VISCOELASTIC MHD FLUID FLOW BETWEEN NONPARALLEL PLATES: 
ANALYTICAL INVESTIGATION

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ABSTRACT
The present study discusses the velocity profile of the steady 2-dimensional flow of a MHD Viscoelastic fluid between two nonparallel plates. Firstly, a similarity transformation is used to reduce the partial differential equations of modeling the flow, to single third-order nonlinear differential equations containing the semi angle between the plates, Reynolds number, the magnetic field strength and Weissenberg number as parameters. The analytical method used Galerkin Optimal Homotopy Asymptotic Method to solve the problem. The obtained approximate results are compared with those of numerical solution in some numerical cases.

KEYWORDS: MHD, Viscoelastic Fluid, Nonparallel plates, Analytical Solution, GOHAM.

1. INTRODUCTION

Most industrial fluid processing includes non-Newtonian liquids like multi-grade oils, liquid detergents, paints, polymer solutions and polymer melts. In recent years the analysis of the effect of rotating concentric cylinders using non-Newtonian liquids is a popular area of research, not only due to its geophysical and technological importance but also in view of the interesting mathematical features presented by the equations governing the flow [1-6].

The study of flow for an electrically conducting fluid has applications in many engineering problems such as MHD power generators, MHD pumps, accelerators, plasma studies, geothermal energy extractions, the boundary layer control, aerodynamic heating, electrostatic precipitation, etc. One of the early works was done by Hughes et al. [7] performing a CFD analysis for simple parabolic and elliptic MHD flows in presence of constant magnetic field. Serizawa et al. [8] investigated the MHD effects on NaK-Nitrogen two phase flow and heat transfer in vertical round tubes. Unsteady MHD film flow over a rotating infinite disk was studied by Kumari and Nath [9]. In a study by Xu et al. [10] a series solution of unsteady three-dimensional MHD flow and heat transfer in the boundary layer for the case of impulsive stretching plate was given. Analytical solution was given to the problem of the unsteady MHD flow of a viscous fluid between moving parallel plates by Sweet et al.[11]. Li et al. [12] solved two-dimensional steady MHD flow in ducts using a numerical scheme.

Most scientific problems in fluid mechanics such as MHD Viscous fluid flow over the stretching sheet are inherently nonlinear by nature and except for a limited number of cases, most of them do not have exact solutions. Accordingly, the nonlinear equations are usually solved using other methods including numerical techniques or using analytical methods. Some of these methods are Homotopy Perturbation Method (HPM) [13, 14], Reconstruction of Variational Iteration Method (RVIM) [15], Glarkin Optimal Homotopy Asymptotic Method (GOHAM) [16] and others [17, 18].

The aim of this study is to discuss the analytic solutions of the two-dimensional MHD viscoelastic flow between converging/diverging flow.

2. Problem Description

Consider a steady, incompressible and non-Newtonian fluid in a 2D converging/diverging channel flow as shown in Fig. 1
Clearly a viscous fluid is governed by continuity and Navier Stokes equations and when the fluid is considered to be incompressible, isothermal and without gravitational force, the conservation of momentum and total mass are as follows [19]:

\[ \nabla \cdot V = 0 \]  
\[ (1) \]

where \( \rho \) is the fluid density, \( p \) is pressure, \( V \) is velocity vector, and \( f_B \) denotes the force

\[ f_B = \sigma(E + V \times B) \times B \]  
\[ (2) \]

in which \( E \) is the electric field. In the last term of Eq. (3) on the right-hand side is \( B = B_0 + b \) the total magnetic field, \( \sigma \) is the electrical conductivity. For small magnetic

\[ f_B = -\sigma \left[ (v_r \cos \theta) \cos \theta B^2 \hat{r} + (v_r \cos \theta) \sin \theta B^2 \hat{r} \right] \]  
\[ (3) \]

Reynolds number the induced magnetic field is neglected and hence it can be easily written [9,10] as:

