Effect of non-affine motion on the centrifugal instability of circular Couette flow

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A B S T R A C T
In the present work, the effect of non-affine motion of the constituents in a complex fluid system (say, a polymeric liquid) is theoretically investigated on the centrifugal instability of circular Couette flow. To achieve this goal, use was made of the linearized Phan-Thien/Tanner (LPTT) model thanks to its allowing non-affine deformation for the polymer strands in a temporary network of junctions through invoking a slip parameter. Knowing the basic-flow velocity and stress fields from the literature, they were subjected to infinitesimally-small, normal-mode perturbations and their time-evolution was monitored using a linear stability analysis for both the axisymmetric and non-axisymmetric modes. An eigenvalue problem was obtained in this way which was solved numerically using the pseudo-spectral, Chebyshev-based, collocation method. Based on the results obtained in this work, it is concluded that the non-affine motion can have a stabilizing or destabilizing effect on circular Couette flow depending on the Weissenberg number and the sign/magnitude of the angular velocity ratio.

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1. Introduction

Exact solution are indispensable tools in the field of fluid mechanics. This is because they always provide a better insight onto the physics of any particular problem at hand. They are also extremely useful for checking the performance of computer codes and algorithms. Unfortunately, exact solutions are quite rare in the field of fluid mechanics, and this particularly so for any given non-Newtonian fluid. The chance of finding an exact solution becomes less probable when the degree of sophistication of the model is increased. For example, while for Giesekus fluids an exact solution does exist in plane stagnation-point flow [1], no such a solution could be found for the Phan-Thien/Tanner (PTT) fluids. This is rather unfortunate because the PTT model is one of the best rheological models when it comes to representing polymeric melts and concentrated polymer solutions. Having said this, it should be conceded that simplified forms of this robust rheological model render themselves to an exact solution in certain geometries [2–5]. One can particularly mention the exact solution found recently by Mirazazadeh et al. [5] in circular Couette flow for the linearized Phan-Thien/Tanner model (LPTT).

The problem with the exact solution found in Ref. 5 in circular Couette flow is that they have reported results for $\varepsilon W^2$ as large as 10, where $\varepsilon$ is the extensional parameter of the LPTT model (see Section 2) and $W$ is the Weissenberg number. Since in their work $\varepsilon$ takes on values in the range of 0.01 to 0.1, one can conclude that in Ref. 5, the Weissenberg number has been allowed to assume values as large as 30. In practice, however, the purely-azimuthal solution found in Ref. 5 may have lost its stability long before such a high Weissenberg number can be reached. Obviously, having found an exact solution does not necessarily mean that the flow can materialize in the real world for arbitrary set of parameters. It is only through a (linear) stability analysis that a threshold can be obtained for the critical Weissenberg number below which the flow can exist in the real world. In the present work, we try to figure out the range of applicability of the exact solution reported in Ref. 5 for the linear LPTT model in circular Couette flow. Another objective of the work is to investigate the effect of the non-affine motion on the instability picture of circular Couette flow. The non-affine motion (which means internal slip between the constituents and the continuum) has been shown to affect the basic flow for the LPTT model [see Ref. 5] thus it is expected to affect the critical Weissenberg numbers. To the best of our knowledge, however, the effect of non-affine motion has not previously been investigated on the instability picture of circular Couette flow [6–15]. To reach its objectives, the work is organized as follows: knowing the basic flow from Ref. 5 for the LPTT model in circular Couette flow, we proceed with imposing infinitesimally-small, normal mode perturbations to the basic flow and invoke a linear.
normal-mode, temporal stability analysis to determine the effect of non-affine motion on the critical Reynolds/Weissenberg numbers. The pseudo-spectral numerical method of solution is then described briefly before proceeding with presenting our new numerical results. The work is concluded by highlighting its major findings.

2. Governing equations

We consider confined flow between two infinitely long concentric cylinders of radii \( R_1 \) and \( R_2 \) as shown in Fig. 1. Both cylinders are allowed to rotate with angular velocity \( \Omega_1 \) and \( \Omega_2 \), respectively—either in the same or in the opposite directions. We employ cylindrical co-ordinate system for our mathematical development (see Fig. 1). We also assume that the fluid is incompressible and takes no effect from the gravitational force while it is flowing in the gap between the two cylinders.

The equations governing the flow are the Cauchy equations of motion together with the continuity equation [16]:

\[
\rho \frac{D \mathbf{V}}{D t} = -\nabla p + \nabla \cdot \mathbf{\tau} \quad (1)
\]

\[
\nabla \cdot \mathbf{V} = 0 \quad (2)
\]

where \( \rho \) is the density, \( \mathbf{V} \) is the velocity vector, \( p \) is the isotropic pressure, and \( \mathbf{\tau} \) is the extra stress tensor. We assume that the liquid of interest obeys the single-mode Phan-Thien/Tanner model (PTT) as its constitutive equation [17]. In its most general form, in this robust viscoelastic fluid model the stress tensor is related to the deformation field by the following relationship [18,19],

\[
\left( \exp \left( 1 + \xi \frac{\lambda}{\eta_0} t \mathbf{r} \cdot \mathbf{\tau} \right) \right) \mathbf{\tau} + \lambda \mathbf{\tau} + \xi \lambda (\mathbf{\tau} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{\tau}) = 2\eta_0 \mathbf{D} \quad (3)
\]

