On Fuzzy Gamma Hypermodules

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Abstract

Let $R$ be a $\Gamma$-hyperring and $M$ be an $\Gamma$-hypermodule over $R$. We introduce and study fuzzy $R_\Gamma$-hypermodules. Also, we associate a $\Gamma$-hypermodule to every fuzzy $\Gamma$-hypermodule and investigate its basic properties.

Key words: $\Gamma$-hyperring, $\Gamma$-hypermodule, fundamental relation, fuzzy $\Gamma$-hypermodule.


1 Introduction

Hyperstructure theory was born in 1934 when Marty [13] defined hypergroups, began to analysis their properties and applied them to groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Zadeh [18] introduced the notion of a fuzzy subset of a non-empty set $X$, as a function from $X$ to $[0, 1]$. Rosenfeld [15] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. In [16], Sen, Ameri and Chowdhury introduced the notions of fuzzy hypersemigroups and obtained a characterization of them. Then in [10], Leoreanu-Fotea and Davvaz introduced and analyzed the fuzzy hyperring notion and in [11], Leoreanu-Fotea introduced the fuzzy hypermodule notion and obtained a connection between hypermodules and fuzzy hypermodules (for more information about fuzzy hypersrtuctures see [1]-[6]). The notion
of a $\Gamma$-ring was introduced by N. Nobusawa in [14]. Recently, W.E. Barnes [7], J. Luh [12], W.E. Coppage studied the structure of $\Gamma$-rings and obtained various generalization analogous of corresponding parts in ring theory. In [3] Ameri, Sadeghi introduced the notion of $\Gamma$-module over a $\Gamma$-ring.

Now in this paper we introduced and study fuzzy $\Gamma$-hypermodules as generalization of $\Gamma$-hypermodule as well as fuzzy modules. The paper has been prepared in 5 sections. In section 2, we introduce some definitions and results of $\Gamma$-hypermodules and fuzzy sets which we need to developing our paper. In section 3, we introduced and study fuzzy $\Gamma$-hypermodules and obtain its basic results. In section 4, we study fundamental relation of fuzzy $\Gamma$-hypermodules.

2 Preliminaries

In this section, we present some definitions which need to developing our paper. As it is well known a hypergroupoid is a set together with a function $\circ : H \times H \rightarrow P^*(H)$, which is called a hyperoperation, where $P^*(H)$ denotes the set of all nonempty subsets of $H$. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ is called a semihypergroup. A hypergroup is a semihypergroup such that for all $x \in H$ we have $x \circ H = H = H \circ x$ (called the reproduction axiom). We say that a hypergroup $H$ is canonical hypergroup if it is commutative, it has a scalar identity, every element has a unique inverse and it is reversible (for more details of hypergroups see [9]).

Definition 2.1. The triple $(R,+,.)$ is a hyperring (in the sense of Krasner) if the following hold: (i) $(R,\cdot)$ is a commutative hypergroup;
(ii) $(R,+)$ is a semihypergroup;
(iii) the hyperoperation ”.” is distributive over the hyperoperation ”+”, which means that for all $r,s,t$ of $R$ we have: $r.(s + t) = r.s + r.t$ and $(r + s).t = r.t + s.t$ (for more about hyperrings see [9] and [11]).

Definition 2.2. Let $(R,\uplus,\circ)$ be a hyperring. A nonempty set $M$, endowed with two hyperoperations $\oplus, \circ$ is called a left hypermodule over $(R,\uplus,\circ)$ if the following conditions hold:
(1) $(M,\oplus)$ is a commutative hypergroup;
(2) $\circ : R \times M \rightarrow P^*(M)$ is such that for all $a,b \in M$ and $r,s \in R$ we have
(i) $r \circ (a \oplus b) = (r \circ a) \oplus (r \circ b)$;
(ii) $(r \uplus s) \circ a = (r \circ a) \oplus (s \circ a)$;
(iii) $(r \circ s) \circ a = r \circ (s \circ a)$.

