Brittle fracture assessment of engineering components in the presence of notches: a review

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ABSTRACT Brittle fracture of notched components has been widely investigated in recent decades both experimentally and theoretically. This is because of designers’ concern about catastrophic failure in notched engineering components made of brittle or quasi-brittle materials. Up to now, extensive studies have been performed on brittle fracture analysis of engineering components weakened by notches of various features under mode I, mode II, mode III and mixed mode loading conditions. In the present paper, the attempt is made to review the research articles published in the open literature on brittle fracture assessment of notched components by means of notch fracture mechanics concepts. The main focus of this paper is on the stress-based fracture criteria, which are the basis of authors’ experience in recent years.

Keywords brittle fracture; notch; notch fracture mechanics; stress-based fracture criteria.

NOMENCLATURE

- CZM = cohesive zone model
- \( d_c \) = critical distance of the mean stress criterion measured from the notch tip
- \( d_c' \) = critical distance of the mean stress criterion measured from the coordinate origin
- \( E \) = Young’s modulus
- \( e_1, e_2, e_3 \) = modes I, II and III functions in the strain energy density expression for sharp V-notches
- \( f_c \) = cohesive strength
- FFM = finite fracture mechanics
- FNR = fictitious notch radius
- FVSD = flattened V-notched semi-disc
- \( G_F \) = cohesive fracture energy
- \( H(\frac{a}{p}) \) = function in the strain energy density expression for U-notches
- \( I_1 \) = mode I function in the strain energy density expression for sharp V-notches
- ICM = imaginary crack method
- \( K_{IC} \) = material toughness
- \( K_{V,I}^{V}, K_{V,II}^{V} \) = modes I and II stress intensity factors of a sharp V-notch
- \( K_{U,I}^{U}, K_{U,II}^{U} \) = modes I and II stress intensity factors of a U-notch
- \( K_{V,O}^{V}, K_{U,O}^{U} \) = modes I and II stress intensity factors of a V-notch with end hole
- \( K_{key}^{V}, K_{key}^{U} \) = modes I and II stress intensity factors of a keyhole notch
- \( L \) = characteristic length
- LM = line method
- \( P_{\max} \) = failure load

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INTRODUCTION

The knowledge of material failures and the ability to prevent such failures have improved considerably since World War II. Catastrophic failures, which occurred in tankers and cargo ships mainly built in the United States during World War II, are the most well-known cases of structural failure.\(^1\) In a report released in 1954, extensive cases of brittle fracture have been described, which took place in a variety of steel structures, including bridges, pressure vessels and pipelines.\(^2\) Nowadays, successful design of engineering components and structures has been facilitated by using the fracture mechanics principles.

Brittle failure in structures, especially when weakened by stress concentrators like cracks and notches, is a topic of great concern because of its catastrophic nature. Indeed, effective theories are essentially required for estimating brittle fracture in components containing different types of notches and cracks.

One of the earliest attempts to study the strength of defective materials is the experimental work performed by Leonardo da Vinci several centuries ago describing some critical causes of fracture.\(^3\) Griffith’s work published in 1920 was an attempt to make a relation between the fracture stress and the flaw size.\(^4\) In this work, Griffith used the stress solution for an elliptical hole presented by Inglis\(^5\)(several years earlier), to investigate the unstable propagation of a defect. Later, Irwin extended the applications of Griffith’s method to metals by considering the energy dissipated by local plastic flow.\(^6\)

In an independent work, Orowan\(^7\) proposed a similar modification to the Griffith theory. Mott\(^8\) also extended the Griffith theory to the case of a rapidly propagating crack. After several years, in 1956, Irwin\(^9\) proposed the energy release rate theory by extending the Griffith method. Irwin’s theory was more applicable for solving engineering problems.

Notched components made of brittle materials are prone to catastrophic failure because of the stress concentration at the notch vicinity. Brittle fracture of components weakened by various types of notches has been widely studied by many researchers and the results have been published in numerous papers. In recent years, brittle fracture assessment of notched domains has been carried out predominantly by means of notch fracture mechanics (NFM), which is an extension of the classical fracture mechanics from cracks to notch problems. The fracture criteria in the NFM context, including the strain energy density, have been developed to predict the fracture behavior of notched components.
energy density (SED),\textsuperscript{10–14} the point stress (PS)/maximum tangential stress (MTS) and mean stress (MS),\textsuperscript{15–28} the cohesive zone model (CZM),\textsuperscript{29–31} the generalized J-integral,\textsuperscript{32,33} and the finite fracture mechanics (FFM).\textsuperscript{34} are very similar to those in the classical fracture mechanics considering the key difference that brittle fracture occurs in a notched component by the initiation of crack from the notch border while in a cracked component by propagation of a pre-existing crack. It should be noted that the Griffith–Irwin criterion\textsuperscript{4,35} cannot be applied directly to notch problems because it is limited only to crack problems.\textsuperscript{36}

In the field of NFM, the first step for brittle failure analysis of a notched component is usually to determine the displacement, strain and stress distributions around the notch. As an early attempt, Fadle\textsuperscript{37} considered a pointed notch with different boundary conditions along the notch edges and applied the eigenfunction expansion method to find the eigenvalues for a notch.\textsuperscript{37} Williams’ papers on two-dimensional stress field problems\textsuperscript{38,39} are still the main references in recent studies. As a basic work, Williams used the Airy stress function satisfying the boundary conditions along the edges of a sharp V-notch to determine the stress field in the vicinity of the notch tip.\textsuperscript{38} He used the Kolosov–Muskelishvili complex function to obtain the stress asymptotic factors. Williams also presented an exact solution of the stress field for sharp V-shaped notches.\textsuperscript{38} Filippi et al.\textsuperscript{38} developed some closed-form expressions for the stress distributions around blunt V-notches. To obtain the stress field, they applied a conformal mapping in an auxiliary system of curvilinear coordinates ‘u and v’, which are related to the Cartesian coordinates ‘x and y’. By using the stress distributions reported in Ref. [38], Novozhilov\textsuperscript{41} proposed a brittle fracture criterion for sharp notches. Based on this concept of failure, some other brittle fracture criteria were later developed by several researchers for various notch features under different loading conditions.

In the present paper, theoretical and experimental research studies reported in the literature on brittle fracture assessment of notched components are reviewed. The review is performed on those papers in which the failure concepts of NFM have been included. Therefore, a large portion of the present paper focuses on local approaches, mainly on the stress-based brittle fracture criteria.

**Linear Elastic Stress Field Around a Notch Under In-Plane Loading Conditions**

Stress field around a sharp V-notch

Several researchers have investigated the stress field and/or the onset of fracture in sharp V-notches for brittle and quasi-brittle materials under static and monotonic loading conditions, for example, Williams,\textsuperscript{38} Seweryn\textsuperscript{32} and Gomez and Elices.\textsuperscript{29,43} As mentioned in the Introduction, Williams applied the Airy stress function satisfying the boundary conditions along the edges of notch to obtain the elastic stress field at the vicinity of a sharp V-notch.\textsuperscript{38} Figure 1 depicts a sharp V-notch together with its polar coordinate system. In Fig. 1, the parameter $2\alpha$ denotes the notch angle.

For a notch under in-plane loading, the biharmonic equation of elasticity ($\nabla^4 \Phi = 0$) should be solved to extract the related elastic stress field. In this equation, $\Phi$ is the Airy stress function. Williams proposed Eq. (1) as suitable stress function for sharp V-notches:

$$ \Phi(r, \theta) = r^{4\lambda+1} f(\theta) $$

By substituting Eq. (1) into the biharmonic equation, an ordinary differential equation was obtained. The general solution of this equation can be expressed as

$$ f(\theta) = C_1 \cos(\lambda - 1)\theta + C_2 \sin(\lambda - 1)\theta + C_3 \cos(\lambda + 1)\theta + C_4 \sin(\lambda + 1)\theta $$

Appropriate boundary conditions should be applied to Eq. (2) to determine the constants $C_i (i=1, 2, 3$ and 4). By applying such boundary conditions and performing some mathematical calculations, Eqs (3) and (4) were obtained:\textsuperscript{38}

$$ \sin(\lambda_1 \omega) + \lambda_1 \sin \omega = 0 $$

(3)

$$ \sin(\lambda_2 \omega) - \lambda_2 \sin \omega = 0 $$

(4)

Equations (3) and (4) are the characteristic equations in which $\lambda_1$ and $\lambda_2$ are the eigenvalues that depend on the V-notch angle. The parameter $\omega$ is also equal to $\pi - \alpha$. Table 1 summarizes the values of $\lambda_1$ and $\lambda_2$ for different notch angles.\textsuperscript{38}

