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Radially inhomogeneous bounded plasmas

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Received 25 May 2015, revised 29 February 2016
Accepted for publication 18 April 2016
Published 10 June 2016

Abstract

On the basis of kinetic theory along with self-consistent field equations, the expressions for dielectric tensor of radially inhomogeneous magnetized plasma columns are obtained. The study of dielectric tensor characteristics allows the accurate analysis of the inhomogeneous properties, beyond limitations that exist in the conventional method. Through the Bessel–Fourier transformation, the localized form of material equations in a radially inhomogeneous medium are obtained. In order to verify the integrity of the model and reveal the effect of inhomogeneity, a special case of a cylindrical plasma waveguide completely filled with inhomogeneous magnetized cold plasma was considered. The dispersion relation curves for four families of electromagnetic (EH and HE) and electrostatic (SC and C) modes are obtained and compared with the findings of the conventional model. The numerical analysis indicates that the inhomogeneity effect leads to coupling of electromagnetic and electrostatic modes each having different radial eigen numbers. The study also reveals that the electrostatic modes are more sensitive to inhomogeneous effects than the electromagnetic modes.

Keywords: kinetic theory, dielectric tensor, inhomogeneous magnetized plasma, Bessel–Fourier transformation, cylindrical plasma waveguide, electromagnetic and electrostatic modes

(Some figures may appear in colour only in the online journal)
interesting for researchers dealing with high-power plasma electronics [16], helicon discharges [17, 18], the active action on the ionosphere plasma [19], and physics of electromagnetic sources in inhomogeneous plasmas [20, 21]. In this paper, the inhomogeneity effect on four families of electromagnetic (EH and HE) and electrostatic (SC and C) modes are investigated to verify the integrity of the presented model. In section 2, the dielectric tensor of radially inhomogeneous magnetized plasma column is presented. The dispersion relation of the cylindrical waveguide filled with radially inhomogeneous magnetized cold plasma is obtained in section 3. Sections 4 and 5 are devoted to the discussions and conclusion.

2. Dielectric tensor of inhomogeneous plasma

The concept of homogeneous and inhomogeneous plasma is associated with the comparison between the wavelength $\lambda$ and the inhomogeneous characteristic length $L_0$. The characteristic length scale of the inhomogeneity of laboratory plasmas usually points to the smallest dimension of the experimental set-up. For example, in controlled thermonuclear plasmas usually points to the smallest dimension of the experimental set-up. For example, in controlled thermonuclear plasmas usually points to the smallest dimension of the experimental set-up. For example, in controlled thermonuclear plasmas usually points to the smallest dimension of the experimental set-up. For example, in controlled thermonuclear plasmas usually points to the smallest dimension of the experimental set-up. For example, in controlled thermonuclear plasmas usually points to the smallest dimension of the experimental set-up.

The spatial dispersion arises because the particles contributing to the displacement field at the point $\mathbf{X}$ and time $t$ retain some memory of the fields they experienced along their previous trajectories to the present field point. The presence of the spatial dispersion is the clear indication that the kernel functions of the material equation are dependent upon the response and the coordinates in a separate form. As an example, for a plasma column with inhomogeneity in the radial direction, the material equation can be investigated for three different situations. First, in the simplest model the spatial dispersion can be ignored. In this case for the homogeneous plasma the tensor of complex dielectric permittivity, $\varepsilon_{ij}(t - t')$ does not depend on $r - r'$, $\psi - \psi'$, and $z - z'$ (here $r$, $\psi$, and $z$ represent cylindrical coordinates at time $t$ and $t'$, respectively). The indices $i$ and $j$ refer to three space dimensions. However, for the inhomogeneous plasma the operator $\varepsilon_{ij}(t - t', r)$ can only depend on $r$. This indicates that the kernel function has a stronger dependence on $r$ than $(r - r')$ and $t'$. The material equation can then be expressed as:

$$D_i(r, \psi, z, t) = \int_{-\infty}^{t'} \varepsilon_{ij}(t - t', r)E_j(r, \psi, z, t') \, dt'. \tag{1}$$

Here $D_i$ and $E_j$ represent the components of the displacement $\mathbf{D}$ and electric field $\mathbf{E}$ vectors, respectively. The summation symbol over the $j$ index is removed according to Einstein notation and summation is applied over repeated indices.

Second, in the case of weakly spatial dispersion, the dependence of the kernel function on $(r - r')$ is stronger than $r$ and $r'$. Therefore, the material equation can be written as:

$$D_i(r, \psi, z, t) = \int_{-\infty}^{t'} \int_{-\infty}^{t'} \varepsilon_{ij}(t - t', r - r', \psi - \psi', z - z', r, \psi, \psi')E_j(r, \psi, \psi', z, t') \, dt'. \tag{2}$$

Finally, the last case is when plasma column experiences strong spatial dispersion. The kernel of the material equation is neither dependent upon $(r - r')$ nor $(\psi - \psi')$; however, it depends on both $r$ and $r'$, as well as $\psi$ and $\psi'$, separately,

$$D_i(r, \psi, z, t) = \int_{-\infty}^{t'} \int_{-\infty}^{t'} \int_{-\infty}^{t'} \varepsilon_{ij}(r, \psi, \psi', z, t - t')E_j(r, \psi', \psi', t'). \tag{3}$$

