FUZZY CONTINUOUS POLYGROUPS

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Abstract. We introduce and study fuzzy (quasi) continuous polygroup as a fuzzy polygroup such that its membership function is (quasi) continuous. Also, we investigate the relationship between the existence of a nontrivial fuzzy (quasi)polygroup and connectness the underlying topological polygroup.

Keywords: hypergroup, polygroup, topological hypergroup, fuzzy continuous polygroup, connectness

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1 Introduction

The concept of hypergroup was introduced by F. Marty [17]. Hypergroups are the largest class of multivalued systems that satisfy the axioms similar to group. Since then many researchers have studied in this field and developed it, for example see [1, 2, 5, 6, 7, 8, 9, 10]. An important special kind of hypergroups are polygroup (or quasi canonical hypergroups). Polygroups are studied in [2, 5, 10], where connections with color scheme, relation algebras and finite permutation groups, geometry and can be found.

The notion of a fuzzy subset of a nonempty set was introduced by L. Zadeh in 1965 [20] as a function from a nonempty set $X$ to interval $[0, 1]$. R. Rosenfeld defined the concept of a fuzzy subgroup of a given group $G$, see [18]. Since then many researchers work in this area.


In this paper we consider continuous membership functions to define fuzzy subpolygroup as a fuzzy subpolygroup such that its membership function is continuous on underlying topological polygroup (in our case the carrier is endowed with usual non fuzzy topology). Finally, we obtain some results, in particular we prove that if a nontrivial fuzzy subpolygroup exists then the underlying topological polygroup must be totally disconnected.
2 Preliminaries

Definitions and assertions from this section can be found e.g. in [1], [2], [10], [7]. Throughout this paper $H$ denotes a nonempty set. A map $\cdot : H \times H \rightarrow P_*(H)$ is called a hyperoperation or a join operation. The join operation is extended to subsets of $H$ in natural way, so that $A \cdot B$ or $AB$ is given by

$$AB = \bigcup \{ab | a \in A, b \in B\}.$$  

In $H$ two hypercomposition right $/$ and left extension $\backslash$ each inverse to $\cdot$ are defined by

$$a/b = \{x \in H | a \in xb\}$$

and

$$b\backslash a = \{x \in H | a \in bx\}.$$  

**Definition 2.1.** A hypergroup is a structure $(H, \cdot)$ that satisfies two axioms,

- **(Reproduction)** $aH = H = Ha$ for all $a \in H$;
- **(Associativity)** $a(bc) = (ab)c$ for all $a, b, c \in H$.

A polygroup or a quasi canonical hypergroup is a hypergroup containing a scalar identity, that is there exists an element $e$ such that $ea = a = ae$ for each $a \in H$. Furthermore, for each $b$ there exists $b^{-1}$ such that $a/b = ab^{-1}$, where $b\backslash a = b^{-1}a$.

**Definition 2.2.** (i) Let $H$ be a hypergroup and $\mu$ be a fuzzy subset of $H$. Then $(H, \mu)$ is called a fuzzy hypergroup of $H$ if

$$\inf_{z \in xy} \mu(z) \geq \min(\mu(x), \mu(y)), \forall x, y \in H.$$  

(ii) Let $H$ be a polygroup and $\mu$ be a fuzzy subset of $H$. Then $(H, \mu)$ is called a fuzzy subpolygroup if $(H, \mu)$ is a fuzzy hypergroup and $\mu(x^{-1}) \geq \mu(x), \forall x \in H$.

**Proposition 2.3.** [5] Let $H$ be a polygroup. Then for all $x, y \in H$ the following statements are satisfies:

(i) $(x^{-1})^{-1} = x$;
(ii) $e = e^{-1}$;
(iii) $e$ is unique;
(iv) $(xy)^{-1} = y^{-1}x^{-1}$;
(v) $a/b = ab^{-1}$ and $b\backslash a = b^{-1}a$.

The proof can be found in [5].