![Fig. 1 A sharp V-notch and its coordinate system.](image-url)
Table 1 The values of $\lambda_1$ and $\lambda_2$ for different notch angles$^{38}$

<table>
<thead>
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<th>$2\alpha$ (°)</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.5</td>
</tr>
<tr>
<td>30</td>
<td>0.5014</td>
<td>0.5982</td>
</tr>
<tr>
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<td>0.5050</td>
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</tr>
<tr>
<td>60</td>
<td>0.5122</td>
<td>0.7309</td>
</tr>
<tr>
<td>90</td>
<td>0.5448</td>
<td>0.9085</td>
</tr>
<tr>
<td>120</td>
<td>0.6157</td>
<td>1.1489</td>
</tr>
<tr>
<td>135</td>
<td>0.6736</td>
<td>1.3021</td>
</tr>
<tr>
<td>150</td>
<td>0.7520</td>
<td>1.4858</td>
</tr>
</tbody>
</table>

After determining the stress function, the stress components in polar coordinate system could be obtained as follows.$^{38}$

\[
\begin{align*}
\sigma_{rr} &= \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ \sigma_r - r \frac{\partial \sigma_\theta}{\partial r} \right] \right\} = K_V^r r^{\lambda_1 - 1} \\
\sigma_{r\theta} &= \left\{ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \sigma_r}{\partial \theta} \right) \right\} = K_V^r r^{\lambda_2 - 1} \\
\sigma_{\theta\theta} &= \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} \right] \right\} = K_V^r r^{\lambda_1 - 1}
\end{align*}
\]

The parameters $K_V^r$ and $K_V^{\lambda_2}$ are the mode I and mode II notch stress intensity factors (NSIFs), respectively, which could be expressed by Eq. (6):

\[
K_V^r = \lim_{r \to 0} r^{1-\lambda_1} \sigma_{r0}(\theta = 0) \\
K_V^{\lambda_2} = \lim_{r \to 0} r^{1-\lambda_2} \sigma_{r\theta}(\theta = 0)
\]

Equation (5) is a general and accurate equation for the linear elastic stress field at the vicinity of a sharp V-shaped notch. It should be noted that for a sharp V-notch with zero angle $\alpha$, Eqs (5) and (6) are simplified to the stress field and the stress intensity factors (SIFs) related to a sharp crack, respectively.$^{38,44}$
Stress field around a blunt V-notch

As an approximate closed-form expression, the linear elastic stress distributions around a blunt V-notch under in-plane mixed mode loading conditions were developed first by Lazzarin and Tovo. Filippi et al. later proposed more accurate stress distributions for the same problem. Figure 2 shows a blunt V-notch and its Cartesian and polar coordinate systems. It is seen in Fig. 2 that the origin of the coordinate systems is located at the distance \( r_0 \) behind the notch tip on the notch bisector line. Such a distance is considered for matching the mathematical derivations of the curvilinear and the Cartesian coordinate systems. The notch angle and the notch radius are denoted in Fig. 2 by \( 2\alpha \) and \( \rho \), respectively.

In Ref. [40], a conformal mapping in an auxiliary system of curvilinear coordinates \( 'U \) and \( 'V \) was utilized to develop the stress field. The Cartesian and curvilinear coordinates are related to each other by \( (X+iY)=(U+iV)^{\rho} \). In this relationship, the power \( q \) is a real positive value between 1 (for a flat edge) and 2 (for a crack).

Under in-plane loading conditions, the elastic stresses in the polar coordinate system can be expressed as

\[
\begin{align*}
\sigma_{\theta\theta} &= \frac{K_{1}' \rho}{\sqrt{2\pi r^3}} \left[ \begin{array}{c}
m_{00}(\theta) \\
m_{rr}(\theta) \\
n_{00}(\theta)
\end{array} \right]^{(I)} + \left( \frac{r}{r_0} \right)^{\rho - 1} \left[ \begin{array}{c}
m_{00}(\theta) \\
m_{rr}(\theta) \\
n_{00}(\theta)
\end{array} \right]^{(I)} \\
\sigma_{r\theta} &= \frac{K_{1}' \rho}{\sqrt{2\pi r^3}} \left[ \begin{array}{c}
m_{0\theta}(\theta) \\
m_{r\theta}(\theta) \\
n_{0\theta}(\theta)
\end{array} \right]^{(I)} + \left( \frac{r}{r_0} \right)^{\rho - 1} \left[ \begin{array}{c}
m_{0\theta}(\theta) \\
m_{r\theta}(\theta) \\
n_{0\theta}(\theta)
\end{array} \right]^{(I)}
\end{align*}
\]

where \( \sigma_{\theta\theta}, \sigma_{r\theta} \) and \( \tau_{r\theta} \) are the elastic tangential, radial and in-plane shear stresses in the polar coordinate system, respectively, and \( r_0 \) is the distance between the origin of the coordinate system and the notch tip. The functions \( m_{ij}(\theta) \) and \( n_{ij}(\theta) \) are defined as

\[
\begin{align*}
\frac{m_{00}(\theta)}{1 + \lambda_1 + \lambda_2} &= \frac{1}{1 + \lambda_1 + \lambda_2} \left[ \begin{array}{c}
(1 + \lambda_1) \cos(1 - \lambda_1) \theta \\
(3 - \lambda_1) \cos(1 - \lambda_1) \theta + \lambda_2(1 - \lambda_1) \\
(1 - \lambda_1) \sin(1 - \lambda_1) \theta
\end{array} \right]^{(I)} \\
\frac{n_{00}(\theta)}{1 + \lambda_1 + \lambda_2} &= \frac{1}{4(\rho - 1)(1 + \lambda_1 + \lambda_2)} \left[ \begin{array}{c}
\lambda \lambda_1 \\
2 \lambda_1 \sin(1 - \lambda_1) \theta
\end{array} \right]^{(I)} \\
\frac{m_{0\theta}(\theta)}{1 - \lambda_2} &= \frac{1}{1 - \lambda_2 + \lambda_2} \left[ \begin{array}{c}
(1 + \lambda_2) \sin(1 - \lambda_2) \theta \\
(3 - \lambda_2) \sin(1 - \lambda_2) \theta + \lambda_2(1 - \lambda_2) \\
(1 - \lambda_2) \cos(1 - \lambda_2) \theta
\end{array} \right]^{(II)} \\
\frac{n_{0\theta}(\theta)}{1 + \lambda_1 + \lambda_2} &= \frac{1}{4(\rho - 1)(1 - \mu_2 + \lambda_2)} \left[ \begin{array}{c}
(1 + \mu_2) \sin(1 - \mu_2) \theta \\
(3 - \mu_2) \sin(1 - \mu_2) \theta + \lambda_2(1 + \mu_2) \\
(1 - \mu_2) \cos(1 - \mu_2) \theta
\end{array} \right]^{(II)}
\end{align*}
\]
The parameters $K_{I}^{V}$ and $K_{II}^{V}$ are the mode I and mode II NSIFs, which can be computed for different notch angles and various notch radii by using Eqs (9) and (10).^{10}

$$K_{I}^{V} = \sqrt{2\pi} \left( \frac{\sigma_{\theta\theta} \theta_{0=0} R^{1-j_{1}}}{1 + \omega_{1} \left( \frac{R}{r} \right)^{j_{1}} + \omega_{2} \left( \frac{R}{r} \right)^{j_{2}}} \right)$$ \hspace{1cm} (9)

$$K_{II}^{V} = \sqrt{2\pi} \left( \frac{\sigma_{\rho\rho} \theta_{0=0} R^{1-j_{2}}}{1 + \omega_{1} \left( \frac{R}{r} \right)^{j_{1}} + \omega_{2} \left( \frac{R}{r} \right)^{j_{2}}} \right)$$ \hspace{1cm} (10)

where $(\sigma_{\theta\theta})_{\theta = 0}$ and $(\sigma_{\rho\rho})_{\theta = 0}$ are the tangential and in-plane shear stresses at the notch bisector line, and $\omega_{i}$ ($i=1, 2$) are two auxiliary parameters presented in Eq. (11).

$$\omega_{1} = \frac{q}{4(q-1)} \left[ \frac{X_{d_{1}}(1 + \mu_{1}) + X_{c_{1}}}{1 + \lambda_{1} + X_{b_{1}}(1 - \lambda_{1})} \right]$$

$$\omega_{2} = \frac{q}{4(\mu_{2} - 1)} \left[ \frac{X_{d_{1}}(1 + \mu_{2}) + X_{c_{1}}}{1 + \lambda_{2} + X_{b_{1}}(1 - \lambda_{2})} \right]$$ \hspace{1cm} (11)

