumps, volcanic eruptions, and explosions can generate various types of waves which can have devastating consequences in marine and coastal areas. Tsunamis are the most distinct, recognizable, and hazardous type of surface wave in this category. It is vital for reasons of public safety to provide methods for predicting the amplitude and arrival time of such gravity waves. Given the repetitive occurrence of tsunamis throughout human history, they have been the target of intensive research for years. Well-known early contributions in this regard include Stoneley [1963] and Hammak [1973]. In recent times, largely due to the increase in the impetus to understand tsunami manifestations and improve Tsunami Early Warning Systems (TEWS), extensive theoretical investigations have been undertaken. Some notable publications are those by Todorovska and Trifunac [2001], Dutykh et al. [2012], Chierici et al. [2010], LeVeque et al. [2011], Godin [2012], Tang et al. [2012], Kadri and Stiassnie [2012], Watada et al. [2014], Coustou et al. [2015], and Abdolali et al. [2015a, 2015b]. Other types of wave are generated as a result of the subsea ground motion along with tsunamis. This includes acoustic-gravity (or sometimes called hydroacoustic) waves in the fluid column, P, SV, and SH waves in the solid medium beneath the ocean floor, and Rayleigh and Scholte (Stoneley) waves at the fluid-seabed interface. Acoustic-gravity waves (abbreviated as AGW) travel at significantly higher velocities than tsunamis and cover very large distances, although not all the energy associated with acoustic waves travels away from the generation area. The components of motion having frequencies above the cutoff are evanescent and therefore die out rapidly. The lower frequency component acoustic waves can therefore be utilized to predict the impending hazard and enable evacuation procedures in threatened areas in a timely manner. These waves, however, only appear in the
In a wide range of published works, it is often assumed that the seabed is rigid (i.e., of infinite stiffness). However, since the tsunami and associated acoustic-gravity waves are in fact the result of disturbances generated within the interior of the solid seabed and transmitted into the ocean, and given the fact that all waves (including elastic) interact with each other, it is more realistic to consider the effect of the elastic properties of the seabed. Some methods allow for the interaction of the elastic bed with the overlying fluid layer by assuming certain models for the complete system. Among such methods is the scheme of Mohapatra and Sahoo [2011] who applied flexible plate theory for the bottom of the ocean, although they neglected the compressibility of the fluid. Nosov et al. [2008] included the seabed elasticity by means of an estimated exponential decay time for energy while investigating the contribution of nonlinearity in tsunami generation. Abdolali et al. [2015a, 2015b] studied AGWs in an ocean due to a sudden bottom displacement with a weakly compressible sediment layer. They found that a notable correction was needed due to this underlying layer compared to that for a rigid bottom. Eyov et al. [2013] investigated the elastic isotropic bed interaction with the progressive acoustic-gravity waves, and showed how such waves recorded far from the source could be used to detect tsunamis in advance.

Isotropy of the seabed is a convenient and simplifying assumption to make but numerous observations (both in the laboratory and in the field) reveal the existence of significant elastic anisotropy of materials which form the ocean floor. Transverse isotropy is a particular and prevalent form of anisotropy in which there is no directional dependence of the elastic properties in one particular plane, referred to as the plane of isotropy. In all other directions, the wave speed is a function of angle. The normal to this plane of isotropy is the axis of symmetry of the material. It is often vertical (a VTI medium) and this is a common assumption made in liquid-solid models involving anisotropy, such as in a layered earth and oceanic crustal models [Crampin, 1981]. Anisotropy in these cases is generally a direct consequence of sediment layering as materials are deposited in the water (rather than following a random orientation) or the existence of horizontally aligned microcracks, or preferred mineral orientation, such as in a hard ocean floor.

Wave propagation in transversely isotropic solids has been studied in detail by many scientists. The reader may refer to early works such as Stoneley [1949] and some more recent ones by Ruger [1997], Khojasteh et al. [2008, 2011, 2013], Eskandari-Ghadi et al. [2008], Zhu and Tsankin [2006], Zhou and Greenhalgh [2008], and You et al. [2016]. Sharma [1999] and Bagheri et al. [2015] investigated the dispersive characteristics of interface elastic waves in a model of a water column with a transversely isotropic seabed. Reflection of waves at an anelastic ocean bed has been studied both numerically and analytically by Sidler and Carcione [2007].

A key parameter in the generation and propagation of waves due to a subsea excitation is the focal depth of the source. To the best of the authors’ knowledge, no explicit theoretical formulation has yet been presented for surface gravity waves (abbreviated as SGW) and AGWs when the source is located within and below the seabed. The majority of the papers in this field assume a specified displacement at the seafloor itself. Other important and effective parameters to consider include, but are not limited to, the water depth (shallow or deep), the distance of the observation point from the source, the seabed bathymetry [Nosov and Kolesov, 2003], the shape of the earthquake fault (slip plane) at the focal point of the excitation and the contribution of nonlinearity in the response [Nosov et al., 2008]. Dutykh et al. [2012] and Tanioka and Satake [1996] pointed out the considerable effect of the horizontal deformation of the ocean bottom due to faulting.