Similarly, the components of the induced current can be represented as:

$$J_i'(r, \psi, z, t) = \int_{-\infty}^{t'} \int_{-\infty}^{t'} \int_{-\infty}^{t'} \sigma_{ijm}(r, \psi, \psi', z - z', t - t')E_j(r, \psi', \psi', t') \tag{4}$$

where $\sigma_{ij}$ is the electric conductivity tensor and the indices $i$ and $j$ again refer to three space dimensions. In the present paper, the plasma column is considered radially inhomogeneous, bounded, and strongly spatial dispersed (the last case). It should be noted that the nonlocality of the material equation cannot be removed by using the Fourier transformation alone. Therefore, the Fourier–Bessel expansion is used for the radial coordinate along with the Fourier series for the angle $\psi$ and the Fourier integral for the $z$-coordinate. Through the use of the Fourier–Bessel expansion, the local material equation in the Fourier space can be written as:

$$D_{im}^\prime(\omega, k) = \sum_m \sum_l \varepsilon_{jim}^\prime(\omega, k)E_{jlm}^\prime \tag{5}$$

$$J_{im}^\prime(\omega, k) = \sum_m \sum_l \sigma_{jim}^\prime(\omega, k)E_{jlm}^\prime \tag{6}$$

where $\varepsilon_{jim}^\prime$ and $\sigma_{jim}^\prime$ are dielectric and conductivity tensor in Fourier–Bessel space, and the indices $m$, $m'$ and $l$, $l'$ are azimuthal and radial numbers, respectively. The detailed derivation of the above relations is listed in appendix A. Note that the presence of indices $l$, $l'$; $m$, $m'$ in the Fourier–Bessel space is due to the consideration of strongly spatial dispersion. The kernels of material equations (3) and (4) are dependent upon $r$, $r'$, $\psi$, $\psi'$, separately; therefore, for the independent coordinates number, independent indices in the Fourier Bessel space must be used.

The kinetic equation with self-consistent fields (Vlasov equation) for particles of type $\alpha$ reads as:

$$\frac{\partial}{\partial t}f_{\alpha} + \mathbf{V} \cdot \nabla f_{\alpha} + e [\mathbf{E} + \mathbf{V} \times \mathbf{B}] \cdot \nabla_{p_{\parallel}} f_{\alpha} = 0. \tag{7}$$

The differential operators $\nabla$ and $\nabla_p$ are gradient operators in ordinary and velocity space, respectively. Due to the small
fluctuations of the electric and magnetic field, the distribution function \( f_{00} \) experiences a small deviation \( \delta f_{00} \) from its equilibrium state \( f_{00} \).

\[
f_{00}(r, \psi, z; V_\psi, \phi, V_\phi, l) = f_{00}(r; V_\psi, \phi, V_\phi) + \delta f_{00}(r, \psi, z; V_\psi, \phi, V_\phi, l) \tag{8}
\]

It is assumed that the equilibrium distribution function \( f_{00} \) only includes the inhomogeneity along the radial direction. The Vlasov equation for the equilibrium state of plasma particles of type \( \alpha \) can then be written as:

\[
V_{\alpha} \cos(\phi - \psi) \frac{\partial f_{00}}{\partial r} - \Omega_\alpha \frac{\partial f_{00}}{\partial \phi} + \sin(\phi - \psi) \frac{V_{\alpha}}{r} \frac{\partial f_{00}}{\partial \psi} = 0. \tag{9}
\]

Here the cylindrical system of coordinates is considered for both the velocity \( \mathbf{V} = (V_\psi, \phi, V_\phi) \) and space \( \mathbf{r} = (r, \psi, z) \) vectors, where \( V_\psi, \phi, V_\phi \) and \( V_\psi \) are particles transverse, polar, and longitudinal velocity component, \( \Omega_\alpha = \left( \frac{eB_0}{m_\alpha} \right) \) is the Larmor frequency of particles of type \( \alpha \), \( e \) and \( m_\alpha \) are charge and mass of particle of type \( \alpha \), and \( B_0 \) is the applied external magnetic field. Introducing a new variable, \( C_\alpha \)

\[
C_\alpha = V_{\psi} \sin(\phi - \psi) + \Omega_\alpha r,
\]

and assuming that the characteristic dimension of plasma inhomogeneity \( L_0 \) is larger than the Larmor radius of particles \( V_{\psi_0} \), then the equilibrium distribution function can be expanded with respect to \( \left( \frac{V_{\psi}}{L_0} \right) \) to second order, the modified Maxwellian distribution function, \( g_{en}(r) \) the inhomogeneous density profile, and \( F_{00}(\alpha, r) \) the ordinary Maxwellian distribution function,

\[
f_{00}(\epsilon_{\alpha}, r, C_\alpha) = \left( 1 + \frac{V_{\psi}}{L_0} \sin(\phi - \psi) \right) F_{00}(\epsilon_{\alpha}, r) \tag{10}
\]

where

\[
F_{00}(\epsilon_{\alpha}, r) = g_{en}(r)F_{00,\alpha}(\epsilon_{\alpha}).
\]

Here \( \epsilon_{\alpha} \) is the mean energy of plasma particles, \( F_{00,\alpha}(\epsilon_{\alpha}, r) \) the modified Maxwellian distribution function, \( g_{en}(r) \) the inhomogeneous density profile, and \( F_{00,\alpha}(\epsilon_{\alpha}) \) the ordinary Maxwellian distribution function,

\[
F_{00,\alpha}(\epsilon_{\alpha}) = \frac{N_{00,\alpha}}{(2\pi m_\alpha T_0)} e^{-\epsilon_{\alpha}/kT_0}
\]

with \( T_0 \) and \( N_{00,\alpha} \) being the temperature and density of particles of type \( \alpha \). It should be noted that the plasma inhomogeneity \( (L_0) \) is defined as \( L_0 = \frac{V_{\psi_0}}{\epsilon_{\alpha}} \). The linearized Vlasov equation for a small deviation, \( \delta f_{00} \), of the distribution function, can be expressed as:

\[
\frac{\partial \delta f_{00}}{\partial t} + V_{\psi} \frac{\partial \delta f_{00}}{\partial z} - V_{\psi} \sin \chi \frac{\partial \delta f_{00}}{\partial \phi} - \Omega_\alpha \frac{\partial \delta f_{00}}{\partial r} + \frac{V_{\psi}}{r} \frac{\partial \delta f_{00}}{\partial \psi} = e_\alpha F_{\psi} \frac{\partial}{\partial \psi} F_{00,\alpha} \times \left( 1 + \frac{V_{\psi}}{L_0} \sin(\phi - \psi) \right) g_{en}(r) \tag{11}
\]

where \( \chi = \phi - \psi \). Any well behaved behavior of \( (r, \psi, z, t) \) can be expanded in terms of orthogonal functions as [11]:

\[
\psi(r, \psi, z, t) = \sum_{m} \int d\omega \int d\omega_{m}(r, t) Y_{m}^{(l)}(r, \psi, z) e^{-i\omega t}. \tag{12}
\]

The orthonormal set \( Y_{m}^{(l)}(r, \psi, z) \) is complete in the intervals \( 0 \leq r \leq a, \ 0 \leq \psi \leq 2\pi, \ -\infty \leq z \leq +\infty, \)

\[
Y_{m}^{(l)}(r, \psi, z) = \frac{\sqrt{2} J_{m}(\rho_{00} r)}{\pi \alpha_{m} + i(\epsilon m_{\alpha})} \left( \begin{array}{c} \cos m_{\psi} \sin \psi \\ \sin m_{\psi} \cos \psi \end{array} \right). \tag{13}\]

\[
m = 0, 1, 2 \ldots; l = 1, 2, 3 \ldots; \ -\infty \leq k \leq +\infty \text{ where } m \text{ is the azimuthal number; } \alpha \text{ is the waveguide radius, } l \text{ is the radial number}, J_{m}(X_{m}) \text{ are the Bessel functions of first kind of order } m, P_{ml} \text{ are the radial wave numbers determined by the boundary condition (} J_{m}(P_{ml}) = 0, \text{ i.e. } P_{ml} = X_{ml} \text{ where } X_{ml} \text{ are the zeros of } J_{m}(X_{ml}) = 0). \]

For the present analysis, the dielectric tensors with even and odd parities are equal; therefore, those terms that are related to even parity and with \( m = 0 \), are only considered. Using the expansion relation (12), in equation (11) leads to,

\[
\begin{align*}
&\left[ \frac{i(\omega - kV_{\psi})}{L_0} \right] \frac{\partial \delta f_{00}}{\partial \psi} \int d\omega_{m}(r, t) \left[ C_{\psi,m} \sin^{2} \psi \frac{\partial \delta f_{00}}{\partial \psi} + V_{\psi}^{2} C_{\psi,m} \sin \psi \frac{\partial \delta f_{00}}{\partial \psi} + V_{\psi}^{2} C_{\psi,m} \cos \psi \frac{\partial \delta f_{00}}{\partial \psi} \right] \\
&+ \int d\omega_{m}(r, t) \left[ V_{\psi}^{2} C_{\psi,m} \sin \psi \frac{\partial \delta f_{00}}{\partial \psi} + V_{\psi}^{2} C_{\psi,m} \cos \psi \frac{\partial \delta f_{00}}{\partial \psi} \right] \left( 1 + \frac{V_{\psi}}{L_0} \sin(\phi - \psi) \right) g_{en}(r) \times \exp \left[ \frac{1}{L_0} \int d\omega \left( i(\omega - kV_{\psi}) \right) \frac{\partial \delta f_{00}}{\partial \psi} \right]
\end{align*}
\]

\[
\int d\omega \int d\omega_{m}(r, t) Y_{m}^{(l)}(r, \psi, z) e^{-i\omega t} = \sum_{m} \sum_{l} Y_{m}^{(l)}(r, \psi, z) e^{-i\omega_{m} t}. \tag{15}
\]

where \( Q_{ml} = \sum_{m} \sum_{l} \left[ A_{00}^{(l)} B_{00}^{(l)} - A_{10}^{(l)} B_{10}^{(l)} \right] \)

The induced current density in the Fourier–Bessel expansion is given by:

\[
J_{\psi,ml}(\omega, k) = \sum_{m} \sum_{l} e_{\psi} Y_{m}^{(l)}(r, \psi, z) e^{-i\omega_{m} t}. \tag{16}
\]

The plasma conductivity \( \sigma_{ijm}^{(l)} \) can then be obtained by substitution of \( \delta f_{00} \) from relation (15) into above equation. The dielectric permittivity and the conductivity in the Fourier–Bessel expansion are related through the following relation

\[
e_{ijm}^{(l)}(\omega, k) = \delta_{ij} \delta_{ml} \omega + \frac{14\pi}{\omega} \sigma_{ijm}^{(l)}(\omega, k). \tag{17}
\]

This leads to the final relation for plasma dielectric permittivity tensor.
\[ e_{l_{ijm}}^{\text{nom}}(\omega, k) = \delta_{ij} \sum_{l} C_{l_{ijm}}^{l_{ijm}} + \sum_{l} C_{l_{ijm}}^{l_{ijm}} + \sum_{l} C_{l_{ijm}}^{l_{ijm}} \]

2.1 The homogeneous plasma approximation

In homogeneous plasma, the wavelength \( \lambda \) is much smaller than the characteristic length \( L_0 (g_\alpha(r) \to 1) \), and

\[ C_{l_{ijm}}^{l_{ijm}} = \delta_{ij}, \quad C_{l_{ijm}}^{l_{ijm}} = 0. \]