The parameters $r_{0}$ and $q$ are also defined as

$$r_{0} = \frac{q - 1}{q} \rho$$

$$q = \frac{2\pi - 2\alpha}{\pi}$$ \hspace{1cm} (12)

$X_{d_{1}}, X_{c_{1}}, X_{b_{1}}, \lambda_{i}$ and $\mu_{i}$ ($i=1, 2$) are auxiliary parameters that depend on the notch angle. The values of these parameters are presented in Table 2 for three widely used notch angles.^{40}

### Stress field around a U-notch

Geometrically, a U-shaped notch is a special case of a blunt V-notch for which the notch angle is 0. So, Eqs (12) and (7) can be simplified to\(^{15}\)

$$r_{0} = \frac{\rho}{2}$$ \hspace{1cm} (13)

$$\sigma_{\theta\theta}(r, \theta) = \frac{1}{2\sqrt{2\pi}r} \left\{ K_{I}^{U} \left[ \left( \frac{3}{2} \right)^{1/2} \cos \frac{\theta}{2} + \frac{1}{2 \cos^{2} \frac{\theta}{2}} \right] \right\} + K_{II}^{U} \left[ \left( \frac{3}{2} \right)^{1/2} \sin \frac{\theta}{2} + \frac{3 \sin^{2} \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \right]$$

$$\sigma_{rr}(r, \theta) = \frac{1}{2\sqrt{2\pi}r} \left\{ K_{I}^{U} \left[ \left( \frac{5}{2} - \frac{\rho}{r} \right)^{1/2} \cos \frac{\theta}{2} - \frac{1}{2 \cos^{2} \frac{\theta}{2}} \right] \right\} + K_{II}^{U} \left[ \left( \frac{5}{2} - \frac{\rho}{r} \right)^{1/2} \sin \frac{\theta}{2} - \frac{3 \sin^{2} \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \right]$$

$$\sigma_{\rho\rho}(r, \theta) = \frac{1}{2\sqrt{2\pi}r} \left\{ K_{I}^{U} \left[ \left( \frac{1}{2} + \frac{\rho}{r} \right)^{1/2} \sin \frac{\theta}{2} + \frac{1}{2 \sin^{2} \frac{\theta}{2}} \right] \right\} + K_{II}^{U} \left[ \left( \frac{1}{2} - \frac{\rho}{r} \right)^{1/2} \sin \frac{\theta}{2} + \frac{1}{2 \sin^{2} \frac{\theta}{2}} \right]$$ \hspace{1cm} (14)

The NSIFs for a U-notch can be expressed as\(^{45-47}\)

$$K_{I}^{U} = \frac{\sqrt{\pi \rho}}{2} \sigma_{\theta\theta}(\frac{\rho}{2}, 0)$$

$$K_{II}^{U} = \lim_{\rho \to 2R} \frac{\sqrt{2\pi r}}{2 \pi} (\sigma_{\rho\rho})_{\theta=0}$$ \hspace{1cm} (15)

### Stress field around a V-notch with end hole

The linear elastic stress field around a V-notch with end hole (VO-notch) has been developed by Zappalorto and Lazzarin\(^{48}\) for mode I, mode II and mode III loading conditions. Because the mathematical derivations are relatively complicated and the detailed expressions of the stress distributions are massive, just some selected expressions are presented herein as example. Figure 3 shows a VO-notch and its polar and Cartesian coordinate systems.

For pure mode I loading, the tangential stress distribution on a VO-notch bisector line ($\theta=0$) is\(^{48}\)

$$\sigma_{\theta\theta}(r) = \frac{K_{IV}}{\sqrt{2\pi}r} \left[ \frac{R^{j_{1}-1}}{\alpha_{1} + \theta_{1}(\gamma)}^{1/2} \right] \left[ 1 + \lambda_{1} + \left( -2\lambda_{1}^{2} + 5\lambda_{1} - 1 \right) \left( \frac{\rho}{r} \right)^{2j_{1}} + \left( 2\lambda_{1}^{3} - 6\lambda_{1} + 4 \right) \left( \frac{\rho}{r} \right)^{2j_{1}+1} \right]$$

$$+ \theta_{1}(\gamma) \left[ 1 + (1 - \lambda_{1}) \left( \frac{\rho}{r} \right)^{2j_{1}} + (2 + \lambda_{1}) \left( \frac{\rho}{r} \right)^{2j_{1}+1} \right]$$ \hspace{1cm} (16)
In Eq. (16), $\sigma_{\theta\theta}$, $K_{IV}^{V}$, and $\rho$ are the tangential stress, the mode I NSIF and the notch radius, respectively. The parameters $\lambda_1$ and $\theta_1(\rho)$ are the eigenvalue and the auxiliary expression, which both depend on the notch angle.\(^{48}\) The tangential component of the stress field for pure mode II loading can also be expressed as\(^{48}\)

$$
\sigma_{\theta\theta}(r, \theta) = \frac{K_{IV}^{V}}{\sqrt{2\pi r}} \left[ R \sin(Q\theta) - AT_{21} \sin(Q\theta) \left( \frac{\rho}{r} \right)^S \right. \\
\left. - AT_{22} \sin(Q\theta) \left( \frac{\rho}{r} \right)^T + \theta_2 \sin(R\theta) \right] + \theta_2 \sin(R\theta)Q \left( \frac{\rho}{r} \right)^S + \theta_2 \sin(R\theta)U \left( \frac{\rho}{r} \right)^S$$

where

$$
AT_{21}(\theta) = \left[ Q \sin(Z\theta) - 2 \cos(\lambda_2\theta) \sin(Q\theta) \right] / \sin(\theta) \quad (17)
$$

$$
AT_{22}(\theta) = \frac{2V}{\tan(\theta)} - 1 \quad (18)
$$

The parameter $K_{IV}^{V}$ is the mode II NSIF. The parameters $\lambda_2$ and $\theta_2$, which both depend on the notch angle, have been defined and given in Ref. [48], and the other auxiliary parameters in Eqs (17) and (18) are presented in Table 3.\(^{48}\)

**Table 3** The auxiliary parameters in Eqs (17) and (18)\(^{48}\)

<table>
<thead>
<tr>
<th>$O$</th>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
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<td>$1 + \lambda_2$</td>
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<tr>
<td>$S$</td>
<td>$T$</td>
<td>$U$</td>
<td>$V$</td>
</tr>
<tr>
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<tr>
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<td>$Z$</td>
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<td>$-$</td>
</tr>
<tr>
<td>$3$</td>
<td>$2\lambda_2 - 1$</td>
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</tr>
</tbody>
</table>

**Stress field around a keyhole notch**

A keyhole notch with its Cartesian and polar coordinate systems is depicted in Fig. 4.

A keyhole notch is a special type of VO-notch in which the notch angle is 0. Therefore, the formulas reported in Ref. [48] for VO-notches could be reduced for 0 notch angle to achieve the stress distributions for keyhole notches. For instance, the tangential stress distribution around a keyhole notch under mixed mode I/II loading can be expressed as

$$
\sigma_{\theta\theta}(r, \theta) = \frac{K_{I}^{key}}{2\sqrt{2\pi r}} \left[ \left( \begin{array}{c} 3 \cos^2 \frac{\theta}{2} + \cos \theta \frac{\rho}{r} \left( \frac{\rho}{r} \right)^{2} \\ + \frac{1}{2} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{\theta}{2} \left( \frac{\rho}{r} \right) + \frac{5}{4} \cos \frac{\theta}{2} \left( \frac{\rho}{r} \right)^{1} \end{array} \right) \right] \\
+ \frac{K_{II}^{key}}{2\sqrt{2\pi r}} \left[ \left( \begin{array}{c} 3 \sin^2 \frac{\theta}{2} + \sin \theta \frac{\rho}{r} \left( \frac{\rho}{r} \right)^{2} \\ + \frac{3}{2} \sin^2 \frac{\theta}{2} + \frac{3}{4} \sin \theta \frac{\rho}{r} \left( \frac{\rho}{r} \right) \end{array} \right) \right]$$

In Eq. (19), $\sigma_{\theta\theta}$, $K_{I}^{key}$, $K_{II}^{key}$, $\rho$, $\theta$ and $r$ are the tangential stress, mode I and mode II NSIFs, notch radius and polar coordinates, respectively.