In this paper, the first of a two part series, a theoretical method is given for analyzing the three-dimensional propagation and dispersive characteristics of waves of different type in a compressible single oceanic layer model overlying a transversely isotropic seabed. A closed-form integral expression is presented for the water surface displacement due to the subsea fault. Furthermore, the wavefield components (displacements and stresses) can be calculated at any point in the field. The method of potentials, accompanied by a special integration method and the Residue Theorem, are utilized to solve the linear equations of motion.
and develop the numerical results. The waves are generated due to an earthquake source (finite fault plane) which is assumed to occur at an arbitrary depth within the seabed’s interior or—as a special case—at the seafloor. As a result of the transformations of the stress-displacement components, there will be vertical displacements at the seafloor due to the horizontal motion in the interior of the seabed, which affects the water column. The model not only covers the vertical ground displacement, but also includes the waves generated due to the horizontal displacement of the area undergoing faulting due to the earthquake. The zone can generally be of arbitrary shape in the derived formulation, although the numerical results presented here are for a circular area of finite radius. The interaction of the SGW and AGW with the Rayleigh or Scholte waves and the various body waves are taken into account. The results also allow for the contribution of any number of acoustic modes which are possible for the specific circumstances of the problem. Velocity dispersion curves for each wave type are also presented for such a model. These curves have applications in other different fields such as marine seismology and hydrocarbon exploration in addition to the tsunami knowledge, but are used here in determining the number of contributing modes in the generation of the surface waves. The method is valid in both the far-field and the near field and for an ocean (liquid layer) of arbitrary thickness. An isotropic seabed can simply be modeled as a special case of the formulation.

In the forthcoming sequel, Part II of the research, we investigate the surface and interface waves in a stratified ocean and ocean bed system, where the major concentration will be on internal gravity waves.

2. Statement of the Problem

2.1. Model Description

The three-dimensional model being analyzed here consists of a compressible inviscid ocean (liquid layer) overlying a homogeneous elastic transversely isotropic seabed (solid half-space). The cylindrical coordinate system \((r, z, \theta)\) with the positive \(z\) axis downward is selected such that the seafloor is located at \(z = 0\) and the free surface of the liquid lies at \(z = -h\) (Figure 1). The formulation allows for any arbitrary shape of the zone in which the original disturbance (subsea offshore earthquake) occurs. Here for example, the time-harmonic asymmetric disturbance in a circular area (as assumed by Hendin and Stiassnie [2013]) at a depth \(s\) within the seabed is considered. The ground motion is decomposed into the vertical (axisymmetric) and horizontal (antisymmetric) motions. The disturbance can be specified with the aid of the Heaviside or unit step function \(H(t)\) and the Dirac delta function \(\delta(t)\). The axisymmetric part of the excitation with amplitude \(U_V\) can be described as:

\[
\omega_z(r, \theta, z) = U_V H(R-r) \delta(z-s) e^{i\omega t},
\]

and the horizontal part having amplitude \(U_H\) can be defined as:

\[
\omega_r(r, \theta, z) = U_H H(R-r) \delta(z-s) \cos \theta \ e^{i\omega t},
\]

\[
\omega_\theta(r, \theta, z) = -U_H H(R-r) \delta(z-s) \sin \theta \ e^{i\omega t}.
\]
Here \(u_x, u_y, \) and \(u_z\) are the displacement components in the three coordinate directions, \(R\) is the radius of the circular zone, and \(\omega\) is the frequency. Hereafter, the time factor \(e^{i\omega t}\) is assumed and suppressed in the equations. It worth mentioning that a point source can be described as a limiting case when \(\lim R \to 0\).

### 2.2. Governing Equations and Solutions

The time-harmonic equations of motion for a homogeneous, transversely isotropic elastic solid presented by \textit{Lekhnitskii} [1963] can be expressed in terms of the two complete potential functions \(F\) and \(\chi\) as follows [\textit{Eskandari-Ghadi}, 2005]:

\[
\left( \nabla^2 \nabla^2 + \delta \omega^2 \frac{d^2}{dz^2} \right) F(r, \theta, z) = 0, \\
\nabla^2 F(r, \theta, z) = 0,
\]

where

\[
\nabla^2 = \nabla^2 + \frac{1}{s_i^2} \frac{d^2}{dz^2} + \frac{1}{\mu_i} \frac{c_{66}}{s_i^2}, \quad i = 0, 1, 2,
\]

\[
\delta = \frac{1}{c_{44} s_2^2} - \frac{1}{c_{11} s_1^4} + \frac{1}{1 + \frac{c_{33}}{c_{44}}},
\]

In these expressions, \(c_{ij}\) are the five elastic constants which characterize the anisotropic solid medium, and \(\rho_s\) is the mass density of the solid. Quantities \(s_1\) and \(s_2\) are the roots of the strain energy function and \(s_0 = c_{66} / c_{44}\).

Applying the Hankel transform of order \(m\) in the radial direction \((r \to \kappa)\) and using a Fourier series expansion in the azimuthal direction, the partial differential equation for \(\tilde{F}^m_\kappa(r, z)\) and \(\tilde{\chi}^m_\kappa(r, z)\) can be obtained:

\[
\left( \nabla^2_{1m} \nabla^2_{2m} + \delta \omega^2 \frac{d^2}{dz^2} \right) \tilde{F}^m_\kappa(r, z) = 0, \quad \nabla^2_{0m} \tilde{\chi}^m_\kappa(r, z) = 0,
\]

where

\[
\nabla^2_{im} = \frac{\rho_s \omega^2}{\mu_i} - \kappa^2 - \frac{1}{s_i^2} \frac{d^2}{dz^2}, \quad i = 0, 1, 2.
\]

The Hankel transform parameter is indicated by \(\kappa\) which is also associated with the radial wavenumber. The superscript on the potentials indicates the order of the Hankel transform and the subscript represents the component order relative to the Fourier expansion series. The relations between the wavefield components and the potential functions both in the physical and Hankel transform domains are given in Appendix A.