For more details about hypermodules see [8], [9], [?] and [18]).
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Definition 2.3. ([7]) Let \( R \) and \( \Gamma \) be additive abelian groups. We say that \( R \) is a \( \Gamma \)-ring if there exists a mapping
\[
\cdot : R \times \Gamma \times R \longrightarrow R
\]
\[
(r, \gamma, r') \longmapsto r.\gamma.r' (= r\gamma r')
\]
such that for every \( a, b, c \in R \) and \( \alpha, \beta \in \Gamma \), the following conditions hold:

(i) \((a + b)\alpha c = a\alpha c + b\alpha c;\)
(ii) \(a(\alpha + \beta)c = a\alpha c + a\beta c;\)
(iii) \(a\alpha(b + c) = a\alpha b + a\alpha c;\)
(iv) \((a\alpha b)\beta c = a\alpha(b\beta c).\)

Definition 2.4. Let \( R \) be a \( \Gamma \)-ring. A (left) gamma module over \( R \) is an additive abelian group \( M \) together with a mapping
\[
\cdot : R \times \Gamma \times M \longrightarrow M
\]
\[
(\text{the image of } (r, \gamma, m) \text{ being denoted by } r\gamma m),
\]
such that for all \( m, m_1, m_2 \in M \) and \( \gamma, \gamma_1, \gamma_2 \in \Gamma \) and \( r, r_1, r_2 \in R \) the following conditions are satisfied:

\[
(GM_1) \quad r.\gamma.(m_1 + m_2) = r.\gamma.m_1 + r.\gamma.m_2;
\]
\[
(GM_2) \quad (r_1 + r_2).\gamma.m = r_1.\gamma.m + r_2.\gamma.m;
\]
\[
(GM_3) \quad r.(\gamma_1 + \gamma_2).m = r.\gamma_1.m + r.\gamma_2.m;
\]
\[
(GM_4) \quad r_1.\gamma_1.(r_2.\gamma_2.m) = (r_1.\gamma_1.r_2).\gamma_2.m.
\]

A right gamma module over \( R \) is defined in analogous manner. In this case we say that \( M \) is a left(or right) \( R\_\Gamma \)-module (for more details about gamma modules see [2]).

Let \((H, \circ)\) be a hypergroupoid. If \( \{A, B\} \subseteq P^*(H) \) and \( \rho \) is an equivalence relation on \( H \), then we denote \( A \bar{\rho} B \) if
\[
\forall a \in A, \exists b \in B : a\rho b, \text{ and, } \forall b \in B, \exists a \in A : a\rho b.
\]
We denote \( A \bar{\rho} B \) if \( \forall a \in A, \forall b \in B \) we have \( a\rho b \).

An equivalence relation \( \rho \) on \( H \) is called regular (strongly regular) if for all \( a, a', b, b' \) of \( H \). The following implication holds:
\[
a\rho b, a'\rho b' \implies (a \circ a')\bar{\rho}(b \circ b')
\]
\[
(a\rho b, a'\rho b' \implies (a \circ a')\bar{\rho}(b \circ b')).
\]

Theorem 2.1. ([17]) Let \((M, +, .)\) be a hypermodule over a hyperring \( R \), let \( \delta \) be an equivalence relation on \( M \) and let \( \rho \) be an strongly regular relation on \( R \). The following statements hold:

(1) if \( \delta \) is strongly regular on \( M \) and \( \forall x, y \in M \) and \( \forall r \in R \) the hyperoperations:
\[ \delta(x) \oplus \delta(y) = \{ \delta(z) \mid z \in x + y \} \quad \text{and} \quad \rho(r) \odot \delta(x) = \{ \delta(z) \mid z \in r.x \}, \]

is define a module structure on \( M/\delta \) over \( R/\rho \);

(2) if \((M, \delta, \oplus, \odot)\) is a module over \( R/\rho \), then \( \delta \) is strongly regular on \( M \).

The relation \( \delta^* \) is the smallest strongly regular relation on the hypermodule \((M, +, \cdot)\) such that \((M/\delta, \oplus, \odot)\) the quotient structure \((M/\delta, \oplus, \odot)\) is a module over the ring \( R/\rho \), and it is called the fundamental relation over hypermodule \( M \).

Hence, \( \delta^* \) is the smallest equivalence relation on \( M \), such that \( M/\delta^* \) is a module over the ring \( R/\rho^* \), where \( \rho^* \) is fundamental relation on \( R \).