**Some of the current approaches for brittle fracture assessment of notches**

**Averaged strain energy density criterion**

The averaged strain energy density (ASED) criterion was formulated and used first for sharp V-notches and later extended to U-notches and blunt V-notches by Lazzarin and Berto.\(^{10,49}\) The ASED criterion states that brittle failure takes place when the average of the SED over a defined control volume, which embraces the notch edge, is equal to a critical SED, $W_c$.\(^{10,12}\) This approach is based both on a precise definition of the control volume and on
the fact that the critical SED does not depend on the notch sharpness. Sheppard\textsuperscript{50} applied the concept of averaging over a finite size volume and examined it in his study. As a material property, the critical energy, varies for different materials. In principle, this value is determined from uniaxial tests, so, it depends on the type of uniaxial loading (i.e. tension or compression) meaning that the critical value $W_c$ under compression is absolutely different from that under tension. If the material has an ideally brittle or quasi-brittle behaviour, $W_c$ can be evaluated by applying the conventional tensile strength, $\sigma_t$, so that it can be computed by the following expression:\textsuperscript{49}

$$W_c = \frac{\sigma_t^2}{2E}$$  \hspace{1cm} (20)

As shown in Fig. 5, for in-plane problems, the control volume becomes a circular sector with a radius $R_0$ centred at the tip of the V-notches in mode I (and mixed mode) loadings. For a blunt V-notch under mode I loading, the control volume is assumed to have a crescent shape along the notch bisector line. Under mixed mode loading conditions, the control volume rigidly rotates with respect to the notch bisector line, and it is centred on the point where SED (or alternatively the maximum principal stress) attains its maximum value.\textsuperscript{51-53}

The total radius of the crescent shape is equal to $R_0 + r_0$, where $r_0$ depends on the angle of V-notch ($2\alpha$), and it can be computed by the following expression:\textsuperscript{44}

$$r_0 = \rho \frac{\pi - 2\alpha}{2\pi - 2\alpha}$$  \hspace{1cm} (21)

Two useful expressions for the radius $R_0$ surrounding the control volume have been provided for the crack case under plane-strain and plane-stress conditions\textsuperscript{12,54} as

$$R_0 = \frac{(1 + \nu)(5 - 8\nu)}{4\pi} \left( \frac{K_{IC}}{\sigma_t} \right)^2 \quad \text{(plane-strain)}$$

$$R_0 = \frac{(5 - 3\nu)}{4\pi} \left( \frac{K_C}{\sigma_t} \right)^2 \quad \text{(plane-stress)}$$  \hspace{1cm} (22)

In Eq. (22), the parameters $\nu$, $K_{IC}$ and $K_C$ are the Poisson’s ratio, the plane-strain fracture toughness and the plane-stress fracture toughness of material, respectively. For sharp notches in which the notch tip radius ($\rho$) is equal to 0, the total strain energy over the area of radius $R_0$ can be determined and the average value of the elastic SED under pure mode I loading conditions can be computed by the following relationship:\textsuperscript{10}

$$W_1 = \frac{I_1}{4E\lambda_1(\pi - \alpha)} \left( \frac{K_1}{R_0^{1-\gamma_1}} \right)$$  \hspace{1cm} (23)

where $\lambda_1$ is the eigenvalue of the Williams solution for mode I and $K_1$ is the related NSIF. The parameter $I_1$ depends on both the notch angle and the Poisson’s ratio having different values under plane-stress and plane-strain conditions. The values of $I_1$ can be found in some previous articles.\textsuperscript{54,55} Equation (23) can be extended to different combinations of mode I, mode II and mode III loading as a function of the corresponding NSIFs ($K_I$, $K_{II}$ and $K_{III}$) as follows:\textsuperscript{10,56}

$$W^\gamma = \frac{\epsilon_1}{E} \left( \frac{K_I}{R_0^{1-\gamma_1}} \right)^2 + \frac{\epsilon_2}{E} \left( \frac{K_{II}}{R_0^{1-\gamma_2}} \right)^2 + \frac{\epsilon_3}{E} \left( \frac{K_{III}}{R_0^{1-\gamma_3}} \right)^2$$  \hspace{1cm} (24)

where $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ are functions of the notch angle.\textsuperscript{55}

The expressions for the averaged SED of blunt V-notches in terms of the NSIFs are very complex and hence are not presented herein. They can be found in some papers published by Berto and Lazzarin (see for instance Ref. [55]).

Like blunt V-notches, the critical volume in U-notches is centred on the notch bisector line for mode I loading conditions (Fig. 6a). Under mixed mode loading, the crescent shape volume rotates rigidly without any change in shape and size and is centred on the point in which the principal stress attains its maximum value along the edge of the notch (Fig. 6b).\textsuperscript{57,58}
The averaged value of the SED for U-notches over the specified control volume can be defined as

$$\mathcal{W} = H \left( \frac{R_0}{\rho} \right) \frac{\pi}{4} \frac{\sigma_{\text{max}}^2}{E}$$

(25)

where \( H \) is a function that depends on the Poisson’s ratio \( v \) and the radius value \( R_0 \) that is normalized by the notch radius \( \rho \). Various values of \( H \) have been presented by Lazzarin and Berto.49

Torabi and Berto applied the SED approach over a control volume, which embraces the notch edge, to predict the fracture loads of some U-notched Brazilian disc (UNBD) specimens made of polycrystalline graphite. These graphite specimens were tested under pure mode I, pure mode II and mixed mode I/II loading conditions.59 They also made use of the SED criterion to predict the experimental results on mode I fracture of blunt V-notched components made of the same polycrystalline graphite.60

Cohesive zone model

In a research paper, Elices et al.61 provided a review of the CZM and mentioned some improvements in this field. Based on the CZM, Gomez and Elices29,43,62,63 proposed a criterion to predict brittle fracture in cracked and notched components. A specific characteristic of the CZM is that it joins a strength-based failure criterion (cohesive strength) to an energy-based fracture criterion (toughness) for the material ahead of the crack (or notch) tip. For performing failure analyses, cohesive-zone elements should be embedded along the fracture plane, which deform based on a traction–separation law.64

According to the CZM criterion, a cohesive crack initiates at the point where the maximum principal stress attains a critical value, namely, the cohesive strength \( f_c \). The direction of the cohesive crack initiation is perpendicular to the maximum principal stress. After the formation of the cohesive crack, it opens by transferring stress from one face to the other.64 Because the failure assessment of notches in mode I conditions was the primary concern in CZM, the rectangular softening function has been usually utilized for this purpose. This simplest softening function depends only on two parameters: the fracture energy \( G_F \) and the cohesive stress \( f_c \). In some works, the cohesive stress \( f_c \) has been taken to be equal to the tensile strength of unnotched specimens. The fracture energy has also been computed by using the plane-strain fracture toughness of material as expressed in the following:40

$$G_F = \frac{K_{IC}^2}{E'}$$

$$E' = \frac{E}{1 - v^2}$$

(26)

Because the fracture path is initially unknown under mixed mode loading conditions, the problem becomes more complex. The procedure of applying this criterion to mixed mode loading conditions has been elaborated in some papers published in the literature.51,65

J-integral

Definitions of the displacement, strain and stress fields ahead of a notch/crack tip and a characterization of the parameters of fracture mechanics are necessary to predict the behaviour of a notch/crack. The energy J-integral is one of the most common fracture parameter utilized for studying crack initiation and propagation in engineering components and structures. Numerical analysis and engineering estimation methods can be applied to evaluate the J-integral approach.33 The J-integral method, as a line integral, was introduced by Rice66 and evaluated for cracks and notches. In this approach, a contour integral between two points that stay on the opposite free borders of the notch was applied to avoid the problem of stress singularity at the notch tip.32

In an interesting research, Livieri32 developed a path-independent J-integral for blunt V-shaped notches under mode I loading. In this work, numerical and analytical solutions were obtained for a material obeying a purely linear elastic behaviour.32 Matvienko and Morozov33 also calculated the J-integral for components containing notches and cracks. They obtained approximate solutions for computing the J-integral in a specimen weakened by cracks and notches under elastic–plastic loading conditions. The effect of strain-hardening exponent on the J-integral and the influence of the applied critical stress (or the crack length) on the strain fields ahead of the crack tip were discussed in their work.33

Theory of critical distances

The theory of critical distances (TCD) has been proposed to describe a group of approaches used to predict brittle fracture of the specimens in which stress concentrators are present.67 The TCD was originally developed by Neuber68 and Peterson69 to estimate the notch failure limits of metallic materials in terms of stresses. According to TCD, failure of mechanical component takes place when the MS over a line (line method)68 or the stress obtained at a certain distance from the stress raiser apex (point method, PM)69 becomes equal to the plain fatigue limit of material. Based on the PM, which is the simplest formulation of TCD, under uniaxial fatigue loading, the structural integrity of the assessed component is in a critical condition as long as the maximum principal stress calculated at a distance \( L/2 \) from the notch tip attains the inherent material strength \( \sigma_0 \). The parameter \( L \) is called
the characteristic length, which is a material property. This commonly used length scale is given by

$$L = \frac{1}{\pi} \left( \frac{K_{IC}}{\sigma_0} \right)^2$$  \hspace{1cm} (27)