The general solutions to equation (6) are:

\[
\tilde{F}^m_\kappa(r, z) = A_m(\kappa)e^{i\lambda_1 z} + B_m(\kappa)e^{-i\lambda_1 z} + C_m(\kappa)e^{i\lambda_2 z} + D_m(\kappa)e^{-i\lambda_2 z}, \\
\tilde{\chi}^m_\kappa(r, z) = E_m(\kappa)e^{i\lambda_3 z} + F_m(\kappa)e^{-i\lambda_3 z},
\]

where

\[
\lambda_1 = \sqrt{\alpha \kappa^2 + b + \frac{1}{2} \sqrt{\alpha \kappa^2 + d \kappa^2 + e}}, \\
\lambda_2 = \sqrt{\alpha \kappa^2 + b - \frac{1}{2} \sqrt{\alpha \kappa^2 + d \kappa^2 + e}}, \\
\lambda_3 = s_0 \sqrt{s_0^2 - \frac{\rho_s \omega^2}{c_{66}}}
\]

In the above terms,
\[
a = \frac{1}{2} (s_1^2 + s_2^2), \quad b = -\frac{1}{2} \rho_i c^2 \left( \frac{1}{\epsilon_{33}} + \frac{1}{\epsilon_{44}} \right), \\
c = (s_2^2 - s_1^2)^2, \\
d = -2\rho_i c^2 \left[ \left( \frac{1}{\epsilon_{33}} + \frac{1}{\epsilon_{44}} \right) (s_1^2 + s_2^2) - 2\epsilon_{15} \left( \frac{1}{\epsilon_{11}} + \frac{1}{\epsilon_{44}} \right) \right], \\
e = \rho_i c \left( \frac{1}{\epsilon_{33}} - \frac{1}{\epsilon_{44}} \right)^2.
\]

Two sets of potential functions shall be defined for the two parts of the seabed separated by the plane \( z = s \) (i.e., above and below the source level). The two subdomains are denoted by superscripts \( \text{II} \) and \( \text{III} \) whereas the superscript \( I \) stands for the liquid ocean layer. Applying the Hankel transform-Fourier series expansion procedure, the transformed components of the stress and displacement fields can be written in terms of the potential functions [Khojasteh et al., 2008] (see also Appendix A).

For a liquid medium, the displacement potential \( \varphi \) is governed by the equation:

\[
\frac{\partial^2 \varphi}{\partial r^2} = \frac{K}{\rho_i} \nabla^2 \varphi(r, \theta, z),
\]

where \( K \) is the bulk modulus and \( \rho_i \) is the mass density of the liquid. If the same procedure of Hankel transform-Fourier series expansion is also applied here, the governing equation takes the following form in the transformed domain:

\[
\left( \frac{\rho_i c^2}{K} - \kappa^2 + \frac{d^2}{dz^2} \right) \tilde{\varphi}_m^0 = 0.
\]

The general solution of this equation is:

\[
\tilde{\varphi}_m^0(\kappa, z) = S_m(\kappa)e^{-i\lambda_3 z} + R_m(\kappa)e^{i\lambda_3 z}.
\]

Here \( \lambda_3 = \sqrt{K/\rho_i} \) is the velocity of sound in the liquid, and:

\[
\lambda_3 = \sqrt{\kappa^2 - \frac{\omega^2}{c_i^2}}.
\]

Note that \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) are multivalued functions and are made single-valued by specifying the branch cuts emanating from the branch points \( \kappa_1 = \pm \omega \sqrt{\rho_i/c_{11}}, \kappa_2 = \pm \omega \sqrt{\rho_i/c_{44}}, \kappa_3 = \pm \omega \sqrt{\rho_i/c_{15}}, \) and \( \kappa_4 = \pm \omega \sqrt{\rho_i/K} \) in the complex \( \kappa \)-plane, such that the real parts of \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \) are non-negative for all values of \( \kappa \). The positive sign of these branch points denotes the wavenumbers of \( P, \) SV, and SH waves in the solid and the \( P \) wave in the liquid, respectively. The transformed hydrodynamic pressure and the vertical displacement of the liquid particles are expressed in terms of the potential as \( \rho_i c \tilde{\varphi}_m^0 \) and \( \partial \tilde{\varphi}_m^0 / \partial z \), respectively.