where \( \xi \) is the extensional parameter, \( \lambda \) is the zero-shear relaxation time, \( \eta_0 \) is the zero-shear viscosity, \( \xi \) is the slip factor, and \( 2\mathbf{D} \) is the rate-of-deformation tensor. In this equation, \( \nabla \) refers to the upper-convected derivative which is defined by:

\[
\frac{\mathbf{V}}{\tau} = \frac{D \mathbf{r}}{D t} - (\mathbf{r} \cdot \nabla) \mathbf{V} + \nabla \cdot (\mathbf{r} \mathbf{V}) \quad (4)
\]

where \( D/Dt \) is the material derivative. It is worth-mentioning that the \( \xi \)-term in Eq. (3) allows the relaxation time of the model to become a non-linear function of the extra stress tensor. In practice, this term accelerates the rate of stress decay at high stresses while it ensures that the stress vanishes when the strain approaches zero. The \( \xi \)-term, on the other hand, alters the rate at which the stress builds up in the fluid, which is why it is allowed to be a function of the rate-of-deformation tensor and the stress itself. Now, before proceeding any further, we try to make these equations dimensionless by substituting:

\[
\mathbf{r}' = \frac{r}{R_2}, \quad \mathbf{z}' = \frac{z}{R_2}, \quad \mathbf{t}' = \frac{t}{\rho (d^2/\eta_0)} , \quad \mathbf{V}' = \frac{V_1}{R_1 \Omega_1}, \quad \mathbf{\tau}' = \frac{\mathbf{\tau}}{\eta_0 (R_1 \Omega_1 / d)}, \quad \mathbf{p}' = \frac{p}{\eta_0 (R_1 \Omega_1 / d)}, \quad \mathbf{r}_{ij}' = \frac{r_{ij}}{\Omega_1^2} \quad \mathbf{t}' = \frac{t}{\rho (d^2/\eta_0)} , \quad \mathbf{V}' = \frac{V_1}{R_1 \Omega_1}, \quad \mathbf{\tau}' = \frac{\mathbf{\tau}}{\eta_0 (R_1 \Omega_1 / d)}, \quad \mathbf{p}' = \frac{p}{\eta_0 (R_1 \Omega_1 / d)} \quad (5)
\]

where \( R_2 \) is the radius of the outer cylinder, and \( d = R_2 - R_1 \) is the gap spacing. It is worth-mentioning that the reason why \( R_1 \Omega_1 \) has been used as the velocity scale is the fact that in rheometry the outer cylinder is often held fixed (\( \Omega_2 = 0 \))—even when it is not fixed, one often replaces the bob instead of the cup for altering the gap setting which is another reason why \( R_2 \) is commonly used in place of \( R_1 \) as the length scale. Also, we would like to stress that although we have relied on \( \rho d^2/\eta_0 \) (the so-called viscous time scale) for making the time dimensionless, it is also possible to rely on \( 1/\Omega_1 \) (the so-called inertial time scale) for this purpose. As discussed by Drazin and Reid [30], they are both equally good as far as data reduction is concerned. In the present work, we have decided to rely on \( \rho d^2/\eta_0 \) simply because in our previous work
dealing with the Giesekus model [see Ref. 14] the same idea has
been used for making the time dimensionless. And, we intend to
compare the results for these two fluid models at a later stage of
the present work.

In dimensionless form (having dropped the asterisk above
dimensionless parameters for convenience) the governing equations
decome [38],
\[ \frac{\partial \mathbf{V}}{\partial t} + \text{Re}(1 - \varphi)(\nabla \cdot \mathbf{V}) \mathbf{V} = (1 - \varphi) \left( -\nabla p + \nabla \cdot \mathbf{r} \right) \]  \hspace{1cm} (6)
\[ \nabla \cdot \mathbf{V} = 0 \]  \hspace{1cm} (7)
\[ \mathbf{r} + \frac{\mathbf{W}_e}{\text{Re}} \left( \frac{\partial \mathbf{r}}{\partial t} \right) + \mathbf{W}_e(1 - \varphi) \left( \nabla \cdot \mathbf{V} \right) \mathbf{V} - \nabla \cdot \mathbf{V} \mathbf{r} - \nabla \mathbf{V} \right) \]  \hspace{1cm} (8)
where \( \varphi = R_1/R_2 \) is the radii ratio, \( \text{Re} = \rho R_1 \Omega_1 d/\eta \) is the Reynolds
number, and \( \mathbf{W}_e = \lambda R_2 \Omega_1 / d \) is the Weissenberg number. It is to be
noted that in this equation we have linearized the exponential
term in Eq. (3) with the presumption that either the Weissenberg
number and/or the extensional parameter are sufficiently
small. The model so obtained is often referred to as the linear
Phan-Thien/Tanner model (LPTT). It is the model used in Ref. 5 for
obtaining their basic flow, and is the focus of the present stability
study.