In Eq. (27), $K_{IC}$ is the plane-strain fracture toughness, which is a material property. For brittle materials like ceramics, $\sigma_0$ is normally equal to the ultimate tensile strength of material, but for materials with a certain amount of plasticity, $\sigma_0$ is higher than the plain material strength, and it can be determined by performing experiments.\cite{70}

Taylor and his co-researchers made successfully use of the TCD for the static failure assessment of notched brittle components under mode I loading conditions.\cite{71,72,73}

There are four different approaches in TCD that use the parameter $L$ and a linear elastic analysis in the failure assessments. Two of these methods are the line method and PM, which are the stress-based approaches. The third method is the imaginary crack method in which a crack is considered to exist at the root of the notch, and fracture occurs when the SIF for this crack attains the fracture toughness $K_I$.\cite{74} The fourth is a method developed by Taylor et al.,\cite{73} namely, the FFM. In this method, similar to that of Griffith, an energy balance is used to obtain the failure condition, but a finite amount of crack extension is assumed.\cite{73}

**Finite fracture mechanics**

According to the FFM criterion, fracture is supposed to propagate by finite steps. This finite crack extension is determined by the contemporaneous fulfilment of two conditions. These conditions, the stress requirement and an energy balance, are defined as\cite{75,77}

$$\left\{ \begin{array}{c}
\int_0^l \sigma(x) dx \geq \sigma_c l \\
\int_0^l G(\gamma) d\gamma \geq G_c l
\end{array} \right.$$

(28)

The first condition implies that the average of the tensile stress $\sigma(x)$ over the crack advancement $l$ should be equal to or higher than the material tensile strength $\sigma_c$. The latter one ensures that the integral of the crack-driving force $G$ should be greater than the necessary energy for creation of the new fracture surfaces, $G_c l$. By applying these two conditions at the incipient of the failure, the crack extension angle and the generalized fracture toughness can be obtained.\cite{77}

Cornetti et al.\cite{78} investigated the effects of the $T$-stress on mixed mode brittle fracture of cracked components through FFM by applying a semi-analytical approach. The critical SIFs and the critical crack kinking angle were reported as the main results. Some experimental data reported in the literature on polymethyl-methacrylate (PMMA) were also used to verify the theoretical predictions.\cite{78} Moreover, Carpinteri et al.\cite{79} applied the FFM criterion to structures with sharp notches. The results of their work demonstrated the ability of this criterion to estimate the experimental data regarding the effect of notch angle on the failure of sharp notches.\cite{34} The coupled FFM approach was used by Sapora et al.\cite{79} to study brittle fracture in blunt V-notches under mode I loading. To validate the results of this approach, numerous experimental data related to notched AISI 4340 steel specimens as well as alumina and PMMA notched samples with varying root radius were considered.\cite{79} Sapora et al.\cite{77} also applied the FFM criterion to investigate brittle fracture of V-notched elements under mixed mode loading. The theoretical results obtained from their FFM criterion were compared with some experimental data available in the literature and also with some theoretical results obtained from other critical distance approaches to validate this criterion.\cite{77}

**STRESS-BASED CRITERIA FOR BRITTLE FRACTURE ANALYSIS OF V-SHAPED AND U-SHAPED NOTCHES**

**Point stress/maximum tangential stress criterion**

One of the well-known criteria for brittle fracture assessment of engineering components is the MTS criterion that has been originally proposed by Erdogan and Sih.\cite{80} This criterion is commonly used for investigating brittle failure in components containing a sharp crack under mixed mode loading conditions. It is based on the so-called local mode I concept, which is the fundamental idea in the fracture assessment of brittle materials.\cite{21} In their work on the cracked plates within-plane loading and transverse shear, the direction of the crack propagation was in radial direction from the crack tip and perpendicular to the MTS.\cite{80} According to the MTS criterion, failure happens when the tangential stress at a critical distance $r_\gamma$ ahead of the crack tip reaches its critical value. The onset of brittle fracture can be simply written in a mathematical form as

$$\sigma_{\theta\theta}(r = r_\gamma) = (\sigma_{\theta\theta})_c$$

(29)

The critical stress $(\sigma_{\theta\theta})_c$ is a material property and is usually assumed to be equal to the ultimate tensile strength, $\sigma_u$, for brittle and quasi-brittle materials. The MTS criterion implies that final fracture for a component under tensile loading will occur only when the
molecular bonds of the material are broken. This is expected to be true for both weakened (cracked or notched) and non-weakened components. Because the stress gradient at the neighbourhood of a defect is high, the local stress attains $\sigma_u$ under lower values of load.

For better understanding of the critical distance $r_c$, the concept of a core region surrounding the crack tip proposed by Sih$^{81,82}$ should be introduced. According to this idea, the critical distance is a scale size of continuum level approach, which means that the continuum mechanics analysis is reliable until the distance $r_c$ from the crack tip. This distance is the radius of the core region. Because of the simplicity of this concept, it has been widely used by several researchers in their experimental and analytical fracture investigations, such as the research studies performed by Tian et al.$^{83}$ who extended this concept to three-dimensional domains, Yokobori et al.$^{84}$ Hyde and Chambers,$^{85}$ Mahajan and Ravi-Chandar,$^{86}$ Gdoutos,$^{87}$ Abdel-Mageed and Pandey$^{88}$ and Chambers et al.$^{89}$

Recently, several authors have considered the MTS criterion for sharp cracks as a stress-based criterion; modified it by considering the contribution of the T-stress and applied the generalized MTS criterion to fracture prediction of different brittle and quasi-brittle materials.$^{90,91}$ The MTS criterion has been extended in recent years to notched domains for predicting the onset of brittle fracture in notches of various features under different loading conditions. Some of the fracture criteria in this context are briefly described in the next subsections.

**Sharp V-notch maximum tangential stress criterion**

The sharp V-notch maximum stress (SV-MTS) criterion has been proposed by Ayatollahi et al.$^{19}$ to predict brittle fracture in engineering components weakened by a sharp V-notch and loaded under mixed mode I/II. According to this criterion, failure takes place when the tangential stress at a critical distance ahead of the notch tip, $r_{c,V}$, reaches a critical value, $(\sigma_{\theta\theta})_c$. The details of failure model and its associated failure curves can be found in Ref. [19].

Ayatollahi et al.$^{19}$ performed a set of fracture tests on a disc-type test specimen, called sharp V-BD specimen (Fig. 7), made of PMMA. The specimens were loaded under mixed mode conditions to evaluate the validity of the SV-MTS criterion. Very good agreement was found between the experimental results obtained from the PMMA sharp V-BD specimens and the theoretical predictions of the SV-MTS criterion.$^{19}$

**U-notch maximum tangential stress criterion**

Ayatollahi and Torabi$^{15,92}$ proposed a failure criterion for brittle fracture in U-notched components under mixed mode I/II and pure mode II loading conditions. This criterion, called U-notch maximum tangential stress (UMTS), was developed by using simultaneously two distinct criteria: the MTS criterion as the main foundation and an auxiliary criterion based on the critical value of the NSIF proposed by Gomez and Elices$^{60}$ for mode I failure of blunt V-notches and by Minor et al.$^{93}$ for that of U-notches.

According to the UMTS criterion, brittle fracture initiates from the U-notch border when the tangential stress at the critical distance ahead of the notch tip, $r_{c,U}$, attains its critical value $(\sigma_{\theta\theta})_c$. To evaluate the validity of the UMTS criterion, the theoretical results of this approach were compared with the experimental data reported by Gomez et al.$^{57}$ and also with the theoretical results of the local SED criterion.$^{57}$ The glassy polymer PMMA was used in Ref. [74] for fracture experiments. The experiments were performed at $-60^\circ$C to suppress plastic deformation and provide linear elastic load–displacement curve. The geometry and loading conditions of the U-notched specimens are shown in Fig. 8.$^{15}$

It has been shown in Ref. [15] that the UMTS criterion could successfully predict the experimental results related to the mixed mode NFT and the fracture initiation angle.