### 3. Boundary Conditions and the Dispersion Equation

The dispersion equation, which relates the wave speed (or wavenumber) to frequency, is obtained by applying the boundary conditions. At the free surface \( z = -h \), the kinematic boundary condition is:

\[
\eta = \frac{\partial \varphi}{\partial z} \bigg|_{z=-h},
\]

where \( \eta \) is the unknown vertical displacement of the surface. The dynamic boundary condition from Bernoulli’s equation is:

\[
\frac{\partial^2 \varphi}{\partial z^2} \bigg|_{z=-h} - g \eta = 0.
\]

In this equation, \( g \) is the acceleration due to gravity. Combining equations (15) and (16), the free surface boundary condition can be obtained. Following the previously introduced Hankel transform-Fourier series
expansion procedure, the time-harmonic free surface boundary condition in the transformed domain can be expressed as:

\[ \partial^2 \tilde{\varphi}_m (z) \bigg|_{z = -h} + g \frac{\partial \tilde{\varphi}_m}{\partial z} \bigg|_{z = -h} = 0. \]  

(17)

At the seafloor, the boundary conditions are continuity of the normal displacement and the vanishing of the shear stress components \((\sigma_{z1}, \sigma_{z2})\). Furthermore, the continuity of the normal component of stress \(\sigma_{zz}\) at the interface implies:

\[ \rho_0 \omega^2 \tilde{\varphi}_m \bigg|_{z = -h} + \rho_0 g \frac{\partial \tilde{\varphi}_m}{\partial z} \bigg|_{z = -h} = -\tilde{\sigma}_{zz}^m. \]

(18)

Note that due to the nonviscous nature of the liquid layer, the perturbation can only be vertically transmitted into the liquid layer. The dispersion equation for the surface/interface waves can now be developed by means of the boundary conditions. The dispersion equation is \(f(\kappa, \omega) = 0\), where:

\[ f(\kappa, \omega) = c_{33} \lambda_3 (\zeta_1 v_1 - \zeta_2 v_2) (\sinh (\lambda_3 h) g \lambda_4 - \cosh (\lambda_3 h) \omega^2) \]

\[ + \sinh (\lambda_3 h) \rho_0 (\zeta_2 v_1 - \zeta_1 v_2) \left( g^2 \lambda_4^2 - \omega^2 \right). \]

(19)

In the above equation:

\[ \zeta_i = (x_2 - x_1) \lambda_i^2 + \kappa^2 (1 + x_1) - \frac{\rho_0 \omega^2}{c_{66}}, \quad \psi_i = x_2 \lambda_i^2 - \zeta_i; \]

\[ v_i = \left( \zeta_i - x_2 \frac{c_{13}}{c_{33}} \kappa^2 - x_3 \lambda_i^2 \right) \lambda_i, \quad i = 1, 2. \]

(20)

The velocity of all kinds of generated waves (SGW, AGW, Rayleigh, and Scholte) can be calculated by means of the derived dispersion equation. Dispersion curves can also be plotted by successively solving this equation.

For any values of the mentioned parameters, there exist at least two real roots \(\kappa_T\) and \(\kappa^*\). The first one \((\kappa_T)\) belongs to the SGW class (also called tsunami mode) and is the largest root. The latter \((\kappa^*)\) belongs to the Rayleigh class if \(\kappa^* < \omega \sqrt{\rho_0 / K}\) and otherwise it is classified as the Scholte wave. It is worth noting that if the seabed is rigid, \(\kappa^*\) would not arise. Dependent upon the frequency and the properties of both the seabed and the seawater, several further roots associated with the acoustic-gravity modes \((\kappa_{AGW})\) can arise. Acoustic-gravity wavenumbers fall in the range \(\omega \sqrt{\rho_0 / c_{44}} < \kappa_{AGW} < \omega \sqrt{\rho_0 / K}\) and reveal a higher velocity compared to the Rayleigh and Scholte modes. Wavenumbers \(\kappa_T\) and \(\kappa_{Scholte}\) (if there is any) are the only roots greater than \(\kappa_T\); even though, the Scholte wavenumber usually falls slightly after \(\kappa_T\) whereas the surface gravity wavenumber is significantly further removed, which implies this mode is considerably slower than the \(P\) wave in water. The Rayleigh, Scholte, and higher modes have a travel time much shorter than the SGW. It should be mentioned that if the water compressibility and finite thickness of the sea column were neglected, for any values assigned to the other parameters, there would arise exactly one root \(\kappa^* > \omega \sqrt{\rho_0 / K}\) associated with the Scholte mode.

Other boundary conditions are pertinent to the depth at which the faulting occurs, i.e., \(z = s\). At this level, the continuity of the stress components is preserved whereas there are discontinuities of displacement components within the faulting zone as described in section 2.1. To solve the problem in the transformed domain, equations (1–3) should also be Hankel-transformed and expanded by Fourier series. From the Sommerfeld radiation condition, the potential functions \(\tilde{F}_m^\ell\) and \(\tilde{\gamma}_m^\ell\) must vanish when \(z \to \infty\).

4. Closed-Form Solution for \(\eta(r, \theta, z)\)

The main objective here is to develop expressions for \(\eta\) and hence for \(\varphi\). The boundary conditions are combined to form the matrix equation as follows:
The load vector at the right side of the equation contains terms associated with both horizontal ($\mathbf{\tilde{H}}_m$) and vertical ($\mathbf{\tilde{V}}_m$) ground motions. In the load vector:

$$\mathbf{\tilde{H}}_{m} = (\pm 1) \frac{e^{\pm jkL}}{2\kappa^2} \mathbf{U}_R \mathbf{J}_1(\kappa x),$$