Implementing the above definition in the obtained dielectric tensor components and comparing the results with corresponding components of the unbounded homogeneous plasma listed in [8] indicates that the present analysis agrees well with the findings of the cited reference. It should be noted that in the limit of unbounded plasma, the propagation angle in \( x-y \) plane \( \xi \) and the ratio of squared thermal to transverse velocity \( Z_\alpha \) are given as:

\[ \xi = \tan^{-1} \frac{k_y}{k_z}, \quad Z_\alpha(k_\alpha) = \left[ \frac{k_x^2 V^2_{T_\alpha}}{\Omega_\alpha^2} \right]. \]

While for bounded plasma, they are modified to

\[ \xi_m = \tan^{-1} \frac{Q_{Pml}}{Q_{Mml}}, \quad Z_{ml} = \left[ -\frac{Q_{Pml}^2 V^2_{T_\alpha}}{\Omega_\alpha^2} \right]. \]

In the symmetric situation \( \xi_{ml} = \pi/4 \) and the dielectric tensor is symmetric.
2.2. The inhomogeneous plasma approximation without density gradient effects

The density gradient effects for the inhomogeneous magnetized plasma is influenced by the frequency of Larmor drift oscillations \( \omega_{dr} \). The Larmor drift frequency can be approximated as:

\[
\omega_{dr} \approx Q_{PM} \frac{V_T^2}{\Omega_L}. \tag{20}
\]

In the limit of \( \frac{\omega_{dr}}{\omega} \ll 1 \) the effect of particles density gradient can be ignored (i.e. \( C_{lam} \approx 0 \)) and the longitudinal dielectric tensor components can be expressed as:

\[
\varepsilon_{zzm}^l(\omega, k) = 1 + \sum_n \frac{\omega_{Pl}^2}{k^2 V_T^2} [1 - I_n(\beta_{0a})] A_n(z_{cmf}). \tag{21}
\]

Here \( \omega_{Pl}^2 = \omega_{Pi}^2 \sum l \varepsilon_{0mm}^l \) is the discrete plasma Langmuir frequency for inhomogeneous plasma. The comparison of the above components of the dielectric tensor with those of the unbounded inhomogeneous plasma obtained through the optical geometric method reveals that in the limit of strong magnetic field, the effect of boundaries is insignificant (the derivatives of perturbed distribution terms in the linearized equation are neglected). For \( n = 0 \) the longitudinal dielectric tensor components are

\[
\varepsilon_{zzm}^l(\omega, k) = 1 + \sum_n \frac{\omega_{Pl}^2}{k^2 V_T^2} [1 - I_n(\beta_{0a})]
\]

which are in complete agreement with findings of Diaz [11].

2.3. The inhomogeneous plasma approximation with density gradient effects

In the limit of \( \frac{\omega_{dr}}{\omega} \gg 1 \), the portion of the dielectric response related to the particles density gradient is not negligible. In this case the most important factor is the consideration of the Larmor drift, which leads to a current in the direction perpendicular to the magnetic field, as well as, the path of the plasma inhomogeneity. In the present analysis, the current is in \( \phi \) direction and for the stable state is not due to the real motion of the charge particles but it is caused by the diamagnetic effect. Therefore, \( J_{eb} \) is interpreted as the differential diamagnetic current in the inhomogeneous plasma. The consideration of the density gradient effect implies that the components of the dielectric tensor are complex. The imaginary part refers to the presence of the drift instability in the inhomogeneous plasma.

The components of the dielectric tensor obtained through the geometric optics method are a function of the inhomogeneous coordinate. Thus the analysis of the range of plasma frequency leads to the eikonal equation in the geometric optics approximation and the dispersion frequency equation is obtained by the use of Somerfield quantization [8]. However, in the present study, there is no need for quantization rule due to the use of Bessel–Fourier transformation.

3. Dispersion equation of inhomogeneous magnetized plasma waveguide

The effects of inhomogeneity on dispersion characteristics of cylindrical waveguide filled with magnetized plasma are analyzed. The plasma is assumed uniform along the \( z \)-axis with arbitrary radial density profile. The applied static magnetic field is also pointed along the \( z \) axis. For frequency range of \( \omega \gg \omega_{dr} \), the diamagnetic effect is neglected. Here, ions are taken to be stationary, and electrons are considered to possess small thermal motion. Considering the material equation (equation (3)), Maxwell’s equations for the radially inhomogeneous magnetized plasma can be written as:

\[
\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot D = 0, \quad \nabla \times H = \frac{\partial D}{\partial t} + \nabla \cdot B = 0. \tag{22}
\]

In order to obtain the wave equations for the electric and magnetic fields in the direction of propagation, two different Fourier–Bessel expansions were considered. For the \( z \)-component of the electric field wave equation with the boundary condition \( E_z|_{r=a} = 0 \), the Fourier–Bessel expansion of the orthogonal group formed on the roots of the Bessel function of the first kind was used. While for the \( z \)-component of the magnetic field wave equation with the boundary condition \( \frac{\partial B_z}{\partial r}|_{r=a} = 0 \), the orthogonal group formed on the roots of the derivative of the Bessel function of the first kind was utilized. The orthonormal set

\[
Y_{ml}^k(r, \psi, z) = \frac{\sqrt{2(1 - m^2/p_{ml}^2)}}{\pi a m_{-1}(X_{ml}^2)} \exp \left( i \frac{mk}{\cos m\psi} \frac{\sin m\psi}{\sin \psi} \right)
\]