In another work, Torabi$^{20}$ utilized the UMTS criterion for predicting the experimentally obtained mixed mode I/II fracture toughness of U-notched graphite plates reported in literature. Again very good agreement was found between the experimental and theoretical results. Moreover, extensive brittle fracture curves have been presented by Torabi,$^{94}$ which are capable of predicting mixed mode I/II brittle fracture in engineering components weakened by U-notches of different radii and made of various brittle materials. Different values...
of the critical distance were used for obtaining the fracture curves.\textsuperscript{20,94}

**Round-tip V-notch maximum tangential stress criterion**

Ayatollahi and Torabi\textsuperscript{16} recently extended the classical MTS criterion to blunt V-notched domains and developed a stress-based criterion, called the round-tip V-notch MTS (RV-MTS) criterion, to predict brittle fracture in blunt V-notched components. According to RV-MTS criterion, brittle fracture occurs in a blunt V-notched element when the tangential stress along the fracture initiation angle $\theta_0$ and at a radial critical distance $r_c$ from the origin of the polar coordinate system (Fig. 2) reach the critical stress ($\sigma_{\theta0}$).

For verifying the validity of the RV-MTS criterion, numerous mixed mode fracture experiments were performed on a disc-type specimen containing round-tip V-notch, called the round-tip V-BD specimen, made of PMMA.\textsuperscript{16} A good agreement was shown to exist between the theoretical and experimental results for both the fracture initiation angle and mixed mode I/II NFT.\textsuperscript{16}

Ayatollahi and Torabi\textsuperscript{18} also verified the RV-MTS criterion experimentally by performing a set of mixed mode brittle fracture tests on the round-tip V-BD specimens made of soda–lime glass (Fig. 9). Comparing the theoretical results with the experimental ones indicated very good correlation for the values of NFT as well as the fracture initiation angles.\textsuperscript{18}

In a purely theoretical work, Torabi\textsuperscript{95} developed numerous failure curves based on the MTS criterion to predict the onset of sudden fracture in V-notched brittle components under mixed mode I/II loading.
In a separate research, the fracture toughness and the fracture initiation angle of the V-BD specimens made of polycrystalline graphite were experimentally measured by Ayatollahi and Torabi under combined tensile–shear loading. The V-notch MTS fracture criterion was then employed to predict the experimental results of the V-BD graphite specimens and very good agreement was found.

Mean stress criterion for sharp V-shaped notches

The MS criterion was first suggested by Wieghardt in 1907. According to this criterion, brittle fracture occurs when the mean value of the tensile stress over a specified critical distance ahead of the notch tip attains the critical stress. The MS criterion can be written in a mathematical form as

\[ \frac{\int_0^d \sigma_{00} \, dr}{\int_0^d dr} = (\sigma_{00})_c \]  \tag{30} \]

In Eq. (30), \( r, \, d_c, \, \sigma_{00} \) and \( (\sigma_{00})_c \) are the distance from the notch tip in a polar coordinate system, the critical distance, the tangential stress and the critical stress, respectively. It is worth mentioning that Novozhilov and Seweryn have also applied Eq. (30) to predict brittle fracture in sharp V-notched components.

Mean stress criterion for round-tip V-shaped notches

In recent years, two failure criteria have been proposed by Ayatollahi and Torabi for brittle failure of blunt V-notches under pure mode I loading. These criteria have been developed based on the MS and PS criteria, which both are well-known criteria for investigating brittle failure in components containing stress concentrators like cracks and notches.

A blunt V-notch and its critical distances in the MS criterion are shown in Fig. 10. The parameters \( d_c \) and \( d'_c \) depicted in Fig. 10 are the critical distances of the MS criterion measured from the notch tip and from the coordinate system origin, respectively. To formulate the MS criterion, the MS over the critical distance should first be computed. Then, the MS should attain the critical stress of material for brittle fracture to take place. The MS criterion can be presented by the following expression, which is also the basis of the Neuber’s fictitious notch rounding concept:

\[ \frac{1}{d_c} \int_{r_0}^{d'_c} \sigma_{00}(r, 0) \, dr = (\sigma_{00})_c \]  \tag{31} \]

For mode I loading, as the specimen geometry and loading conditions are symmetric, fracture initiates from the notch tip and propagates along the notch bisector line (i.e. \( \theta = 0 \) in Fig. 2). The tangential stress distribution over the notch bisector line can be written as terms of \( r \) as

\[ \sigma_{00}(r, 0) = \frac{K_{1}^{V, \rho}}{\sqrt{2\pi}r^{1-\frac{\rho}{2}}} \left( 1 + \frac{r}{r_0} \right)^{\frac{\rho}{2}} n_{00}(0) \]  \tag{32} \]

The entire parameters of Eq. (32) were defined earlier in the section on Linear Elastic Stress Field around a Notch under In-plane Loading Conditions. By substituting Eq. (32) into Eq. (31), we have

\[ \frac{1}{d_c} \int_{r_0}^{d'_c} \frac{K_{1}^{V, \rho}}{\sqrt{2\pi}r^{1-\frac{\rho}{2}}} \left( 1 + \frac{r}{r_0} \right)^{\frac{\rho}{2}} n_{00}(0) \, dr = (\sigma_{00})_c \]  \tag{33} \]

Simplifying Eq. (33) gives

\[ \frac{K_{1}^{V, \rho}}{\sqrt{2\pi}} \left( \int_{r_0}^{d'_c} \frac{dr}{r^{1-\frac{\rho}{2}}} + \frac{n_{00}(0)}{r_0^{\rho-1}} \int_{r_0}^{d'_c} r^{\rho-1} \, dr \right) = d_c (\sigma_{00})_c \]  \tag{34} \]

After performing the integrations and by equating the NSIF \( K_{1}^{V, \rho} \) with the notch fracture toughness (NFT) \( K_{1c}^{V, \rho} \) and equating the tangential stress \( \sigma_{00} \) with the critical
stress ($\sigma_{00}$), Eq. (35) can be obtained for the onset of brittle fracture as

$$\frac{K_{Ic}^N}{\sqrt{2\pi}} \left[ d_c - r_0' \right] + \frac{K_{Ic}^N}{\sqrt{2\pi}} \left[ \frac{n_{00}(0)}{\mu_0} d_0' + \frac{n_{00}(0)}{\mu_0} d_0' - r_0' \right] = d_c (\sigma_{00})_c,$$

(35)

Finally, the NFT can simply be extracted from Eq. (35) as

$$K_{Ic}^N = \sqrt{2\pi (\sigma_{00})_c d_c},$$

(36)

According to Fig. 10, the following expression is valid:

$$d_c^* = r_0 + d_c$$

(37)

The critical distance $d_c$ can be considered as Eq. (38), which is, in fact, the critical distance for sharp cracks, and it has been previously used also by Seweryn[22] for sharp V-notches.

$$d_c = \frac{2}{\pi} \left[ \frac{K_{Ic}}{(\sigma_{00})_c} \right]^2$$

(38)

In Eq. (38), $K_{Ic}$ is the plane-strain fracture toughness of material. To compute the value of NFT for blunt V-shaped notches, the critical distances, the eigenvalues and the other known parameters should be substituted into Eq. (36).

To verify the proposed criteria, fracture tests have been carried out in Ref. [21] on a notched test specimen called the round-tip V-notched semi-circular bend (RV-SCB) specimen (Fig. 11). Note that the semi-circular bend specimen weakened by a sharp crack has been previously used in the experimental studies of brittle fracture,99-102 but this was the first time that a V-notched version of the semi-circular bend specimen (i.e. the RV-SCB specimen) was suggested and used for performing fracture experiments on V-notches.[21] The RV-SCB specimens have been made of PMMA and prepared for various notch angles and different notch radii. The fracture tests have been carried out under three-point bend loading conditions.[21] The experimental results reported by other researchers from the three-point bend samples of PMMA and four-point bend samples of alumina–7% zirconia ceramic have also been utilized for additional verifications.[21]

As the main results of this study, the variations of the NFT predicted by the MS and PS criteria versus the notch radius were obtained for different notch angles together with the experimental results. For instance, Fig. 12 represents very good agreement that exists between the results of the MS criterion and the experimental results obtained from the RV-SCB specimens.[21]

Moreover, Ayatollahi and Torabi[22] studied the effects of specimen geometry on brittle fracture in blunt V-notches experimentally and theoretically. As shown in Fig. 13, three different round-tip V-notched specimens made of polycrystalline graphite were considered in experiments. The MS criterion was employed to predict the NFT of the V-notched graphite specimens. It was demonstrated that the experimental results obtained from these three different specimen shapes were in a good agreement with the theoretical results of the MS criterion.[22]

U-notch mean stress criterion

The U-notch mean stress (UMS) is a criterion that uses the MS failure concept to generally predict brittle fracture in all U-notched components regardless of the type of loading.