$$\mathbf{\tilde{V}}_0 = \frac{\mathbf{U}_R \mathbf{J}_0(\kappa x)}{\kappa},$$

where $\mathbf{\tilde{H}}_m$ and $\mathbf{\tilde{V}}_m$ are zero for the rest of the values of $m$. Note that a two-layer seabed can be easily modeled by slightly modifying the matrix at the left side of equation (21). By solving equation (21), the unknown coefficients and potential functions are determined. Hence, the expression for $\mathbf{\tilde{H}}_m$ can be developed now. The expression is decomposed into separate parts attributed to the vertical and horizontal ground motions. For the vertical ground motion:

$$\mathbf{\tilde{H}}_{m,v} = -\frac{c_{33} \mathbf{\tilde{A}}_4(f(-2i\kappa v_1 - e^{2i\kappa v_1} v_2) + 2f(\omega^2, \kappa v_1) \mathbf{\tilde{V}}_m)}{f(\kappa, \omega)};$$

and for the horizontal ground motion:

$$\mathbf{\tilde{H}}_{m,h} = \frac{c_{33}(e^{-2i\kappa v_1} + e^{2i\kappa v_1}) \mathbf{\tilde{A}}_4(\omega^2 \mathbf{\tilde{H}}_m)}{2\kappa \mathbf{\tilde{J}}_2 v_1 - \mathbf{\tilde{J}}_1 v_2 f(\kappa, \omega)}.$$  

In addition to the water surface displacement, all the stress (pressure) and displacement components within both the sea and the solid substratum can also be obtained in terms of the determined potential functions by means of the equations presented in Appendix A. The explicit expressions for these wavefield components are not mentioned here for brevity but they are exploited in developing the numerical results.

The expressions for the transformed surface displacement (and also other displacement/pressure components) are returned back to the physical domain by means of an inverse Hankel transform integral and substituting in the Fourier series coefficients as:

$$Y(r, \theta) = \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} k \mathbf{\tilde{Y}}_m(k) J_m(kr) dk e^{im\theta}. \tag{25}$$

The expressions are obviously too complicated to be analytically integrated and special methods must be applied for evaluating the numerical results. Moreover, the roots of the dispersion equation function as the singular points during the integration process and should be precisely handled. The wavenumbers associated with the surface waves are the poles whereas those associated with the body waves in the liquid ($P$, $S$) and solid substratum ($P$, $S$, and $SH$) are called the branch points. For the numerical evaluation of the results, the procedure introduced by Khojasteh et al. [2008] is modified due to the dispersive characteristics and various types of existing waves. The path of integration is divided into separate subintervals falling between the starting point ($\kappa = 0$), the poles, and the infinity (large enough values of $\kappa$). To account for the entire domain, the residue of integration due to the infinitesimal semicircle over each pole should be added. Since the poles are the singular points of first order, the integrands can be rewritten in the form of $q(\kappa \omega)/f(\kappa, \omega)$ where $q(\kappa \omega)$ is analytic. Thus, from the Residue Theorem the integral over the limiting small semi-circle at the pole is equal to $-\pi i \text{Res}(\kappa \omega)$, where $\text{Res}(\kappa \omega) = \lim_{\kappa \rightarrow \kappa_0} q(\kappa \omega)/(f(\kappa, \omega)/d\kappa)$. The residues at the branch points are zero. A computer code was developed in Mathematica software applying an adaptive quadrature approach for the
integration between the poles and also after the pole having the largest abscissa \( \kappa \) out to a sufficiently large value of \( \kappa \).

5. Asymptotic and Special Cases

Some special cases can now be developed by simplifying the derived expressions.

Case 1. For the deep water approximation; i.e., \( \kappa h \gg \kappa \), the dispersion equation is separated into two parts; one is attributed to the persistent Scholte wave as:

\[
\varepsilon_{33} \dot{\lambda}_4 (\zeta_2 \nu_1 - \zeta_1 \nu_2) + \rho_l (\zeta_2 \theta_1 - \zeta_1 \theta_2) (\dot{\lambda}_4 + \omega^2) = 0, \tag{26}
\]

and the other part is associated with the SGW, which is not affected by the ocean bed:

\[
\omega^2 = g \dot{\lambda}_4. \tag{27}
\]

Note that if the compressibility of the water is not considered, the latter dispersion equation reduces to the well-known equation \( \omega^2 = g \kappa \).

Figure 2. Phase velocity dispersion curves for the first four modes for a model with an isotropic seabed.

Figure 3. Phase and group velocity dispersion curves for (a) Rayleigh, Scholte, and the first four acoustic-gravity modes for an ocean of anisotropic bed. (b and c) SGW for an ocean with an anisotropic seabed when (b) \( h = 4000 \) and (c) \( h = 100 \) m.
Case 2. For the shallow water approximation, when $\frac{j h}{C_{28}} \ll 1$, the dispersion equation is again separated into two parts, one associated with the nondispersive Rayleigh wave on a transversely isotropic half-space not being affected by the water, i.e., $\zeta_2 v_1 - \zeta_1 v_2 = 0$ [Khojasteh et al. 2008] and the other one which is the dispersion equation for the SGW, $\omega = \lambda_4 \sqrt{gh}$. The latter gives the phase velocity (celerity) of the SGW as $c_{SGW} = \sqrt{gh}$, if the compressibility is neglected.