In this study purely radial flow is assumed. By rearranging the equations for polar

\[ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) = 0 \]  
\[ (4) \]

coordinate system with no tangential velocity, the simplified equations can be obtained as:

\[ \rho \left( v_r \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) = -\frac{\partial p}{\partial r} + \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{\theta \theta}) + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial r} - \frac{\tau_{\theta \theta}}{r} \right) - \sigma v_r B^2 \cos^2 \theta \]  
\[ (5) \]

\[ 0 = -\frac{\partial p}{\partial \theta} + \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r \theta}) + \frac{1}{r} \frac{\partial \tau_{\theta \theta}}{\partial \theta} - \frac{\tau_{\theta \theta}}{r} \right) - \sigma v_r B^2 \sin \theta \cos \theta \]  
\[ (6) \]

By directly inserting the magnetic field on the flow domain:

\[ \tau = \alpha_1 D + \alpha_2 D^T + \alpha_3 D \cdot D \]  
\[ (7) \]

where \( \alpha_1, \alpha_2 \), and \( \alpha_3 \) are the material modules and are considered to be functions of

\[ D = \frac{1}{2} \left[ \nabla V + (\nabla V)^T \right] \]  
\[ (8) \]

and upper convexed time derivative \( (\dot{V}) \) is the special time derivative calculated in the form of:

\[ \dot{V} = (V \cdot V) D - (\nabla V)^T \cdot D - D \cdot \nabla V \]  
\[ (9) \]
Eliminating pressure from Eqs.(6, 7) and using Eq.(5) the following nonlinear ODE can be obtained:

\[ F'' + 2 \text{Re} \alpha FF' + 4\alpha^2 F' + Wi (8aFF'' + 32\alpha^3 FF') \]
\[ - H \text{Re}[F' \cos(\alpha \eta) - \alpha F \sin(2\alpha \eta)] = 0 \] (11)

where \( \text{Re}, Wi, \) and \( H \) are the Reynolds, Weissenberg and Hartman numbers respectively. With the boundary conditions as the form:

\[ F(\pm 1) = 0, \quad F(0) = 1 \] (12)

Eq.(11) is defined by considering the following parameters:

\[ \text{Re} = \frac{\rho f_0 \alpha}{\mu_0}, \quad Wi = \frac{\psi_{1,0} f_0}{r^2 \mu_0 \alpha}, \quad H = \frac{\sigma B^2 r^2 \alpha}{\rho f_0} \] (13)

\( \mu_0 \) and \( \psi_{1,0} \) denoting the fluid's viscosity coefficient and the first normal stress coefficient of the fluid's elasticity, respectively calculated in zero shear rate.

\[ f_0 = \frac{Q}{\int_{-1}^{1} \alpha F(\eta) d\eta} \]

In Eq.(13) \( f_0 \) is a dimensional constant that can be obtained from the volumetric flow rate \( Q \), by means of the equation:

3. Application of GOHAM

Following differential equation is considered:

\[ L(u(t)) + N(u(t)) + g(t) = 0, \quad B(u) = 0 \] (15)

where, \( L \) is a linear operator, \( u(t) \) is an unknown function, \( g(t) \) is a known function, \( N(u(t)) \) is a nonlinear operator and \( B \) is a boundary operator. By means of OHAM one first constructs a set of equations:

\[ (1 - p)[L(\phi(\tau, p), g(\tau)) - H(p)[L(\phi(\tau, p)) + g(\tau) + N(\phi(\tau, p))]B(\phi(\tau, p))] = 0 \] (16)

where, \( \tau \) is an independent variable, \( p \in [0, 1] \) is an embedding parameter, \( H(p) \) denotes a non-zero auxiliary function for \( \phi(\tau, 0) = u_0(\tau), \quad \phi(\tau, 1) = u(\tau) \)

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Thus, as \( p \) increases from 0 to 1, the solution \( \phi(\tau, p) \) varies from \( u_0(\tau) \) to the solution \( u(\tau) \),

\[ L(u_0(\tau)) + g(\tau) = 0, \quad B(u_0) = 0 \] (18)

The auxiliary function \( H(p) \) can be chosen in the form:

\[ H(p) = p_1 C_1 + p_2 C_2 + \cdots \] (19)

Where \( C_1, C_2, \ldots \) are constants which can be determined later. Expanding \( \phi(\tau, p) \) in a series with respect to \( p \), one has:

\[ \phi(\tau, p, C_i) = u_0(\tau) + \sum_{k=1}^{i} u_k(\tau, C_i) p_k, \quad i = 1, 2, \ldots \] (20)