The first attempt to use the MS criterion for predicting brittle fracture in U-notches has been made by Torabi[23] who applied this criterion to theoretically estimate the tensile load-bearing capacity of 15 U-notched isostatic graphite plates reported in the literature. The results of his research showed that the MS criterion is an excellent failure criterion for small values of the notch tip radius, while it is a satisfactory criterion for medium and large notch radii.[23] The MS criterion has also been successfully utilized by Torabi et al.[24] for predicting the mode I fracture load of the UNBD specimens made of polycrystalline graphite. Figure 14 represents a UNBD graphite specimen subjected to mode I loading.[24]

Under mixed mode I/II loading, the UMS criterion has been formulated and utilized in Ref. [47] to predict brittle fracture in UNBD graphite specimens having various notch radii. A similar investigation has also been carried out on the same UNBD graphite specimens under pure mode II loading and as a result, the NFT ratio (i.e. the mode II NFT/mode I NFT) has been predicted well by means of the UMS and UMTS criteria.[103]

A UNBD graphite specimen is shown in Fig. 15 under pure mode II loading conditions.

It is worth noting that the V-BD and the UNBD specimens are two new specimens for performing brittle fracture tests on V-notches and U-notches. Because most of the fracture criteria, particularly the stress-based criteria, normally present their results in terms of the critical NSIFs, it is in most cases necessary to convert the experimentally obtained fracture loads of the V-BD and UNBD specimens to the corresponding values of the critical NSIFs, which is normally performed by the finite element (FE) analysis. To make it more convenient for the researchers in the field, Torabi and co-researchers[15,104] analysed the stress distributions at the vicinity of notch in the V-BD and UNBD specimens.
under mixed mode I/II loading conditions. The NSIFs were computed for the V-BD and UNBD specimens from FE analysis for different notch geometries and various loading conditions from pure mode I to pure mode II.\(^{45,104}\) By knowing the value of the fracture load and using the plots reported in Refs [45,104], the critical NSIFs can directly be obtained for the two specimens.

**BRITTLE FRACTURE OF V-NOTCHES WITH END HOLES AND KEYHOLE NOTCHES**

To avoid catastrophic failures, cracks and other types of local damage should be removed from the defective notched members or arrested by means of effective methods during periodic inspections. In engineering applications, a common repairing method for small cracks is to drill a hole and remove the crack from the notch border. The hole removes the crack but changes the initial geometry of the notch as well. Thus, the new notch is geometrically different from the initial notch, and hence, the stress distributions around the new notch are trivially different from those around the primary notch. Consequently, the failure analysis of the new notch is essential because the load-carrying capacity of a notched member depends significantly on the stress distributions at the notch vicinity and hence, on the notch feature. As shown in Fig. 16, the explained repairing method changes the

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**Fig. 11** The RV-SCB specimen.\(^{21}\) RV-SCB, round-tip V-notched semi-circular bend; PMMA, polymethyl–methacrylate.
initial V-notch to a VO-notch, and this new feature may have a mechanical behaviour different from that of the V-notch alone. Therefore, failure analysis of the VO-notch is essential. The most serious failure mode for notched components made of brittle materials is certainly brittle fracture, which occurs abruptly and catastrophically. Thus, it is important to study the fracture resistance of VO-notched members.

A few works have been performed in recent years dealing with determination of the stress distributions and brittle fracture assessment of VO-notches. The first research related to the VO-notches is that published by

![Fig. 13](image13.png) Three different round-tip V-notched specimens made of polycrystalline graphite.22

![Fig. 14](image14.png) The U-notched Brazilian disc graphite specimen under mode I loading.24
Zappalorto and Lazzarin in which the linear elastic stress field around VO-notches was formulated for various notch geometries under mode I, mode II and mode III loading conditions.48 Lazzarin et al.105 later studied the NSIFs for blunt notches. They also investigated the stress distributions and the NSIFs for VO-notches under mode II loading.105 Berto and Zappalorto applied the fictitious notch radius concept in their research to assess brittle fracture in VO-notches.106 A keyhole notch, as a specific kind of VO-notch having zero notch angle, has been investigated by Lazzarin et al.53 for brittle fracture by testing numerous graphite plates weakened by central keyhole notches of various notch radii under mode I and mixed mode I/II loading conditions. The fracture loads of the specimens were recorded in their experimental programme and predicted successfully by applying the SED criterion.53 Kullmer and Richard107 have also investigated brittle fracture in keyhole notches under mixed mode loading. Several fracture tests were carried out on the compact–tension–shear–notch specimens made of PMMA, and a stress-based failure criterion was utilized to predict the experimental results.107 Berto et al.108 applied the local SED criterion to predict the fracture loads of several rectangular isostatic graphite plates weakened by double edge VO-notches and loaded under uniaxial compression. The experimental results reported in Ref. [108] were also used by Torabi and Ayatollahi109 to verify the PS and the MS failure models as two stress-based brittle fracture criteria.

One of the most recent studies on brittle fracture analysis of VO-notches is that published by Torabi and Amininejad110 in which extensive mode I fracture tests on the VO-notched Brazilian disc (VO-BD) specimens made of PMMA have been performed for various notch angles and different notch radii. The PS and the MS criteria were successfully used for predicting the experimentally obtained mode I NFT values.110 Recently, Torabi and Amininejad111 have studied brittle fracture in VO-notches under pure mode II loading. In their study, two brittle fracture criteria, called VO-MTS and VO-MS, have been proposed to predict the onset of brittle fracture in engineering components weakened by VO-notches and loaded under pure mode II loading. These two criteria have been developed based on the MTS and the MS failure concepts. To verify the validity of the proposed criteria, 72 fracture tests were carried out on a disc-type specimen containing a central slit, called VO-BD (Fig. 17), under pure mode I and pure mode II loading conditions. All in all, 12 different VO-notch geometries were considered for preparing the PMMA VO-BD samples; three notch angles (30°, 60° and 90°) and four notch radii (0.5, 1, 2 and 4 mm).111

Figures 18 and 19 represent the theoretical results of the VO-MTS and VO-MS criteria in predicting the notch fracture toughness ratio and the fracture initiation angle, respectively, for different notch geometries together with the experimental results of the VO-BD PMMA specimens.111 It is seen from Figs. 18 and 19 that both criteria could predict the experimental results very well.111

Torabi et al.112 published a paper in which experimental, theoretical and numerical results for brittle fracture of VO-notched components made of PMMA were reported. The fracture experiments were performed at room temperature on the VO-BD specimens made of PMMA under mode II loading for different notch angles.
and various notch radii. The SED criterion was formulated and employed to predict the experimental fracture loads. It was shown that the SED criterion works well on VO-notched components under pure mode II loading conditions.\textsuperscript{112}

A keyhole notch is the special type of a VO-notch in which the notch angle is zero. Recently, Torabi\textsuperscript{113} developed two closed-form expressions for predicting the mode I apparent NFT of brittle materials weakened by keyhole notches based on the PS and the MS criteria. Such expressions have been verified by using the experimental results available in the open literature on tensile fracture of rectangular isostatic graphite plates containing central keyhole notches of various notch radii. A very good agreement has been demonstrated in their research between the results of both criteria and the experimental results.\textsuperscript{113} The mode II NFT of PMMA weakened by keyhole notches has been successfully investigated by Torabi and Abedinasab\textsuperscript{114} both theoretically and experimentally. The NFT tests were carried out by means of the keyhole-notched Brazilian disc specimens, and the theoretical predictions were performed by using the key-MTS and key-MS criteria.\textsuperscript{114} They also performed similar fracture assessments on keyhole notches under pure mode I and mixed mode I/II loadings.\textsuperscript{115,116} The keyhole-notched Brazilian disc specimen utilized in the fracture experiments of Refs. [114–116] is depicted in Fig. 20.

Torabi and Pirhadi\textsuperscript{117} also studied brittle fracture in keyhole notches under mixed mode I/II loading. On
the basis of the MTS and MS failure concepts, they de-
veloped two brittle fracture criteria, namely, the key-
MTS and key-MS criteria for predicting the mixed
mode NFT and the fracture initiation angle. The
criteria were veri-
fi
ed by means of a bulk of experimen-
tal data reported in Ref. [53] on rectangular graphite
plates containing central inclined keyhole notches of
various radii. It was reported in Ref. [117] that
although both criteria provide good predictions to the
experimental results, the key-MS criterion works much
better than the key-MTS criterion particularly for large
notch radii.

As described in the previous sections, the stress-based
brittle fracture criteria including TCD, SV-MTS, RV-
MTS and UMTS are all local approaches that use the
critical distances in their predictions. Apart from TCD,
the other criteria have been formulated based on the
closed-form stress field around the notch in terms of
the NSIFs. This makes such criteria independent of the
overall geometry of the notched component, and hence,
the criteria can be used in designing various notched
structures. Although the criteria mentioned earlier seem
to be different from TCD, their failure concepts are the
same. In addition, they employ the same critical distances
in their predictions, except that for TCD, the critical dis-
tance is always measured from the notch tip, while for the
other criteria, from the coordinate origin. Because the
values of critical distances are very important in brittle
fracture predictions of notches, they are briefly listed in
Table 4 for various criteria.