Case 3. The results reduce to those for an isotropic seabed by appropriately setting the elastic constants to $c_{11} = c_{33} = 2\mu + \lambda$, $c_{12} = c_{13} = \lambda$, $c_{44} = c_{66} = \frac{\mu}{2}$. If the role of gravity at the "liquid-solid interface" is also neglected, which does not depart far from reality, the results developed by Eyov et al. [2013] are obtained. In this case, $\lambda_1 = \frac{\sqrt{\mu}}{\rho}, \lambda_2 = \frac{\sqrt{\mu}}{\rho}, \lambda_3 = \frac{\sqrt{\mu}}{\rho}$, $\lambda_4 = \frac{\sqrt{\mu}}{\rho}$, and $\lambda_5 = \frac{\sqrt{\mu}}{\rho}$ being the wavenumbers associated with P and S waves, and the following expressions should be substituted into the derived expressions:

$$
\begin{align*}
\zeta_1 &= \frac{\lambda_1^2}{\rho} \beta^2, \\
\zeta_2 &= \frac{\lambda_2^2}{\rho} \beta^2, \\
\zeta_3 &= \frac{\lambda_3^2}{\rho} \beta^2, \\
\zeta_4 &= \frac{\lambda_4^2}{\rho} \beta^2.
\end{align*}
$$

(28)

Case 4. If the gravitational terms are neglected, the obtained dispersion equation and wavefield quantities reduce to the dispersive elastic and acoustic waves in an ocean underlain by an anisotropic solid. The dispersion equation in this case is equivalent to the dispersion equation presented by Bagheri et al. [32] who investigated the dispersive characteristics of different "non-gravitational" modes, but not the full response of the system.

Case 5. If the seabed is considered rigid, the dispersion equation and respective curves developed by Yamamoto [1982] can be obtained.
6. Numerical Implementation and Example

As stated earlier, the water surface displacement and propagation pattern of the waves within the liquid-solid system are the main objectives of this research. As a by-product, however, phase velocity \( (V = \omega/k) \) and group velocity \( (U = d\omega/dk) \) dispersion curves which have various applications in different fields are also presented by solving the dispersion equations. The dispersion curves can be utilized here to determine the number of the poles in the integration path and the location of each.

To confirm the results with a similar but simpler case, the phase velocity dispersion curves for the first four AGW modes for an ocean with an isotropic seabed are presented in Figure 2 which shows exact agreement with those presented by Eyov et al. [2013]. In order to produce these curves, the properties of the seawater and the seabed (densities and elastic constants) are set as follows: \( c_{SH} = c_{SV} = 3550 \text{ m/s}, c_{P} = 6300 \text{ m/s}, \rho_{l} = 1450 \text{ m/s}, \rho_{s} = 2750 \text{ kg/m}^3, \) and \( \rho_{l} = 1020 \text{ kg/m}^3 \) where \( c_{SH}, c_{SV}, \) and \( c_{P} \) are the wave speeds of SH, SV and P waves in the seabed, respectively, and \( \omega = 2\pi \text{ Rad/s}. \)

Now, to consider an ocean with a transversely isotropic bed and find the displacement and stress wavefields, first the phase velocity dispersion curves for the SGW, the Rayleigh wave (which turns to the Scholte wave at higher frequencies) and the following four AGWs are derived for a gravitating ocean (Figure 3). The VTI seabed properties are as for the Mesaverde Sandstone [Ben-Menahem and Sena, 1990] having elastic constants: \( c_{11} = 50.01 \text{ GPa}, c_{12} = -8.6 \text{ GPa}, c_{44} = 24.57 \text{ GPa}, c_{55} = 26.59 \text{ GPa}, \) and \( c_{33} = 45.05 \text{ GPa}, \) and density \( 2870 \text{ kg/m}^3. \) The properties of the overlying water layer are \( \rho_{l} = 1000 \text{ kg/m}^3 \) and \( K = 2.2 \text{ GPa}. \) SGW velocity dispersion curves are given for water depths \( h = 4000 \text{ m} \) (Figure 3b) and \( h = 100 \text{ m} \) (Figure 3c). In contrast to SGW, all the AGWs and the Scholte mode exhibit cutoff frequencies. Moreover, except SGW, all the group velocity curves exhibit a minimum value called the Airy phase. Generally, as the frequency increases, more AGW modes arise. The dispersion curves demonstrate the significant difference in velocity between the SGW (tsunami mode) and other types of existing modes.

As explained in section 4, by subdividing the integration path, using the Residue Theorem, and applying the adaptive integration method, the water surface displacement and all other wavefield components at
any point in the domain can now be calculated in the physical domain. For a given dimensionless frequency \( \frac{\omega h}{c_{SV}} \), the number and velocities of the contributing modes are determined by means of the dispersion curves. Since the poles along the integration path are the respective wavenumbers associated with each mode, the number of phase or group velocity curves cut by a vertical line \( x = \frac{c_{SV}}{h} \text{ const} \) indicates the number of the poles and involved modes. For example, the schematic integration path when three AGWs, Scholte, and SGW are involved can be found in Figure 4.

For numerical interpretation of the displacements and pressures, the results are presented for different dimensionless frequencies \( \frac{\omega h}{c_{SV}} \) and also for \( h = 1000 \) m and \( h = 100 \) m.