As to the performance of the PTT model, it can be said that it
is one of the most robust models when it comes to representing
polymeric melts and concentrated polymer solutions [18,19].
Indeed, the model has proven very successful in simulating the flow
of such fluids in complex geometries in which the material is sub-
jected to simultaneous shear and extensional deformations [20-
26]. Having said this, it should be conceded that the model has
its own limitations. For example, extra care should be exercised in
situations where high shear rates are involved as the model might
predict negative stress while the strain and/or the strain rates are
positive [27]. The model also shows spurious oscillations during
start-up of shear flow for the case of \( \xi \neq 0 \) [16]. But, perhaps the most serious
problem with the model is that it rarely renders it-
self to an analytical solution in complex flows. As a matter of fact,
until recently, analytical solutions found for this model were
restricted to fully-developed pipe and channel flows [28,29]. In fact,
the semi-analytical solution found by Mirzazadeh et al. [5] in cir-
cular Couette flow appears to be the most sophisticated solution
found for this model so far [5]. In the present work, the mere
reason for resorting to this cumbersome but versatile model is the
inclusion of the slip factor in this model which is lacking in other
simpler but comparable models such as Giesekus model [17]. Since,
unlike the Giesekus model, the LPTT model has four material prop-
erties (\( \lambda, \eta_0, \tau_s, \xi \)), it better fits the rheological data for complex
fluids. As a matter of fact, the model has an excellent performance
in predicting the shear and extensional flow behavior of most poly-
meric liquids. To show this in better perspective, we resort to the
material functions known for this rheological model in shear and
extension. In simple shear we have [4]:
\[ \eta(\dot{\gamma}) = \frac{\eta_0 - \lambda \xi \tau_{ss}}{1 + 2\lambda(1 - \xi) / \eta_0 (2 - \xi) \tau_{ss}}; \quad \psi_1(\dot{\gamma}) = \frac{2\tau_{ss}}{(2 - \xi)^2 \dot{\gamma}^2}; \quad \psi_2(\dot{\gamma}) = -\frac{\xi}{2} \psi_1 \]  \hspace{1cm} (9)
where \( \eta \) is the apparent (shear) viscosity, \( \psi_1 \) is the first-
normal-stress-difference coefficient, \( \psi_2 \) is the second-normal-
stress-difference coefficient, and \( \dot{\gamma} \) is the shear rate. (In these
equations \( \tau_{ss} \) is obtained through solving a third-order algebraic
equation, as discussed in Ref. 4.) As to the performance of the
model in predicting the extensional viscosity, \( \eta_e \), we originally
re-
sorted to the equation developed by Xue et al. for this purpose
[20], but there appears to be something wrong with the typing of
their equation as it could not reproduce the results shown in their
Fig. 3c. As such, we have decided to derive the extensional viscos-
ity of this fluid model in extension by ourselves from scratch. The
following equation was obtained this way in planar steady ex-

tensional flow [38]:
\[ \eta_e = \frac{(g \lambda \varepsilon + 1) g}{8 \dot{\gamma}^2 \lambda (1 - \xi)} \]  \hspace{1cm} (10a)
where \( g = \tau_{sx} + \tau_{sz} \) is obtained by solving the following third-order
algebraic equation [38]:
\[ -\dot{\gamma}^2 \lambda^2 g^2 - 2 \dot{\gamma} \lambda g^2 + (1 + 4 \lambda^2 \dot{\gamma}^2 (1 - \xi)^2) g + 8 \lambda \dot{\gamma}^2 (1 - \xi) = 0 \]  \hspace{1cm} (10b)
where \( \dot{\gamma} \) is now the strain rate. Fig. 2 shows the effect of on
the shear and extensional flow behavior of the LPTT fluid. As

can be seen in Fig. 2, \( \varepsilon \) strongly controls the extensional behavior of
the fluid, which is why it is called the extensional parameter.
On the other hand, its effect on the shear flow data is rather weak.
That is, although it is true that shear-thinning starts at lower shear
rates when \( \varepsilon \) is increased (see Fig. 2), the severity of shear-
thinning (as judged by the slope of viscosity and normal stress
profiles) remains virtually intact when \( \varepsilon \) is increased.

Fig. 3 shows the effect of \( \xi \) on the shear and extensional be-

havior of the LPTT fluids. As can be seen in this figure, \( \xi \) strongly
affects the degree of shear-thinning of the fluid but its effect on
the extensional viscosity is rather weak. This material property,
which is often called the slip parameter, is also responsible for the
rise of a non-zero (negative) second normal stress difference [31].
As can be seen in this figure, an increase in \( \xi \) decreases \( \psi_1 \), pro-
vided that Weissenberg number is larger than roughly 1. On the
other hand, an increase in \( \xi \) only increases \( \psi_2 \) but also controls its thinning behavior when the Weissenberg number is larger
than roughly 1 (see Fig. 3). Since circular Couette flow is typical
of shear-dominated flows, \( \xi \) is obviously much more important than
\( \varepsilon \) in this particular flow.

As to the physical origin of the slip factor, \( \xi \), we would like to
stress that this material property allows the deformation of, say,
polymer strands embedded in the network to slip past the macro-
scopic medium. Due to such internal slips, the strands transmit
only a fraction of their tension to the surrounding continuum as
it slips, and this is partly responsible for the shear-thinning be-

havior of the fluid (for both viscosity and the two normal stress
differences) as can be seen in Fig. 3. In practice, the inclusion of the
\( \xi \)-term enables the PTT model to better fit data for struc-
tured materials as compared with other comparable models such
as Giesekus [17]. As a matter of fact, non-affine motions appear to
be quite common among complex materials such as polymer hy-
drolys, polymer melts, concentrated polymer solutions, and gran-
uar materials [32–36]. The two parameters \( \varepsilon, \xi \) define the non-
linear behavior. When \( \varepsilon \ll \xi \) the behavior in shear flow is mainly
determined by \( \xi \), and \( \varepsilon \) serves to blunt the singularity in elonga-
tion that otherwise would be present. However, a single value of
the slip parameter \( \xi \) cannot fit both shear viscosity and first nor-
mal stress difference satisfactorily.