**Definition 2.4.** (i) Let $H$ be a polygroup. A nonempty subset $K$ of $H$ is called a subpolygroup of $H$ if $e \in H$ and $K$ is itself a polygroup under the hyperoperation on $H$.

(ii) Let $H$ be a polygroup and $N$ be a subpolygroup of $H$. Then we say that $N$ is normal subpolygroup if $xN \subseteq Nx$ for all $x \in H$.
Theorem 2.5. Let $H$ be a hypergroup (polygroup) and $\mu$ be a fuzzy subset of $H$. Then

(i) if $(H, \mu)$ is a fuzzy polygroup, then

$$\mu(e) \geq \mu(x) \quad \forall x \in H.$$ 

(ii) $(H, \mu)$ is a fuzzy hypergroup (polygroup) if and only if level subset $\mu_t = \{x | \mu(x) \geq t\}$ is a subpolygroup of $H$, for each $t \in \text{Im} \mu$.

Definition 2.6. Let $H$ be a hypergroup and $(H, \tau)$ be a topological spaces. Then:

(i) $(H, \cdot, \tau)$ is called a topological hypergroup (resp. polygroup) if

$$V_* = \{(x, y) \in H \times H | x \cdot y \subseteq V\},$$

(resp. $\{(x, y) \in H \times H | x \cdot y^{-1} \subseteq V\}$)

is open in $H \times H$ with respect to the product topology on $H \times H$.

(ii) $(H, \cdot, \tau)$ is called a weak topological hypergroup (resp. polygroup) if for every open set $V$ in $H$, the set

$$V^* = \{(x, y) \in H \times H | xy \cap V \neq \emptyset\}$$

(resp. $\{(x, y) \in H \times H | xy^{-1} \cap V \neq \emptyset\}$)

is an open subset in $H \times H$.

(iii) $(H, \cdot, \tau, \tau_*)$ is called a strong topological hypergroup (resp. polygroup) if the map

$\alpha : H \times H \rightarrow P_*(H)$, defined by $\alpha(x, y) = xy$, resp. $\alpha(x, y) = xy^{-1}$, is continuous with respect to the product topology on $H \times H$ and $H$, respectively.

Lemma 2.7. Let $(H, \cdot)$ be a polygroup and $\tau$ be a topology on $H$. Then $(H, \cdot, \tau)$ is a topological polygroup if and only if:

(i) for every elements $x, y \in H$ and each open set $U$ containing $xy$ there exists neighborhoods $V$ and $W$ of $x$ and $y$ respectively, such that $VW \subseteq U$,

(ii) for every element $x$ in $H$ and each neighborhood $V$ of it there exists a neighborhood $W$ of $x$ such that $W^{-1} \subseteq V$.

Let $(H, \tau)$ be a topological space and let $V$ be an arbitrary open subset of $H$ (member of $\tau$). Then the family $\mathcal{U}$ consisting of all closed initial segments determined by such $V$,

$$\{U \in P_*(H) | U \subseteq V\} = [\cdot, V]$$

is a base for a topology.

Definition 2.8. The upper topology on $P_*(H)$ is that generated by the base $\mathcal{U}$, where

$\mathcal{U} = \{[\cdot, V] | V \in \tau\}$. This topology is denoted by $\tau_{\mathcal{U}}$.

Theorem 2.9. Let $(H, \cdot)$ be a polygroup. Let $(H, \tau)$ be a topological space. Then $(H, \cdot, \tau)$ is a weak topological polygroup if and only if $(H, \cdot, \tau, \tau_{\mathcal{U}})$ is a topological polygroup.
Proof. Let \( V \) be an arbitrary open subset of \( H \). Then
\[
\phi^{-1}([\cdot, V]) = \{(x, y)|xy^{-1} \in [\cdot, V]\} = \{(x, y)|xy^{-1} \subseteq V\} = V^*.
\]
Therefore \( \phi \) is continuous if and only if \( V^* \) is an open subset of \( H \times H \). Since \([\cdot, V]\) is an arbitrary element of a basis of \( \tau \), the proof is complete. \( \square \)

Definition 2.10. Let \((H, \tau)\) be a topological space and \( G \) be an arbitrary open subset of \( H \). Let
\[
I_G = \{U \in P_*(H)|U \cap G \neq \emptyset\},
\]
and
\[
\mathcal{L} = \{I_G|G \in \tau\}.
\]
Then the lower topology on \( P_*(H) \) is generated by subbase \( \mathcal{L} \). This topology is denoted by \( \tau_{\mathcal{L}} \).