Figure 5a shows the amplitudes of the particle vertical displacements for an ocean of 4000 m depth when the ground motion occurs at \( s = \frac{5h}{2} \), the dimensionless frequency is \( \frac{\omega h}{c_{SV}} = 1 \), and \( R = 2h \). This figure, which shows the water vertical surface displacement due to vertical, horizontal, and the overall nonsymmetric water surface displacement, enables comparison between the contributions of the vertical and horizontal ground motions. The normal displacement of the ocean bottom \( z = 0 \) when both vertical and horizontal ground motions are present.
horizontal ground motions are included is also presented in Figure 5a and can be compared to the water surface displacement due to the same excitation. Real and imaginary parts of the hydrodynamic pressure at the ocean bed due to vertical and horizontal ground motions are displayed in Figure 5b. The curves of the quantities for the vertical ground motion are axisymmetric whereas those for the horizontal ground motion are antisymmetric. Note that as the dispersion curves indicate, for $x_h = c_{SV}^5 1$ only the SGW (tsunami mode) and Rayleigh modes are involved. In this case, the velocities of the surface gravity wave and the Rayleigh wave are 13.4 m/s (48.3 km/h) and 1877 m/s, respectively.

In Figures 6 and 7, the wavefields are also presented for $h = 100$ m, $R = h/2$ and two dimensionless frequencies $\omega h/c_{SV} = 1$ and $6$. In Figure 6, the amplitudes of the normal displacement of the particles are presented at different levels, i.e., within the seabed, $z = h/3$, at the seafloor, $z = 0$, and within the seawater at $z = -h/3$, $z = -2 h/3$, and $z = -h$ to describe the propagation of the disturbance in the radial direction at these levels. Figures 6a and 6b show the displacement at the above specified elevations for unit vertical and horizontal ground motions when $\omega h/c_{SV} = 6$, whereas Figures 6c and 6d present results for $\omega h/c_{SV} = 1$. For $\omega h/c_{SV} = 1$, only the SGW and Rayleigh are contributing whereas for $\omega h/c_{SV} = 6$, three further acoustic-gravity modes are also involved in addition to the Scholte and SGW modes.

To find out further about aspects of the phenomenon, the amplitudes of the hydrodynamic pressure within the seawater and the normal stress within the seabed ($-\sigma_{zz}$) along the levels mentioned in Figure 6 are presented in Figure 7a. The figure shows the considerable stress which arises within the seabed at the points near to the source zone, especially compared to the ocean medium.

Vertical profiles along the $z$ axis are presented in Figures 7b and 7c in order to depict how the generated disturbance due to the vertical ground displacement propagates upward to reach the free surface of the ocean and forms the surface waves and also how it damps out downward into the earth. These figures (7b and 7c) give the real and imaginary parts of the normal transformed displacement and pressure along the $z$ axis. As apparent in Figure 7b, the real part of the normal displacement shows a discontinuity at the point $z = s$ with a gap value equal to unity. This discontinuity is expected due to the earthquake rupture at the focal point and is equivalent to the mathematical description given in equation (1). In the vertical profiles, there are also discontinuities in the gradients at $z = 0$, which arise from the stepwise difference in the medium properties between the solid sea bottom and the seawater.

To investigate the effect of the location of the faulting (rupture) zone and the frequency of excitation on the generated surface wave, we show in Figure 8 the real and imaginary parts of the water surface...
displacement $\eta$ for ground motions at $s = 3h/2$ and $s = h/2$ and two frequencies $\omega h/c_{SV} = 1$ and 6. The surface displacement due to the horizontal ground motions are also compared to that due to a vertical ground motion in the same figure. The displacements for high frequencies appear to be more oscillatory and larger in amplitude. Furthermore, the figure reveals that, as expected, as the source gets close to the interface the larger surface waves arise. The figure also indicates, somewhat surprisingly, that the effect of the horizontal component is fairly comparable to the effect of the vertical motion.

7. Conclusion

Different aspects of wave propagation in an ocean with an anisotropic seabed when a dynamic source excitation occurs at a certain depth within the bed are theoretically studied. The formulation is fairly simple and the introduced procedure to obtain the numerical results is quite straightforward. The analysis can be easily adopted to solve the problems without the common restrictive assumptions about the parameters such as focal depth, deep or shallow water, or seeking the results only at distances far from or near to the source. When utilizing the numerical procedure, special care must be exercised in the number and the exact value of the poles which are pertinent to the waves of different type.

By solving the deduced dispersion (frequency) equations, phase and group velocity curves are presented for all types of possible dispersive waves in the model, which include surface gravity, acoustic-gravity, Rayleigh, and Scholte waves. The velocity dispersion curves indicate the great difference in the velocity of the SGW and other types of the waves which is the basic concept of the TEWS using a network of sensors and solving an inverse problem. The dispersion equation can directly be exploited to forecast the characteristics of the waves while shoaling. Although further details about shoaling are not in the framework of this paper, it can be conveniently handled by extending the current work and modifying the boundary conditions. The bathymetric variations which are not considered in this paper can be handled by dividing the entire region into tracts of relatively uniform depth and again utilizing the dispersion equations to find new velocities and then calculate the water surface displacement.

The role of different parameters such as the focal point location, bed anisotropy, and the frequency of excitation are taken into account. The results reveal that the depth at which the fault rupture is located is undoubtedly one of the dominant parameters of the problem, although it is usually disregarded. The common observation of anisotropy in the ocean bed is also accounted for; where the SV and SH waves have different travel times. This difference in the velocities is important since the higher bond of the AGWs is the SV wave velocity which is, contrary to an isotropic solid, not equal to SH wave velocity.