It is worth-mentioning that the LPTT model reduces to the
upper-convected Maxwell (UCM) model by simply setting \( \xi = 0 \).
On the other hand, for \( \varepsilon = 0 \) the LPTT model reduces to the
Johnson–Segalman (J–S) model [16,17]. The case of \( \xi = 0 \), which is
called the simplified Phan-Thien/Tanner (SPTT) model, renders it-
self more easily to an analytical solution [37]. This is not the case
for the LPTT model, for which the number of exact (analytical or semi-analytical) solutions is quite limited [5].

3. Basic flow

An important step in any stability analysis is obtaining the velocity, pressure, and stress fields under steady conditions—the so-called basic flow. With the assumption that the basic flow is purely-tangential, Mirzazadeh et al. [5] have shown that the LPTT model renders itself to a semi-analytical solution in circular Couette flow. For completeness, a brief discussion of their solution is presented here (see Ref. 5 for more details). After some tedious mathematical manipulations, they have derived the following equation for the tangential velocity from which the pressure and stress field can also be obtained quite straightforwardly:

$$\frac{v_\theta(r)}{r} = \frac{1 + 2/n}{4(1 - \phi)\phi^2 \tau_w We_n \chi} \left[ r^2 \sqrt{r^4 - n^2} + \tan^{-1} \left( \frac{\sqrt{r^4 - n^2}}{n} \right) \right] + \frac{n^2}{4(1 - \phi)\phi^2 \tau_w We_n \chi} \left( \frac{1}{r^2} \right) + C$$  \hspace{1cm} (11)

where C is determined from the velocity boundary condition at the surface of one of the cylinders (see Ref. 5), $n = 2\phi^2 \tau_w We \sqrt{\chi}$, $\chi = \xi(2 - \xi)/\varepsilon(1 - \xi)$, and $\text{We}_n = We \sqrt{\varepsilon(1 - \xi)}$. It should be noted that in this equation $v_\theta$ and $r$ are both dimensionless. As a matter of fact, to make the radial position dimensionless, like the present work, Mirzazadeh et al. [5] have relied on $R_0$ as is customary in the literature [see, for example, Ref. 30]. What is not so customary is that they have relied on $R_0 (\Omega_2 - \Omega_1)$ to make the tangential velocity dimensionless. (In practice, this has forced us to rescale their velocity with ours which has been made dimensionless using $R_0 \Omega_1$.) Fig. 4 shows the effect of the extensional and slip parameters ($\varepsilon, \xi$) on the basic-flow velocity profiles. The velocity profiles are seen to be affected by both parameters (see also Ref. 5). Fig. 5 shows a plot of this equation for $\xi = 0.05$ and $\varepsilon = 0.05$. For comparison purposes, this figure also includes the velocity profiles for other variations of the LPTT model (i.e., SPTT, UCM, J–S, and Newtonian models). As can be seen in this figure, at any given radius, the tangential velocity is largest for the Newtonian fluids and smallest for the LPTT fluids. This figure also shows that the velocity profile for the UCM and Newtonian fluids are the same, as is well-established in the literature [16] (Fig. 5).

Knowing the basic flow velocity profile, one can easily proceed with finding the pressure and stress fields under steady conditions [see Ref. 5 for the details]. In the next section the basic-flow solution obtained this way by Mirzazadeh et al. [5] will be perturbed to see if they are vulnerable to centrifugal instability. As
is customary, we report results for different radii ratio and different angular velocity ratio, $\omega = \Omega_2/\Omega_1$. Also, in order to address the effect of $\Omega_2$ on the critical Reynolds number, $Re$, we can make it dimensionless using the ratio $\eta/\rho R_2 d$. The dimensionless $\Omega_2$ so-obtained will be referred to as the outer-cylinder Reynolds number, $Re_2 = \rho R_2 \Omega_2 d/\eta$. To differentiate the two Reynolds numbers involved in our problem, we will refer to the inner-cylinder Reynolds number by $Re_1$ instead of $Re$ in subsequent sections.

4. Linear stability analysis

Knowing the basic flow velocity field ($V_{ss}$), stress field ($T_{ss}$), and pressure field ($P_{ss}$) we are now ready to proceed with their susceptibility to become unstable when slightly perturbed. To that end, we follow the classical normal-mode analysis which allows a separation-of-variable form for the solution with regard to the spatial and temporal variables [30]. Thus, a non-axisymmetric infinitesimal disturbance is superimposed onto the primary basic
flow solutions \((V_{ss}, T_{ss}, \Pi_{ss})\) in the form of:

\[
\begin{align*}
V(r, z, \theta, t) &= V_{ss}(r) + \text{Real}[\hat{V}(r) \exp (\sigma t + i\alpha z + i\theta)] \\
T(r, z, \theta, t) &= T_{ss}(r) + \text{Real}[\hat{T}(r) \exp (\sigma t + i\alpha z + i\theta)] \\
\Pi(r, z, \theta, t) &= \Pi_{ss}(r) + \text{Real}[\hat{\Pi}(r) \exp (\sigma t + i\alpha z + i\theta)]
\end{align*}
\] (12)