is complete in the intervals \( 0 \leq r \leq a, 0 \leq \psi \leq 2\pi \), and \( -\infty \leq z \leq +\infty \). Here \( m \) is the radial number, \( l \) is the azimuthal number, and \( J_{ml}^k \) is the derivative of the Bessel function of the first kind of order \( m \), \( P_{ml}^k \) are the radial wave numbers determined by the boundary condition. For the symmetric mode \( (m = 0) \), the Maxwell curl equations can be rewritten in terms of transverse and parallel components in the space of Fourier–Bessel transformation (equation (13));

\[
B_{00}^l = -\frac{k}{\omega} E_{00}^l \tag{18}
\]

\[
B_{00}^l J_0(P_{00}r) = \frac{k}{\omega} E_{10}^l J_0(P_{00}r) + \frac{i}{\omega} E_{01}^l \frac{\partial}{\partial r} J_0(P_{00}r) \tag{19}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} r E_{10}^l J_0(P_{00}r) - \frac{1}{r} \frac{\partial}{\partial \psi} E_{10}^l J_0(P_{00}r) = i\omega B_{00}^l J_0(P_{00}r) \tag{20}
\]

\[
B_{00}^l = \frac{\omega}{k c^2} D_{00}^l \tag{21}
\]

\[
-\frac{i\omega}{c^2} D_{00}^l J_0(P_{00}r) = ik B_{00}^l J_0(P_{00}r) - B_{00}^l \frac{\partial}{\partial r} J_0(P_{00}r) \tag{22}
\]
\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} r B_{\phi 0}^f J_0(p_{0fr}) - \frac{1}{r} \frac{\partial}{\partial \theta} B_{\theta 0}^f J_0(p_{0fr}) &= - \frac{i \omega}{c^2} D_{\psi 0}^f J_0(p_{0fr}). \\
\end{align*}
\]  
(23)

Use of equation (5) along with substitution of equations (18) and (21) into (22) and (19) leads to

\[
\sum_f \Pi_{r0}^{lf} E_{r0}^f + \sum_f \Pi_{\psi0}^{lf} E_{\psi0}^f = \frac{ik c^2}{\omega} E^f_0 \frac{\partial}{\partial \psi} J_0(p_{0fr}) J_0(p_{0fr})
\]

(24)

\[
\sum_f \Pi_{r0}^{lf} E_{r0}^f + \sum_f \Pi_{\psi0}^{lf} E_{\psi0}^f = -\frac{ik c^2}{\omega} B_{\psi 0}^f \frac{\partial}{\partial \psi} J_0(p_{0fr}) J_0(p_{0fr})
\]

(25)

where

\[
\Pi_{r0}^{lf} = -(k^2 c^2 \omega^2)^2 \delta_{pfr}^{lf} + \varepsilon_{r0}^{lf}, \quad \Pi_{r0}^{lf} = -(k^2 c^2 \omega^2)^2 \delta_{pfr}^{lf} + \varepsilon_{r0}^{lf}
\]

And

\[
\varepsilon_{r0}^{lf} = \varepsilon_{\psi0}^{lf} = \delta_{pfr}^{lf} - \frac{\omega_p}{\omega} c_{\psi 0}^{lf},
\]

\[
\varepsilon_{r0}^{lf} = -\varepsilon_{\psi0}^{lf} = -i \frac{\omega_p}{\omega} c_{\psi 0}^{lf}.
\]

It is clear from equations (24) and (25) that the inhomogeneity causes coupling between \( \psi \) and \( r \)-components of the electric field. Equations (24) and (25) can be expressed in matrix form as follows

\[
\begin{pmatrix}
\Pi_{r0}^{l1} & \ldots & \Pi_{r0}^{l1} & \varepsilon_{r0}^{l1} & \ldots & \varepsilon_{r0}^{l1} \\
\varepsilon_{r0}^{l1} & \ldots & \varepsilon_{r0}^{l1} & \Pi_{\psi0}^{l1} & \ldots & \Pi_{\psi0}^{l1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\Pi_{r0}^{l1} & \ldots & \Pi_{r0}^{l1} & \varepsilon_{r0}^{l1} & \ldots & \varepsilon_{r0}^{l1} \\
\varepsilon_{r0}^{l1} & \ldots & \varepsilon_{r0}^{l1} & \Pi_{\psi0}^{l1} & \ldots & \Pi_{\psi0}^{l1}
\end{pmatrix}
\begin{pmatrix}
E_{r0}^1 \\
E_{\psi0}^1 \\
\vdots \\
E_{r0}^l \\
E_{\psi0}^l \\
\vdots \\
E_{r0}^t \\
E_{\psi0}^t
\end{pmatrix}
= \frac{ik c^2}{\omega} E_0^f \frac{\partial}{\partial \psi} J_0(p_{0fr}) J_0(p_{0fr})
\]

(26)

The solution of above equations is

\[
E_{r0}^f = \frac{ik c^2}{\omega} \sum_f Z_{r0}^{lf} E_0^f \frac{\partial}{\partial \psi} J_0(p_{0fr}) J_0(p_{0fr})
\]

(27)

where

\[
A_{0fr}^{ll} E_{r0}^f + A_{0fr}^{lf} B_{\psi 0}^f = 0
\]

(28)

The wave equation for the \( z \) component of the electric field can be reached at by substitution of above field components into equation (23)

Here \( Z_{0l}^{lf} \) represents the element on the \( l \)th row and \( f \)th column of the inverse of coefficient matrix in equation (26). The wave equation for the \( z \) component of the electric field can be reached at by substitution of above field components into equation (23)

\[
A_{0fr}^{ll} E_{r0}^f + A_{0fr}^{lf} B_{\psi 0}^f = 0
\]

(27)

where

\[
A_{0fr}^{ll} = -X_{0fr}^{ll} \delta_{lf} + \omega^2 c^2 \sum_f Z_{r0}^{lf} X_{0fr}^{lf} - \sum_f Z_{r0}^{lf} D_{0fr}^{lf} - \sum_f Z_{r0}^{lf} E_{r0}^f
\]