Fig. 19 The theoretical results of the VO-maximum tangential stress (MTS)
and VO-mean stress (MS) criteria in predicting the fracture initiation angle
together with the experimental results of the VO-notched Brazilian disc
specimens. Eqs. (7), (9), (16) and (18) have been given in Ref. [111]

Fig. 20 The key-Brazilian disc specimen.114–116
Under compressive stresses, closure occurs for cracks such that crack faces compress each other resulting in no crack growth. Thus, the crack problems under compression have been rarely studied by the classical fracture mechanics. Although the positive mode I (i.e. opening mode) fracture in notched components is much more significant than the negative one (i.e. closing mode), to have a reliable and successful design, failure assessments should be performed for any types of notches introduced in an engineering structure regardless of the type of loading (e.g. shear, tensile or compressive) that the notches sustain.

A few papers have been published in the literature on brittle fracture investigation of notched components under compression. Some of them published many years ago and some others in recent years have utilized the NFM principles for failure predictions.

Recently, brittle fracture in notched graphite plates under compression has been studied theoretically and experimentally by Berto et al.\textsuperscript{108} Their test specimen was a rectangular graphite plate containing double edge VO-notches subjected to pure compression. The fracture load of the specimens was successfully predicted by means of the SED criterion for various notch angles and different notch radii.\textsuperscript{108}

Based on the experimental results reported by Berto et al.,\textsuperscript{108} Torabi and Ayatollahi\textsuperscript{109} performed a brittle fracture study in which they investigated notched components under compression. In their study, two well-known failure criteria, namely, the PS and the MS criteria were extended to compressive loading conditions, and two closed-form expressions were developed for predicting the negative mode I fracture toughness of VO-notches in brittle materials. They evaluated the validity of the theoretical results of both criteria by using 42 test results given in Berto et al.\textsuperscript{108}

More recently, Torabi et al.\textsuperscript{118} studied the compressive brittle fracture in round-tip V-notched components theoretically and experimentally. In the experiments, a test specimen, called flattened V-notched semi-disc (FVSD) specimen, made of PMMA were tested under compressive loading to measure the compressive NFT for different notch angles and various notch radii. The well-known PS and MS brittle fracture criteria were utilized to predict the experimental results of the FVSD specimens. The FVSD specimen is schematically represented in Fig. 21. In Fig. 21, the parameters $2\alpha$, $\rho$, $a$, $b$, $L$, $t$ and $P$ are the notch angle, the notch radius, the notch length, the flattened length, the disc radius, the specimen thickness and the compressive load, respectively.

![Fig. 21 The flattened V-notched semi-disc specimen including its geometrical parameters](image)
To assess possible influences of the elastic–plastic behaviour of PMMA at room temperature on the validity of the PS and MS criteria, which were developed based on the linear elastic behaviour of material, some elastic–plastic FE analyses were performed on the FVSD specimens for determining the size of plastic zone around the notch. The FE results showed that the size of the plastic zone was small enough in comparison with the specimen dimensions. Therefore, the plastic zone could be ignored and the PS and MS criteria could be applied to predict the experimental results.\[118\]

**CLASSIFICATION OF MATERIALS, NOTCHES AND FRACTURE CRITERIA**

In this short section, the materials and notch features studied by the researchers in the context of linear elastic notch fracture mechanics (LENFM) are classified in a tabulated form and the effectiveness of various brittle fracture criteria is briefly discussed (see Table 5).

As can be seen in Table 5, the effectiveness of the MTS and MS criteria has been well demonstrated in predicting brittle fracture for most notch features made of various materials. Regarding O-notches, only TCD has been examined for brittle fracture in polyurethane\[120\] materials. Moreover, for VO and keyhole notches, only MTS, MS and ASED criteria have been verified mainly by means of the test results on PMMA and graphite. Table 5 clearly indicates that the V-notch has received the most attention by researchers, because its brittle fracture has been frequently investigated for various materials by different criteria. From Table 5, it can be suggested that O-notches, VO-notches and keyhole notches may be good choices to be investigated for a wide range of brittle materials by means of other criteria, like TCD, CZM, FFM, J-integral etc.

**THREE-DIMENSIONAL EFFECTS ON BRITTLE FRACTURE OF NOTCHED MEMBERS**

Most of brittle and quasi-brittle fracture criteria in the context of linear elastic fracture mechanics and LENFM have been developed on the basis of the stress and strain distributions around cracks and notches resulting from plane elasticity theories. Recent findings have revealed considerable differences between the results of actual three-dimensional and simplified two-dimensional brittle and quasi-brittle fracture criteria, particularly for very thick plates and very thin shells. This is because the intensity of the coupled modes increases or decreases as a power function of the plate thickness.\[123\]

It is well-known that the Poisson’s ratio effect in a plate weakened by a sharp crack and subjected to anti-symmetric plane loading leads to the formation of a coupled out-of-plane singular mode.\[124\]–\[129\] The numerical and theoretical analyses have revealed that a similar influence exists also in sharp V-notched plates of finite thicknesses that might play a significant role in fracture initiation of the notched plates loaded under mode II, particularly in the presence of large V-notch angles.\[130\]–\[133\] Regarding blunt notches, for example, U-notches, O-notches and elliptic notches, having large notch radii, it has been more recently shown in Ref. \[134\] that the stress field corresponding to the out-of-plane mode also exists and is of significant influence in brittle fracture assessment of notches. Dealing with coupled modes in V-notched plates of finite thickness, a valuable review paper has been recently published that can be used by those interested in the field.\[135\] From the statement earlier, one may suggest that for brittle and quasi-brittle failure analysis in notched plates of finite thicknesses, the existence and power of the coupled modes should be numerically checked in order to achieve more accurate fracture predictions.

At the end of this review paper, it is essential to highlight that some valuable review articles have also been published in recent years on brittle and quasi-brittle failure assessment of engineering materials by means of local approaches\[55,135\] that can be taken into account together with the present article. Most of the papers reviewed in Ref. \[55\] have not been reviewed again in this paper. This is because the main emphasis in the present paper is on the experiments performed by authors themselves and also on the fracture criteria, which are based on stress-based models for fracture initiation in brittle and quasi-brittle materials. Therefore, the readers can also refer to other review articles in this field as well.

**Table 5** Classification of materials, notches and fracture criteria in the context of LENFM

<table>
<thead>
<tr>
<th>Notch feature</th>
<th>Material</th>
<th>Brittle fracture criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>PMMA, graphite, soda–limestone glass, steel, ceramic, glass ceramic,[119] (\text{PUR (polyurethane),}[120] \text{polycarbonate and plastics})</td>
<td>TCD, SV-MTS, RV-MTS, MS, ASED, CZM and FFM</td>
</tr>
<tr>
<td>U</td>
<td>PMMA, graphite, (\text{PUR}^{121}) and (\text{FGM}^{122}) (functionally graded material)</td>
<td>TCD, UMTS, UMS, ASED, J-integral and CZM</td>
</tr>
<tr>
<td>O</td>
<td>(\text{PUR}^{120})</td>
<td>TCD</td>
</tr>
<tr>
<td>VO</td>
<td>PMMA and graphite</td>
<td>VO-MTS, VO-MS and ASED</td>
</tr>
<tr>
<td>Keyhole</td>
<td>PMMA and graphite</td>
<td>Key-MTS, key-MS and ASED</td>
</tr>
</tbody>
</table>
SUMMARY AND CONCLUSIONS

This review article summarizes most of the theoretical and experimental studies published in the open literature on brittle fracture of engineering components weakened by notches. The major failure models proposed by researchers within the context of LENFM were reviewed for brittle fracture assessment of notched components under various loading conditions. Particular emphasis was on the stress-based fracture criteria, which are the basis of the authors’ research activities in recent years. A survey of the references indicated that V-shaped and U-shaped notches have been studied much more than the other notch features because of their wide range of engineering applications. Meanwhile, as repaired notch shapes, VO-notches and keyhole notches have also been investigated in recent years for brittle fracture because of their practical applications in the repair of notched structures. In addition, fracture under various loading conditions including in-plane (mode I, mode II and mixed mode I/II) and out-of-plane (mode III and mixed mode I/III) loadings was reviewed in this paper. Moreover, some limited investigations on negative mode I brittle fracture in notches were reviewed. This topic has recently received much attention within the context of LENFM. Although it is not claimed that the entire papers published in the field of the LENFM have been mentioned herein, the present paper can provide a useful and concise review for the researchers who are interested in working in the field.

REFERENCES