where \(\hat{V}, \hat{T}, \hat{\Pi}\) are the perturbation amplitudes to the basic-flow velocity, stress, and pressure fields, respectively, \(\sigma\) is the growth rate of the perturbation, \(\alpha\) is the wavenumber of the assumed periodicity in the axial direction, and \(m\) is the azimuthal wavenumber. For a given \(m\), the perturbation involves a sinusoidal variation of the pertinent variables in the \(\theta\)-direction with a frequency equal to \(m\). Thus (unlike \(\alpha\)) \(m\) can only take integer numbers. Substituting the above expressions into the equations of motion and also the constitutive equation leads to a system of 10 ordinary differential equations comprising the continuity, the three momentum equations, and the six stress constitutive equations [38]. Denoting the three components of the perturbation eigenfunction \(V(r)\) by \(\tilde{V}_r, \tilde{V}_\theta, \tilde{V}_z\), the linearized form of the continuity equation becomes:

\[
0 = \tilde{V}_r + \frac{\tilde{V}_\theta}{r} + i\alpha \tilde{V}_z + \frac{1}{r} i m \tilde{V}_\theta
\] (13)

where prime means first derivative in the radial direction. (Here and in the following equations, we drop 'hats' above perturbed variables, for convenience.) In a similar fashion, the \(r, \theta,\) and \(z\) components of the linearized momentum equations become [20]:

\[
\begin{align*}
\sigma \tilde{V}_r &= -\frac{im \tilde{V}_\theta}{r} + \tilde{V}_\theta + \frac{2 \tilde{V}_\theta \tilde{V}_\theta}{r} + \frac{1}{r} \text{Re} \left( \frac{\text{im} T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - \Pi' \\
\sigma \tilde{V}_\theta &= -\frac{im \tilde{V}_r}{r} + \frac{1}{2} \text{Re} \left( \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - i m \Pi \\
\sigma \tilde{V}_z &= i \alpha \tilde{V}_\theta + \frac{im \tilde{V}_\theta}{r} + \frac{1}{r} \text{Re} \left( \frac{T_{zz}}{r} + i\alpha T_{zz} \right) - i \alpha \Pi
\end{align*}
\] (14)

As to the \(r, \theta, z, \beta, \theta,\) and \(\beta\) components of the linearized stress constitutive relationships we have [20]:

\[
\begin{align*}
\sigma T_{rr} &= -2 \epsilon T_{\theta \theta} T_{rr} - i \frac{\tilde{V}_\theta}{r} - \tilde{V}_\theta T_{\theta \theta} - \epsilon T_{zz} T_{rr} - \xi (\tilde{V}_\theta T_{\theta \theta} + \tilde{V}_\theta T_{\theta \theta}) \\
&\quad - \epsilon T_{zz} T_{\theta \theta} + \tilde{V}_z T_{\theta \theta} + 2 \tilde{V}_z T_{\theta \theta} - \frac{1}{r} \text{Re} \left( \frac{\text{im} T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - \Pi' \\
&\quad + \frac{1}{r} \text{Re} \left( i \text{im} T_{\theta \theta} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - i m \Pi \\
\sigma T_{\theta \theta} &= -2 \epsilon T_{\theta \theta} T_{\theta \theta} - \epsilon T_{zz} T_{\theta \theta} - \xi (\tilde{V}_\theta T_{\theta \theta} + \tilde{V}_\theta T_{\theta \theta}) \\
&\quad - \epsilon T_{zz} T_{\theta \theta} - \xi (\tilde{V}_\theta T_{\theta \theta} + \tilde{V}_\theta T_{\theta \theta}) \\
&\quad + \frac{1}{r} \text{Re} \left( \frac{\text{im} T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - \Pi' \\
&\quad + \frac{1}{r} \text{Re} \left( i \text{im} T_{\theta \theta} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - i m \Pi \\
\sigma T_{zz} &= 2 \epsilon T_{\theta \theta} T_{zz} - 2 \epsilon T_{zz} T_{zz} - \xi (\tilde{V}_\theta T_{\theta \theta} + \tilde{V}_\theta T_{\theta \theta}) \\
&\quad + \frac{1}{r} \text{Re} \left( \frac{\text{im} T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - \Pi' \\
&\quad + \frac{1}{r} \text{Re} \left( i \text{im} T_{\theta \theta} + \frac{T_{\theta \theta}}{r} + \frac{T_{\theta \theta}}{r} + i\alpha T_{\theta \theta} \right) - i m \Pi
\end{align*}
\] (17)
Table 1
A comparison between our results and published data for the axisymmetric mode $(m = 0)$ for Newtonian fluids [40].