(26)

The same procedure can be followed for the \( z \) component of the magnetic field; however, the derivatives of the Bessel–Fourier space (equation (17)) must be taken into account

\[
A_{0fr}^{ll} E_{r0}^f + A_{0fr}^{lf} B_{\psi 0}^f = 0
\]

(28)

where

\[
A_{0fr}^{ll} = \frac{\omega^2 c^2}{c^2} \delta_{lf} + \sum_f Z_{r0}^{lf} X_{0fr}^{lf} X_{0fr}^{lf} + \sum_f Z_{r0}^{lf} D_{0fr}^{lf}
\]

(26)

It must be noted that each of the coefficients \( A_{0fr}^{ll} \) and \( A_{0fr}^{lf} \) (\( j = 1, 2 \)) is an \( t \times t \) matrix.
The assumed inhomogeneous dispersion relation can be expressed in the following matrix form:

\[
\begin{pmatrix}
\Lambda_{0e1} & \Lambda_{0e2} & \ldots & \Lambda_{0e3} \\
\Lambda_{1e1} & \Lambda_{1e2} & \ldots & \Lambda_{1e3} \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{l e1} & \Lambda_{l e2} & \ldots & \Lambda_{l e3}
\end{pmatrix}
\begin{pmatrix}
E_{01}^l \\
E_{02}^l \\
\vdots \\
E_{0l}^l
\end{pmatrix}
\begin{pmatrix}
\omega_0^2 \\
\omega_0^2 \\
\vdots \\
\omega_0^2
\end{pmatrix}
= 0.
\]

\[\text{For nontrivial solutions of the above equations, the determinant of the coefficients matrix, } \Lambda \text{ (the above } 2l \times 2l' \text{ matrix) must be zero.}\]

4. Discussion

The effects of inhomogeneity on the dispersion characteristics of cylindrical waveguide filled with magnetized plasma are considered. A numerical analysis has been carried out to investigate the solutions of the dispersion equation for the symmetric modes \((m = 0)\) of cylindrical plasma waveguide. There are four families of modes in a completely filled plasma waveguide, namely, the cyclotron (C), the space-charge (SC), the EH and the HE waveguide modes. The EH and HE waveguide modes are the TM and TE modes of an empty waveguide modified by the presence of the magnetized plasma, respectively. For the present study, the waveguide radii, the applied external magnetic field and the plasma equilibrium number density are assumed: \(a = 1.55 \text{ cm, } B_0 = 0.19546 \text{ T, } n_0 = 1.0745 \times 10^{17} \text{ m}^{-3}[13] \), respectively. All frequencies and wave-numbers are normalized as \(\omega_0 = P_0B_0 \) and \(\omega_0a\). The inhomogeneous radial density profile for the electrons is assumed to have the following profile:

\[g_0(r) = 1/(1 + (3r/a)^2).\]

The assumed inhomogeneous density profile adjusts well with the classic profile obtained from the experiments performed by Malberg and Wharton [24].

The plots of dispersion relation for SC_{01}, SC_{02}, SC_{03} and SC_{04} space-charge modes are presented in figure 1. The solid and dashed curves represent the homogeneous and inhomogeneous cases of cylindrical plasma waveguide, respectively. The inspection of these plots indicates that the inhomogeneous effect shifts the space charge curves downward lower frequencies and the effect is more noticeable at higher wave-numbers. The analysis of different orders of space charge modes shows that the magnitude of the inhomogeneity effect is independent of the perpendicular wave-number \(l\). In other words, for large values of perpendicular wavenumber, the influence of inhomogeneity is almost diminished.

The characteristic dispersion relation for the first four symmetric modes of the cyclotron waves, C_{01}, C_{02}, C_{03} and C_{04} are shown in figure 2. The solid and dashed curves belong to homogeneous and inhomogeneous cases, respectively. The inhomogeneity in plasma column has analogous effect on the cyclotron modes, which is a shift toward lower frequencies. In this case, however, with the increase of the normalized wavenumber, the frequency shift patterns for different cyclotron modes are dissimilar. For C_{01} mode, as the normalized wave number increases, a rather sharp fall in frequency is noticeable. However, the other three modes (C_{02}, C_{03} and C_{04}) behave differently and remain almost steady as the wave number increases.

Figures 3 and 4 show the plots of the dispersion relation for the EH and HE waveguide modes. The solid and dashed curves again represent the homogeneous and inhomogeneous cases, respectively. From these figures, it is evident that the effect of inhomogeneity on waveguide modes is insignificant. There is a small down shift in the normalized frequency band. The comparison of different waveguide modes indicates that the increase of perpendicular wavenumber leads to a less significant effect in magnitude of inhomogeneity. As the radial eigen number \(l\) increases, the ratio of perpendicular wavelength to waveguide radius is reduced and as a result, the propagating wave feels the effect of waveguide boundary (inhomogeneity) less. It should be noted that the plots of the four families of cylindrical plasma waveguide modes for homogeneous plasma column (represented here by solid curves) are completely compatible with those presented in previous studies [13, 14].

In figure 5, the plots of the corresponding frequency shift versus normalized wavenumber for the two states of homogeneous and inhomogeneous plasma column for all four modes (SC_{01}, SC_{02}, EH_{01} and HE_{01}) are presented. In the figure, the choice of radial numbers \(l = 1\) (solid curves) and \(l = 4\) (dashed curves) is arbitrary. The figure indicates that the sensitivity of electrostatic plasma modes to the inhomogeneous effect is more noticeable than that of electromagnetic waveguide modes. Furthermore, between the electrostatic modes, the space charge mode is more sensitive to inhomogeneity than the cyclotron mode. The similar comparison between the EH and HE electromagnetic modes reveals that the EH mode experiences more shift in the frequency band with respect to inhomogeneity than the HE mode. It should be noted that as the radial number \(l\) increases, the variation of inhomogeneity with respect to the longitudinal wavenumber declines. This indicates that the inhomogeneous effect acts independently with respect to the longitudinal wavenumber. This behavior is more noticeable in the electromagnetic modes.