The snapshots that show how the evolving disturbance propagates upward into the seabed indicate the significant alteration in the pattern when it reaches the ocean and propels the water within the ocean. In this regard, the displacement profile indicates that the particle displacements are suddenly amplified when the wave transfers into the water. Depending upon the circumstances, the displacement of the water surface particles may be several times the displacement of the particles in the vicinity of the focal point or at the seafloor.

Comparisons are made also to examine the effect of the frequency of the excitation. The results indicate that as the frequency increases, further modes which travel with a higher wave speed are also possible. Higher frequency causes more disturbance and larger amplitudes whereas a lower frequency results in longer wavelengths.

The contribution of the horizontal ground motion is also examined. Although the horizontal motion, as expected by intuition, causes surface waves with lower amplitudes than vertical motion, the displacement curves reveal that its contribution in forming these waves cannot totally be neglected. In fact, since most previous researches assume the faulting to be a simplified (upward) motion in a limited zone at the water-seabed interface, and enforcing the rest of the area to be fixed at its position, the contribution of the horizontal motion has been disregarded. However, when (horizontal) faulting occurs within the interior of the seabed, there will be vertical displacement at the seafloor resulting in the ocean reacting as an apparent surface displacement. Note that due to the inviscid assumption of the seawater, there is no horizontal interaction between the sea and the seabed.

Expressions derived in this paper for the emergent waves and the velocity dispersion curves are of significant benefit for tsunami hazard assessment, underwater earthquake source localization, and marine
seismology. The speed of the waves which can be extracted from the dispersion curve is an important factor when studying tsunamiic deposits.

The results have also potential benefits for engineering purposes because they can be used to analyze the marine and coastal structures subjected to hydrodynamic forces. In this regard, since the displacements and pressures are determined at any point, one can utilize the Green’s function concept for both the structure and the medium and use the compatibility conditions to derive the respective boundary integral equations. The derived expressions can be also used to verify the numerical methods which are often used to tackle similar problems.

**Appendix A: Inter-Relations Between Potentials, Displacements, and Stresses**

The potentials \( F \) and \( \chi \) are related to the displacement components as follows:

\[
\begin{align*}
\mathbf{u}_i(r, \theta, z) &= -\alpha_3 \frac{\partial^2 F(r, \theta, z)}{\partial \theta \partial z} - \frac{\partial \chi(r, \theta, z)}{\partial \theta}, \\
\mathbf{u}_0(r, \theta, z) &= -\alpha_3 \frac{1}{r} \frac{\partial F(r, \theta, z)}{\partial \theta}, \\
\mathbf{u}_2(r, \theta, z) &= \left( 1 + z_1 \right) \nabla_m + \alpha_2 \frac{\partial^2 F(r, \theta, z)}{\partial z^2} + \frac{\rho_i \alpha_2^2}{c_{66}} F(r, \theta, z),
\end{align*}
\]

The dimensionless \( \alpha \) terms are defined in terms of the elastic moduli through the ratios:

\[
\alpha_1 = \frac{C_{12} + C_{66}}{C_{66}}, \quad \alpha_2 = \frac{C_{44}}{C_{66}}, \quad \alpha_3 = \frac{C_{13} + C_{44}}{C_{66}}.
\]

The stress and displacement components are related to the transformed potential functions as follows. Hankel transform properties have been utilized to derive these equations.

\[
\begin{align*}
\hat{u}_m^n &= \left( 2z_2 \frac{d^2}{dz^2} + \frac{\rho_i \alpha_2^2}{c_{66}} - \kappa^2 \left( 1 + z_1 \right) \right) \hat{u}_m^n, \\
\hat{u}_m^{n+1} + i\hat{u}_m^{n+1} &= \alpha_3 \kappa \frac{dF_m}{dz} - i \kappa \hat{u}_m^n, \\
\hat{u}_m^{n-1} - i\hat{u}_m^{n-1} &= -\alpha_3 \kappa \frac{d^2F_m}{dz^2} - i \kappa \hat{u}_m^n,
\end{align*}
\]

\[
\begin{align*}
\hat{\sigma}_m^n &= \frac{d}{dz} \left( 2z_3 C_{13} \kappa^2 + C_{33} \left( \frac{\rho_i \alpha_2^2}{c_{66}} - \kappa^2 \left( 1 + z_1 \right) \right) + C_{33} \frac{2z_2 d^2}{dz} \right) \hat{r}_m, \\
\hat{\sigma}_m^{n+1} + i\hat{\sigma}_m^{n+1} &= C_{44} \kappa \left( \left( z_3 - z_2 \right) \frac{d^2}{dz^2} + \kappa^2 \left( 1 + z_1 \right) - \frac{\rho_i \alpha_2^2}{c_{66}} \right) \hat{F}_m - C_{44} \kappa \frac{dF_m}{dz}, \\
\hat{\sigma}_m^{n-1} - i\hat{\sigma}_m^{n-1} &= -C_{44} \kappa \left( \left( z_3 - z_2 \right) \frac{d^2}{dz^2} + \kappa^2 \left( 1 + z_1 \right) - \frac{\rho_i \alpha_2^2}{c_{66}} \right) \hat{F}_m - C_{44} \kappa \frac{d^2F_m}{dz^2}.
\end{align*}
\]

**References**


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No data has been utilized in this study.