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>m</th>
<th>$\omega$</th>
<th>$\text{Ref. 40}$</th>
<th>Present work $\text{Re}_c$</th>
<th>$\alpha_c$</th>
<th>Present work $\text{Re}_c$</th>
<th>$\alpha_c$</th>
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<td>0.55</td>
<td>0</td>
<td>0</td>
<td>184.99</td>
<td>3.128</td>
<td>184.58</td>
<td>3.127</td>
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<td>0</td>
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<td>85.775</td>
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<tr>
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<td>0</td>
<td>68.184</td>
<td>3.362</td>
<td>68.186</td>
<td>3.36</td>
<td></td>
</tr>
</tbody>
</table>

\[
-2 \frac{v_0 T_{\theta \theta_0}}{r} - \varepsilon T_{\theta \theta_0} T_{rr} - \varepsilon T_{\theta \theta_0} T_{\theta \theta_0} - 2 \varepsilon T_{\theta \theta_0} T_{\theta \theta_0} - 2 \frac{v_0 T_{\theta \theta_0}}{r} \\
- \alpha_c T_{\theta \theta_0} + \frac{\xi_0 T_{\theta \theta_0}}{r} + \frac{\chi v_0 T_{\theta \theta_0}}{r} - i \xi v_0 \beta T_{\theta \theta_0} - \frac{i \xi v_0 \beta T_{\theta \theta_0} m}{r} + \frac{1}{\text{We}_c} \left( - T_{\theta \theta_0} + 2 \frac{v_0 m}{r} + 2 \frac{v_0}{r} \right)
\]

\[
\sigma T_{zz} = - \frac{1}{2} \xi v_0 \alpha T_{\theta \theta_0} - \frac{1}{2} \frac{v_0 m}{r} - T_{zz} T_{rr} - T_{zz} T_{\theta \theta_0} - \frac{1}{2} \xi v_0 \beta T_{\theta \theta_0} + \frac{1}{2} \xi v_0 \beta T_{\theta \theta_0} m + \frac{1}{\text{We}_c} \left( 2 v_0 T_{\theta} - T_{zz} \right)
\]

where the six components of the perturbation stress tensor (symmetric) tensor, $\tilde{\sigma}(r)$, have been denoted by $\tilde{T}_{rr}, \tilde{T}_{\theta \theta}, \tilde{T}_{z z}, \tilde{T}_{r \theta}, \tilde{T}_{r z}, \tilde{T}_{\theta z}$ before dropping their "hat". This system of first-order differential equations need 10 boundary conditions to be amenable to a numerical solution. We rely on the no-slip and no-penetration conditions at the walls, and also on the periodic boundary condition in the $z$-direction for this purpose [see Ref. 20 for the details]. This linearized boundary-value problem poses an eigenvalue problem in the form of:

\[
A \cdot U = \sigma B \cdot U
\]

where $A$ and $B$ are the coefficient matrices. We have relied on the standard pseudo-spectral, Chebyshev-based collocation method for solving the eigenvalue problem as posed by Eq. (22) [see Ref. 38 for the details]. In fact, $U$ in this equation is the vector comprising the value of unknown variables at the Gauss–Lobatto collocation points [39]. For any given set of parameters: $\phi, \xi, \omega, \phi, \alpha, m, \text{We}_c$, and $\text{Re}_2$, the spectral method renders a spectrum of eigenvalues, $\sigma$, with $\tilde{V}, \tilde{T}, \tilde{F}$ being the eigen-functions. The most dangerous (and so, the most unstable) mode is the one for which the real part of the eigenvalue is maximum. In practice, in an attempt to find the critical Reynolds number, $\text{Re}_c$, it suffices to find the mode for which the real part of the eigenvalue is zero. The critical wavenumber and Reynolds number are then determined by locating the apex of the neutral stability curve so obtained. We have validated the code for the axisymmetric mode $(m = 0)$ and also the non-axisymmetric modes $(m \neq 0)$ using known data for Newtonian fluids [40]. We have reached to eight significant digits of accuracy by using just 21 Gauss–Lobatto collocation points and 17 Chebyshev terms. As can be seen in Tables 1 and 2 the code is doing a nice job for Newtonian fluids for both axisymmetric and non-axisymmetric modes.

To further verify the code, we have at our disposal the linear stability results published in Ref. 42 for the Taylor-Couette flow of upper-convected Maxwell model. For $\phi = 0.95$. We/Re = 0.01, and $\omega = 0.5$ the critical Reynolds number has been reported in this reference to be equal to 201.5 for $m = 0$, and this is virtually the same as that obtained by our code. As a matter of fact, our code could also easily reproduce results for the non-axisymmetric modes reported for UCM fluids in Ref. 43. Having verified the code, we can safely use it to investigate stability of LPTT model in circular Couette flow.

5. Results and discussion

In this section we report our new numerical results obtained for the critical Reynolds number for the LPTT model [38]. Our main objective is to investigate the effect of the slip factor, $\xi$, on the instability picture of the circular Couette flow. A summary of the results are only presented (see Ref. 38 for more results). Although theoretically this parameter can assume values in the range of $0 < \xi < 2$, the rheological data available for polymeric liquids suggest that it is in the range of $0 < \xi < 0.2$ [16]. As such, we have decided to limit our results to the same range.