The cutoff frequency is the most important parameter of plasma waveguides. The influence of inhomogeneity on cutoff frequency of plasma waveguide has never been considered. The effect of inhomogeneity on plasma waveguide electrostatic modes is more visible than the electromagnetic modes. Between the two sets of electrostatic modes (space charge
and cyclotron mode), near the cutoff frequency \( (k \to 0) \), the space charge mode diminishes. Therefore, the present analysis only includes the study of cyclotron modes. Figures 6 and 7 show the plots of the normalized cutoff frequencies of cyclotron modes \((C_{01}, C_{02}, C_{03} \text{ and } C_{04})\) versus normalized plasma and cyclotron frequencies, respectively. Once again the solid and dashed curves correspond to the homogeneous and inhomogeneous case, respectively. As was expected,
the inhomogeneous effect causes a down shift in the cutoff frequency. Furthermore, the increase of the plasma density and applied magnetic field intensifies the influence of the inhomogeneity. The inspection of all plasma modes indicates that the magnitude of the inhomogeneity effect is highly dependent on the radial number \( l \) and increases as the perpendicular wave number gets larger.

The external magnetic field leads to coupling of modified TM\(_{l0}\) and TE\(_{l0}\) modes of plasma waveguide. The present analysis shows that inhomogeneity effect acts in a similar
manner. However, the coupling takes place between modes with different radial numbers (i.e. EH$_{01}$ mode couples to all EH$_{l0}$ modes where $l=2, 3, \ldots$). The inhomogeneous function, $g(r)$, reveals how intense the coupling occurs. The increase of the inhomogeneous characteristic length leads to stronger coupling. Figure 8 shows the plot of the normalized frequency difference between the TM$_{01}$-mode of homogeneous unmagnetized and the corresponding EH$_{01}$ mode of homogeneous magnetized plasma waveguide as a function of normalized wave number (solid curve). This figure also illustrates the plot of the corresponding TM$_{01}$ mode of inhomogeneous unmagnetized plasma.

5. Conclusion

The analytical calculation of the dielectric permittivity of radially inhomogeneous plasma column is presented. To examine the integrity of the model, a special study is undertaken to demonstrate the inhomogeneity effect on the completely filled magnetized plasma waveguide. The dispersion relation curves for EH, HE, SC and C modes in a cylindrical plasma waveguide were obtained through numerical calculations. In the homogeneous limit, the dispersion relation was derived based on the Bessel–Fourier transformation. The consideration of the density gradient leads to the appearance of imaginary part in all of dielectric tensor components, which indicates the existence of the drift instability in the inhomogeneous plasma. The analysis shows that inhomogeneity effect leads to coupling of plasma modes with different radial number $l$. The increase of the inhomogeneous characteristic length points to stronger coupling. The shape and structure of inhomogeneous function, $g(r)$, identifies the intensity of coupling. The investigation of modes shows that the inhomogeneity causes a down shift in frequency response. The sensitivity of electrostatic modes to the inhomogeneity effect is more than that of electromagnetic modes. The mode coupling caused by inhomogeneity along plasma column radial direction is stronger than coupling due to external magnetic field.

Appendix A

The expansion of the dielectric tensor components and electric field in the Bessel–Fourier space can be written as:

$$\varepsilon_i(r, t') = \frac{1}{2\pi} \int \frac{dk' dk}{\omega} \sum_m \sum_{i, l} e^{i(\omega t' - \omega t)} e^{i k(\Delta r)}$$

$$\times \frac{2 J_m(P_{m\rho} r)}{a^2 J_{m+1}(r) J_{m+1}(P_{m\rho} r)} e^{-im\psi_{l, t'}} \varepsilon_{lmm}^{(i)}(\omega, k)$$

(A.1)

inhomogeneity in the plasma column radial direction comprehensively. However, other approximate descriptions of radial inhomogeneity such as geometrical optics method used in the analysis of helicon discharge [25] cannot fully describe the coupling.

$$E_i(r', \psi', z', t') = \frac{1}{2\pi} \int \frac{dk' dk}{\omega} \sum_m \sum_{l} e^{i(k' z' - \omega t')}$$

$$\times \frac{\sqrt{2} J_m(P_{m\rho} r')}{a J_{m+1}(P_{m\rho} r')} e^{im\psi_{l, t'}} \varepsilon_{lmm}^{(i)}(\omega, k).$$

(A.2)
The following equation is obtained through the substitution of the above equations in equation (1)

\[ D_l(r, \psi, z, t) = \frac{1}{2\pi} \int dk d\omega \sum_m \sum_l \frac{\sqrt{2}}{a_{m+l}(X_{ml})} \]

\[ \times J_m(P_{mlr}) e^{iml} e^{i(kz - \omega t)} \sum_l \varepsilon_{ijmm} E'_{i\ell} \]. \hspace{1cm} (A.3)

On the other hand, the expansion of displacement field in the Bessel–Fourier space reads as:

\[ D_l(r, \psi, z, t) = \frac{1}{2\pi} \int dk d\omega \sum_m \sum_l \frac{\sqrt{2}}{a_{m+l}(X_{ml})} \]

\[ \times J_m(P_{mlr}) e^{iml} e^{i(kz - \omega t)} D'_l(\omega, k). \hspace{1cm} (A.4) \]