Theorem 2.11. Let \((H, \cdot)\) be a polygroup and \( \tau \) be a topology on \( H \). Then \((H, \cdot, \tau)\) is a weak topological polygroup if and only if \((H, \cdot, \tau, \tau_{\mathcal{L}})\) is a topological polygroup.

Proof. Let \( G \) be an arbitrary nonempty open subset of \( H \) and consider the map \( \phi : H \times H \rightarrow P_*(H) \), defined by \( \phi(x, y) = xy^{-1} \). Then
\[
\phi^{-1}(I_G) = \{(x, y) \in H \times H|xy^{-1} \in I_G\}
\]
\[
= \{(x, y)|x.y^{-1} \cap G \neq \emptyset\} = V^*.
\]
Therefore \( \phi \) is continuous if and only if \( V^* \) is open in \( H \times H \). \( \square \)

Definition 2.12. The Vietories topology \( \tau_{\mathcal{U}} \) on \( P_*(H) \) is generated by \( \mathcal{U} \) and \( \mathcal{L} \) together. An alternative characterization of \( \tau_{\mathcal{U}} \) is as follows. Let \( G_1, \cdots, G_n \) be a finite collection of open sets and let
\[
\mathcal{B}(G_1, \cdots, G_n) = \{U \in P_*(H)|U \subseteq \bigcup_{i=1}^n G_i, U \cap G_i \neq \emptyset\}
\]
then the family of all such collections \( \mathcal{B}(G_1, \cdots, G_n) \) is a subbase for \( \tau_{\mathcal{U}} \).

Theorem 2.13. Let \((H, \cdot)\) be a polygroup and \( \tau \) be a topology on \( H \). Then \((H, \cdot, \tau)\) is both and weak topological polygroup if and only if \((H, \cdot, \tau, \tau_{\mathcal{U}})\) is a strong topological polygroup.

Proof. The proof is an immediate consequence of Theorems 2.9 and 2.11. \( \square \)

Definition 2.14 (1). Let \( H \) be a hypergroup. A fuzzy subset \( \mu \) of \( H \) is called a fuzzy hypergroup if the following conditions are satisfied:

(i) \( \min\{\mu(x), \mu(y)\} \leq \inf_{a \in xy}\{\mu(a)\} \forall x, y \in H; \)

We denote the fuzzy hypergroup \( \mu \) of \( H \) by \((H, \mu)\).

A fuzzy hypergroup \((H, \mu)\) is called a fuzzy polygroup if \( H \) is a polygroup and satisfies the following condition:

(ii) \( \mu(x) \leq \mu(x^{-1}) \forall x \in H. \)
Remark 2.15. Let \((H, \mu)\) be a fuzzy polygroup. Then the following statements hold:

(i) \(\mu(x) \leq \mu(e)\) for all \(x \in H\),
(ii) \(\mu(x) = \mu(x^{-1})\).

Theorem 2.16. [1] Let \(H\) be a hypergroup (resp. polygroup) and \(\mu\) be a fuzzy subset of \(H\). Then \(\mu\) is a fuzzy hypergroup (polygroup) of \(H\) if and only if every nonempty level subset \(\mu_t\) is a subgroup of \(H\), where

\[ \mu_t = \{x \in H | \mu(x) \geq t\}, t \in [0, 1]. \]

In the sequel of this paper by \(H\) we mean a polygroup by identity \(e\).