5.1. The axisymmetric mode $(m = 0)$

Fig. 6 shows the neutral instability curve for the LPTT model and its different variations for a gap size of $\phi = 0.883$ for the stationary mode (i.e., $m = 0$). As is customary, we have plotted the inner-cylinder Reynolds number $\text{Re}_1 = \rho R_1 \Omega_1 d / \eta_0$, as a function of the outer-cylinder Reynolds number $\text{Re}_2 = \rho R_2 d / \Omega_2 / \eta_0$ under critical conditions. (For a given fluid and a given geometry, this is tantamount to saying that for a given $\Omega_2$ we temper with $\Omega_1$ such that the Taylor vortices first start to appear.) A comparison between Newtonian and UCM results shows that the effect of $\lambda$ is destabilizing. Since for the UCM model $\lambda$ gives rise to a constant first normal stress difference, one can conclude that the effect of first normal stress difference is destabilizing. On the other hand,
a comparison between UCM and J–S results reveals that the effect of $\xi$ is also destabilizing, at least for this set of parameters (to be elaborated later on). The inclusion of $\xi$-term in the J–S model gives rise to non-zero second normal stress difference and a shear-thinning viscosity, as compared with the UCM model. The effect of second normal stress difference is already known to be stabilizing [see Refs. 42 and 43]. As to the effect of shear-thinning, experimental data obtained by Escudier et al. [44] for Lapnité/CMC and also Xanthan polymer solutions suggest that it has a stabilizing effect on the non-axisymmetric mode [44]. As to the axisymmetric mode, there is discrepancy between different theories, some suggesting a stabilizing effect [see, for example, Ref. 45] and some a destabilizing effect [see, for example, Ref. 46]. In a recent work, Alibenyahia et al. [47] proved that the discrepancy can be traced back to the inappropriate use of the reference viscosity in the definition of the Reynolds (or Taylor) number. They have shown that if the Reynolds number is defined based on the inner-wall shear-viscosity then the shear-thinning always has a stabilizing effect on circular Couette flow in the theoretical works cited above. With this in mind, one can conclude that the destabilizing effect of $\xi$ is perhaps due to the first-normal-stress-difference coefficient being shear-thinning (see, also, Fig. 3). A comparison between J–S and SPTT results shown in Fig. 6 suggests that the effect of $\varepsilon$ is also destabilizing. The destabilizing effect of $\varepsilon$ on plane Poiseuille flow is well established in the literature [48]. And, it has been attributed to its effect on the elongational deformations experienced by the fluid elements as soon as perturbed modes start to grow (even though the basic flow is predominantly of shear type). Since an increase in $\varepsilon$ lowers the extensional viscosity, one can conclude that strain-softening has a destabilizing effect on the flow. And, this may also partially contribute to the destabilizing effect of $\xi$ mentioned above because an increase in $\xi$ also gives rise to (albeit weak) strain softening (see Fig. 3). Finally, Fig. 6 shows that, for this set of parameters, the LPTT model is the least stable one in the group. This is not surprising realizing the fact that it includes all three destabilizing parameters $\lambda$, $\varepsilon$, $\xi$ as discussed above.

As mentioned above, for the set of parameters used for plotting Fig. 6, the non-affine motion has a destabilizing effect on circular Couette flow. This can best be seen in Fig. 7 which shows results obtained at three different slip parameters for the stationary mode (i.e., $m = 0$). We have been able to obtain converged results similar to those shown in Fig. 7 for other Weissenberg numbers (up to 1.7) for the same $\phi$ and $\varepsilon$. Quite surprisingly, we have found out that, depending on the Weissenberg number, the effect of $\xi$ can also be stabilizing.

Fig. 8 shows the stabilizing effect of the slip factor on the first mode quite clearly for the case in which the outer cylinder is fixed (i.e., $\omega = 0$). As can be seen in this figure, at Weissenberg numbers smaller that roughly 0.35, the effect of $\xi$ is slightly stabilizing. At larger Weissenberg numbers, however, its effect is clearly destabilizing. To interpret these intriguing result, it should be noted that an increase in the slip factor gives rise to a simultaneous shear-thinning viscosity and shear-thinning normal-stress-difference coefficients if the Weissenberg number is sufficiently large (see Fig. 3)—at low Weissenberg numbers, they are all constant, as can be seen in Fig. 3. As mentioned earlier, while the effect of $\psi_1$ is destabilizing, the effect of $\psi_2$ and $\eta$ are stabilizing. It is the competition between the destabilizing effect of $\psi_1$ and the combined stabilizing effect of $\psi_2$ and $\eta$ which determines which effect eventually prevails. And, because the Weissenberg number has a strong influence on the normal stress distribution, such peculiar behavior is plausible when the net effect is considered.

At this stage, for mere curiosity, we have decided to compare the Taylor–Couette flow of LPTT fluids with that of the Giesekus fluids. The latter fluid model describes how the relaxation time of a molecule (dumbbell) is altered when the surrounding molecules (dumbbells) are oriented. The relaxation behavior becomes anisotropic and results in an additional quadratic term of the stress tensor compared to the Maxwell model. The mobility factor in this model, $\alpha$, is a measure of the severity of the anisotropy. Although the Giesekus model does not account for the non-affine motion, its shear and extensional flow behavior is very similar to the LPTT model provided $\xi = 0$. In fact, the role played by the mobility factor on the rheological behavior of the Giesekus model is very similar to the role played by the extensional parameter in the LPTT model. In our previous work [14] we have already studies Taylor–Couette instability of the Giesekus fluid. Here, we just compare the two models for the case of $\phi = 0.883$, and $\omega = 0$. But, first we would like to stress that we are not in favor of one model at the expense of the other; that is, each model has its own advocates and fits different materials well.