Comparison of equations (A.3) and (A.4) lead to

\[ D'_l(\omega, k) = \sum_m \sum_l \varepsilon_{ijmm}(\omega, k) E'_{i\ell}(\omega, k). \hspace{1cm} (A.5) \]

Following the same procedure, the current density material relation can be written as:

\[ J'_l(\omega, k) = \sum_m \sum_l \varepsilon_{ijmm}(\omega, k) E'_{i\ell}(\omega, k). \hspace{1cm} (A.6) \]

**Appendix B**

\[ A'_{mm'}^{l'} = \left[ \frac{2}{(a^2 J_{m+l} + (X_{ml})) J_{m+l}(X_{ml})} \right] \]

\[ \times \int_0^\pi dr J_m(P_{mlr}) J_m(P_{mlr}) \]. \hspace{1cm} (B.1)

\[ A'_{mm}^{l'} = \left[ \frac{2}{(a^2 J_{m+l} + (X_{ml})) J_{m+l}(X_{ml})} \right] \]

\[ \times \int_0^\pi dr \left\{ \frac{d}{dr} J_m(P_{mlr}) \right\} J_m(P_{mlr}) \]. \hspace{1cm} (B.2)

\[ B'_{mm'}^{l'} = \left( \frac{1}{\pi} \right) \int_0^{2\pi} \sin \psi \left\{ \frac{d}{d\psi} \sin m' \psi \right\} \cos m \psi \]

\[ = \left( \frac{m'}{2} \right) [\delta_{m',m+1} + \delta_{m',m-1}] \hspace{1cm} (B.3) \]

\[ B'_{mm}^{l'} = \left( \frac{1}{\pi} \right) \int_0^{2\pi} \sin \psi \left\{ \frac{d}{d\psi} \cos m' \psi \right\} \cos m \psi \]

\[ = \left( -\frac{m'}{2} \right) [\delta_{m',m+1} + \delta_{m',m-1}] \hspace{1cm} (B.4) \]

\[ B'_{mm'}^{l'} = \left( \frac{1}{\pi} \right) \int_0^{2\pi} \cos \psi \cos m' \psi \cos m \psi \]

\[ = \left( \frac{1}{2} \right) [\delta_{m',m+1} + \delta_{m',m-1}] \hspace{1cm} (B.5) \]
\[ \Pi_{2\omega,k}(\omega,k) = \left\{ \eta_{\text{ind}} \left[ \sin \xi_{\text{ind}} A_n - \cos \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] \right. \\
+ \left[ \sin \xi_{\text{ind}} \eta_{\text{ind}} + \cos \xi_{\text{ind}} \eta_{\text{ind}} \right] \frac{n^2 A_n}{z_{\text{cond}}} \right. \\
- \left. 2 \sin \xi_{\text{ind}} \eta_{\text{ind}} z_{\text{cond}} A_n' = \frac{\eta_{\text{ind}}}{2} \left[ \sin \xi_{\text{ind}} A_n' + \cos \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] \right\} \] (C.6)

\[ \Pi_{2\omega,k}(\omega,k) = \eta_{\text{ind}} \left[ \cos \xi_{\text{ind}} A_n' - \sin \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] \\
+ \left[ \cos \xi_{\text{ind}} \eta_{\text{ind}} + \sin \xi_{\text{ind}} \eta_{\text{ind}} \right] \frac{n^2 A_n}{z_{\text{cond}}} - A_n' \right] \right. \\
- \left. 2 \cos \xi_{\text{ind}} \eta_{\text{ind}} z_{\text{cond}} A_n' \right. \\
- \left. \sin \xi_{\text{ind}} \eta_{\text{ind}} \frac{A_n}{z_{\text{cond}}} - A_n' \right] \left[ \cos \xi_{\text{ind}} A_n' - \sin \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] \right\} \] (C.7)

\[ \Pi_{2\omega,k}(\omega,k) = \eta_{\text{ind}} \left[ \cos \xi_{\text{ind}} A_n' - \sin \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] + \frac{n^2 A_n}{z_{\text{cond}}} \eta_{\text{ind}} \\
- \left. 2 \cos \xi_{\text{ind}} \eta_{\text{ind}} z_{\text{cond}} A_n' \right. \\
+ \left. \frac{\eta_{\text{ind}}}{2} \left[ \cos \xi_{\text{ind}} A_n' - \sin \xi_{\text{ind}} \frac{A_n}{z_{\text{cond}}} \right] \right\} \] (C.8)

where

\[ \eta_{\text{ind}} = \sum_{\ell} \xi_{\ell} \zeta_{\ell} \]

\[ \eta_{\text{ind}} = \frac{1}{Q_{PM}} \sum_{\ell} \xi_{\ell} \zeta_{\ell} \]

\[ \eta_{\text{ind}} = \frac{1}{Q_{PM}} \sum_{\ell} \xi_{\ell} \zeta_{\ell} \]

\[ \eta_{\text{ind}} = \frac{1}{\sqrt{2} Q_{PM}} \left[ \sum_{\ell} \xi_{\ell} \zeta_{\ell} \cos \left( \xi_{\ell} - \frac{\pi}{4} \right) + \sum_{\ell} \xi_{\ell} \zeta_{\ell} - \frac{\pi}{4} \right] \]

\[ \eta_{\text{ind}} = \frac{1}{\sqrt{2} Q_{PM}} \left[ \sum_{\ell} \xi_{\ell} \zeta_{\ell} \sin \left( \xi_{\ell} - \frac{\pi}{4} \right) + \sum_{\ell} \xi_{\ell} \zeta_{\ell} - \frac{\pi}{4} \right] \]

References