3 Fuzzy Continuous Polygroups

Definition 3.1. A fuzzy polygroup \((H, \mu)\) is called a fuzzy topological polygroup if \(H\) is a topological polygroup and \(\mu\) is continuous.

Definition 3.2. A topological polygroup \(H\) is fuzzy trivial if all topological fuzzy polygroups on \(H\) are constant.

Theorem 3.3. For a topological polygroup \(H\) the following statements are equivalent:

(i) \(H\) contains no open subgroup other than \(H\),
(ii) \(H\) is generated by any neighborhood of the identity,
(iii) \(H\) is fuzzy trivial.

Proof. (i) \(\rightarrow\) (ii): Suppose that there is a neighborhood \(U\) of \(e\) that does not generate \(H\). Then \(V = U \cap U^{-1}\) is a symmetric neighborhood of \(e\) that does not generate \(H\) either. It is easy to verify that \(W = \bigcup_{i \geq 1} V^n\) is an open subgroup different from \(H\), which is a contradiction.

(ii) \(\rightarrow\) (iii): Let \((H, \mu)\) be a topological fuzzy polygroup and there exists \(a \in H\) such that \(\mu(a) < \mu(e)\). Consider a neighborhood \(U\) of \(e\) defined by \(U = \{x \in H | \mu(x) \geq \mu(a)\}\). Since \(H\) is generated by \(U\) then, \(a \in x_1 x_2 \cdots x_n\), where all \(x_i \in U\). We have

\[ \mu(a) \geq \inf_{x \in x_1 \cdots x_n} \mu(x) \geq \min_{0 \leq i \leq n} \{\mu(x_i)\} > \mu(a), \]

which is a contradiction.

(iii) \(\rightarrow\) (i): Suppose that \(K\) is an open subgroup of \(H\) and \(K \neq H\). Now consider fuzzy subset \(\mu\) of \(H\) defined by \(\mu(x) = 1\) if \(x \in K\) and \(\mu(x) = 0.5\) otherwise. Then by Theorem 2.11 \((H, \mu)\) is a fuzzy polygroup of \(H\). It is easy to verify that \(K\) is also closed subset of \(H\). Hence \(\mu\) is a continuous function on \(H\). Therefore \((H, \mu)\) is fuzzy nontrivial polygroup which is a contradiction.

Corollary 3.4. If \(H\) is a connected topological polygroup, then \(H\) is fuzzy trivial.

Proof. If \(H\) is connected, then it is easy to see that \(H\) satisfies condition (ii) of previous theorem. Thus \(H\) is fuzzy trivial.
Lemma 3.5. Let \((H, \mu)\) be a continuous fuzzy polygroup. Then the component \(N\) of the point \(e\) in the topological space \(H\) is closed normal subpolygroup of \(H\).

Let 
\[ H^* = H/N = \{xN | x \in H\}, \]
and define \(\mu^*\) on \(H^*\) by 
\[ \mu^*(xN) = \mu(x). \]
We get:

Theorem 3.6. Let \(H\) be a polygroup. Then \((H^*, \mu^*)\) is a continuous fuzzy polygroup.

Proof. It is easy to see that \(\mu\) is constant function on \(N\). Also it is easy to verify that \(\mu^*\) is a fuzzy polygroup.

Let \(\pi\) be the canonical homomorphism from \(H\) onto \(H^*\). Now since \(\mu = \mu^* \circ \pi\), then it is easy to see that \(\mu^*\) is continuous. Clearly \(\mu^*\) satisfies the conditions of Definition 2.3. Therefore \((H^*, \mu^*)\) is fuzzy continuous polygroup.

4 Conclusion

In this paper we introduced and studied fuzzy (resp. quasi) continuous polygroup as a fuzzy polygroup with the continuous membership function. We have investigated the relationship between existence of a nontrivial fuzzy (resp. quasi)polygroup and connectness of the underlying topological polygroup. We have proved that if a nontrivial fuzzy subpolygroup exists then the underlying topological polygroup must be totally disconnected.

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