Fig. 9 shows the critical Reynolds number as a function of the Weissenberg number for the LPTT and Giesekus models. The case of $\varepsilon = 0$ in the LPTT model corresponds to the case of $\alpha = 0$ in the Giesekus model (both models reduce to UCM under this condition), which is why the curves representing these two limiting cases are virtually the same in this figure. And, for the UCM fluid the effect of the Weissenberg number (say, $N_1$) is known to be destabilizing [14]. But, while for the LPTT model an increase in $\varepsilon$ makes the flow to become monotonically less stable at any given Weissenberg number, for the Giesekus fluid there is a peak in the instability curve for all $\alpha \neq 0$ (see Fig. 9). That is to say that, for any non-zero mobility factor in the Giesekus model, the effect of the Weissenberg number can be stabilizing or destabilizing depending on it being smaller or larger than a critical value. In Ref. 14 this behavior was attributed to a competition between the destabilizing effect of $N_1$ on the one hand with the stabilizing effect of $N_2$ on the other hand. The second normal stress difference, $N_2$, is zero for the LPTT model for the case of $\xi = 0$. Therefore, there is no stabilizing mechanism involved for this fluid to compete with the destabilizing effect of $N_1$, and so we are witnessing a monotonic drop in the critical Reynolds number for the LPTT fluid (see Fig. 9).

Fig. 10 shows the neutral stability curves for the LPTT and Giesekus models at a given Weissenberg numbers. This figure shows that while for the LPTT model, the most unstable case (i.e., the least $Re_1$) occurs when the outer cylinder is fixed, for the Giesekus model it occurs when the outer cylinder is rotating in the opposite direction. It is worth-mentioning that to screen out the complicating effect of the shear-thinning viscosity from the comparison, we have obtained our LPTT results at $\varepsilon = 0.37$ which give rise to virtually the same flow curve for $\alpha = 0.2$ in the Giesekus model. This figure also shows the LPTT fluid is less stable than the Giesekus fluid when $\varepsilon = 0$, as discussed above.

Fig. 11 shows a replot of Fig. 7 fluid in the range of $Re_1 = -50$ to $+50$ over which numerical results were available for the Giesekus fluid [14]. This figure shows that the effect of the slip parameter in the LPTT model is qualitatively the same as the effect of the Weissenberg number in the Giesekus models. This is not surprising realizing the fact that the elastic behavior of the LPTT fluid (as represented by $N_1$ and $N_2$) are closely controlled by the slip parameter (see Fig. 3).

5.2. The non-axisymmetric modes ($m \neq 0$)

Fig. 12 shows the neutral instability curve for the LPTT fluid obtained for both the axisymmetric mode ($m = 0$) and two non-axisymmetric modes ($m = 1$ and $m = 2$) for $\phi = 0.7$, $\varepsilon = 0.05$. $We = 1$, $\omega = 0$ at two different slip parameters of: $\xi = 0$ and $\xi = 0.05$. As can be seen in this figure, for any given slip parameter, the non-axisymmetric modes are slightly more stable than the symmetric mode. More importantly, an increase in the slip parameter is seen to decrease the critical Reynolds number for both
Fig. 7. Effect of the slip parameter in the LPTT model on the stability of circular Couette flow (m = 0) obtained for φ = 0.883, ε = 0.1, We = 1.0.

Fig. 8. Effect of the slip parameter, ξ, on the critical Reynolds number obtained for φ = 0.7, ε = 0.3, m = 0, ω = 0.

Fig. 9. A comparison between the effect of Weissenberg number on the critical Reynolds number of LPTT fluid (left plot, ξ = 0) and the Giesekus fluid (right plot) in Taylor-Couette flow obtained for: φ = 0.883, ω = 0, m = 0.
Fig. 10. A comparison between the neutral stability curve of LPTT fluid (left plot, $\epsilon = 0.37, \xi = 0.0, \text{We} = 1.5$) and Giesekus fluid (right plot, $\alpha = 0.2, \text{We} = 1.5$) in Taylor-Couette flow obtained for $\phi = 0.883$.

Fig. 11. A comparison between the LPTT ($\epsilon = 0.1, \text{We} = 1$) and Giesekus fluids ($\alpha = 0.2$) in Taylor-Couette flow at $\phi = 0.883$.

Fig. 12. Effect of the slip parameter on the neutral stability curves obtained for $\phi = 0.7, \epsilon = 0.05, \text{We} = 1, \omega = 0; \xi = 0.0$ (left), $\xi = 0.05$ (right).
types of modes suggesting that the non-affine motion has a destabilizing effect on the Taylor–Couette flow, at least, for this set of parameters. Fig. 13 shows the variation of the critical Reynolds number with the Weissenberg number. The destabilizing effect of the Weissenberg number is evident in this figure.

6. Concluding remarks

In the present work, the effect of non-affine motion was theoretically investigated on the stability picture of circular Couette flow for both axisymmetric and non-axisymmetric modes. To account for the non-affine motion, use was made of the linearized Phan-Thien/Tanner (LPT) model which allows non-affine deformation of (say, polymer strands) through incorporating a convenient slip parameter. Knowing the basic-flow velocity and stress fields from Ref. 5, they were subjected to infinitesimally small, time-dependent, normal-mode perturbations and their time-evolution was monitored using a linear stability analysis. An eigenvalue problem was obtained which was solved numerically using the pseudo-spectral Chebyshev-based, collocation method. Based on the results obtained in this work, it is concluded that the non-affine motion can have a stabilizing or destabilizing effect on the circular Couette flow depending on the Weissenberg number and the angular velocity ratio. The subtle influence of the slip parameter on the stability picture was attributed to a competition between the destabilizing effect of first normal stress difference and the stabilizing effect of shear-thinning and/or second-normal-stress-difference which are all influenced by the Weissenberg number.

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