Substituting Eq.(20) into Eq.(16), collecting the same powers of \( p \), and equating each coefficient of \( p \) to zero, a set of differential equations with boundary conditions can be obtained. Solving these differential equations by boundary conditions, \( u_0(\tau), u_1(\tau, C_1), u_2(\tau, C_2), \ldots \) are obtained.
Generally speaking, the solution of Eq.(4) can be determined approximately in the form:

\[ \phi(\tau, p, C_i) = u_0(\tau) + \sum_{k=1}^{\infty} u_k(\tau, C_i) p_k, \quad i = 1, 2, ..., m \]  \hspace{2cm} (21)

\[ \tilde{u}(m) = u_0(\tau) + \sum_{k=1}^{m} u_k(\tau, C_i) \]  \hspace{2cm} (22)

Note that the last coefficient \( C_m \) can be a function of \( \tau \). Substituting Eq.(19) into Eq.(15), results the following residual:

\[ R(\tau, C_i) = L[\tilde{u}^{(m)}(\tau, C_i)] + g(\tau) + N[\tilde{u}^{(m)}(\tau, C_i)] \]  \hspace{2cm} (23)

If \( R(\tau, C_i) = 0 \) then \( \tilde{u}^{(m)}(\tau, C_i) \) happens to be the exact solution. Generally such a case will not arise for nonlinear problems, but the

\[ w_i = \frac{\partial R(\tau, C_1, C_2, ..., C_m)}{\partial C_i}, \quad i = 1, 2, ..., m \]  \hspace{2cm} (24)

The unknown constants \( C_i (i = 1, 2, ..., m) \) can be identified from the conditions:

\[ J(C_1, C_2) = \int_a^b w_i R(\tau, C_1, C_2, ..., C_m) d\tau = 0 \]  \hspace{2cm} (25)

Where \( a \) and \( b \) are two values, depending on the given problem. With these constants, the approximate solution (of order \( m \)) (Equation. (25)) is well determined. It can be observed that the method proposed in this work generalizes these two methods using the special (more general) auxiliary function \( H(p) \).

4. Results and Discussions

In this article, an analytical solution for magnetohydrodynamic flows of viscoelastic fluids in converging/diverging channels is presented. In figure 2, the effect of Reynolds number on velocity profile in case \( \alpha = -\frac{\pi}{9} \), \( H=10 \), \( Wi=0.1 \). As it can be seen in figure 2, increasing the Reynolds number results in a uniform increase in the velocity. Figure 2 also presents a comparison between analytical results obtained by HPM and numerical solutions achieved from fourth order Runge Kutta method. According to figure, the obtained results have good agreement with numerical solutions.

Fig.(2): Effect of Reynolds number on Velocity profile in case \( \alpha = -\frac{\pi}{9} \), \( H=10 \), \( Wi=0.1 \)

(Line: GOHAM, Symbols: RK4)
In Figure 3, the effect of semi angle on velocity profile in case Re=30, Wi=0.2 and H=10 has been presented for both cases converging and diverging channel. The velocity curves show that the rate of transport is considerably increased with increase of the semi-angle between the two walls especially when two plates move together.

Fig.(3):- Effect of semi angle on Velocity profile in case Re=30, Wi=0.2 and H=10 a- converging channel, b- diverging channel.

In Figure 4 we can see the effect of the magnetic field strength to the velocity in \( r \) direction. As it can be seen by increasing \( H \) the maximum velocity is increasing. Therefore, increasing the electrical conductivity of the fluid or increasing the magnitude of the magnetic field results in a non-uniform increase in \( r \)-direction velocity.

Fig.(4):- Effect of Hartman number on velocity profile in case \( \alpha = -\frac{\pi}{9} \), Re=1, Wi=0.1

Effect of Wiesenber number on the velocity profiles in a converging channel, has been presented in figure 5. As it can be illustrated by increasing Wiesenber number, an increment on velocity profile is occurred.
CONCLUSIONS

In this article, an analytical solution for magnetohydrodynamic flows of viscoelastic fluids in converging/diverging channels is presented. A similarity transform reduces the Navier–Stokes and energy equations to a set of non-linear ordinary differential equations that are solved analytically by means of the GOHAM. The results obtained in this study are compared with numerical results. Close agreement of the two sets of results indicates the accuracy of GOHAM. The effect of key parameters on velocity has been also checked through some plots.

REFERENCES
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