If we denote by \( U \) the set of all expressions consisting of finite hyperoperations either on \( R \) and \( M \) or the external hyperoperation applied on finite sets of elements of \( R \) and \( M \), then we have

\[ x \delta y \iff \exists u \in U, \text{ such that } \{x, y\} \subseteq u. \]

\( \delta^* \) is the transitive closure of \( \delta \). In the fundamental module \((M/\delta^*, \oplus, \odot)\) over \( R/\rho^* \), the hyperoperations \( \oplus \) and \( \odot \) are defined as follows:

\( \forall x, y \in M \) and \( \forall z \in \delta^*(x) \oplus \delta^*(y) \), we have \( \delta^*(x) \odot \delta^*(y) = \delta^*(z) \); \( \forall r \in R \), \( \forall x \in M \) and \( \forall z \in \delta^*(r), \delta^*(x) \), we have \( \rho^*(r) \odot \delta^*(x) = \delta^*(z) \). (for more details about the fundamental relation on hyperstructures see [8] and [9]).

**Definition 2.5.** A multivalued system \((R, +, .)\) is a \( \Gamma \)-hyperring if the following hold:

(i) \((R, +)\) and \( \Gamma \) are canonical hypergroups;

(ii) \((R, \cdot)\) is semihypergroup.

(iii) \((\cdot)\) is distributive with respect to \((+)\), i.e., for all \(x, y, z\) in \( R \) we have \( x.(y + z) = (x.y) + (x.z) \) and \( (x + y).z = (x.z) + (y + z) \).

**Definition 2.6.** Let \((R, \oplus, \odot)\) be a \( \Gamma \)-hyperring and \((\Gamma, \ast)\) be a canonical hypergroup. We say that \((M, +, \cdot)\) is a left \( \Gamma \)-hypermodule over \( R \), if \((M, +)\) be a canonical hypergroup and there exists a mapping

\[ \cdot : R \times \Gamma \times M \longrightarrow P^*(M) \]

\[ (r, \gamma, m) \mapsto r \cdot \gamma \cdot m \]

such that for every \( r, s \in R \) and \( \alpha, \beta \in \Gamma \) and \( a, b \in M \), the following conditions are satisfied:

\((GHM_1)\)

(i) \((r \oplus s), \alpha.a = r.\alpha.a + s.\alpha.a; \)

(ii) \(r.(\alpha \ast \beta).a = r.\alpha.a + r.\beta.a; \)

(iii) \(r.\alpha.(a + b) = r.\alpha.a + r.\alpha.b; \)

\((GHM_2)\)

\(r \circ \alpha \circ s, \beta.a = r.\alpha.(s, \beta.a). \)
A right \( \Gamma \)-hypermodule of \( R \) is defined in a similar way. In this case we say that \( M \) is a \( R_{\Gamma} \)-hypermodule.

### 3 Fuzzy Gamma Subhypermodules

In the sequel \( R \) is a \( \Gamma \)-hyperring and all gamma hypermodules are considered over \( R \). In [16] M.K. Sen, R. Ameri, G. Chowdhury introduced the notion of fuzzy semi\( \Gamma \)-hypergroups, in [10] V. Leoreanu-Fotea, B. Davvaz study fuzzy hyperrings and V. Leoreanu-Fotea in [11] studied fuzzy hypermodules. Now in this section we follows these and introduce and studied fuzzy gamma hypermodules.

Let \( S \) and \( \Gamma \) be two nonempty sets. \( F^*(S) \) denotes the set \( H \) of all nonzero fuzzy subset of \( S \). A fuzzy \( \Gamma \)-hyperoperation on \( S \) is a map \( \circ : S \times \Gamma \times S \rightarrow F^*(S) \), which associates a nonzero subset \( a \circ \gamma \circ b \) for all \( a, b \in S \) and \( \gamma \in \Gamma \). \((S, \circ)\) is called a fuzzy \( \Gamma \)-hypergroupoid.

A fuzzy \( \Gamma \)-hypergroupoid \((S, \circ)\) is called a fuzzy \( \Gamma \)-hypersemigroup if for all \( a, b, c \in S \) and \( \alpha, \beta \in \Gamma \), we have \( a \circ \alpha \circ (b \circ \beta \circ c) = (a \circ \alpha \circ b) \circ \beta \circ c \), where for any \( \mu \in F^*(S) \), we have \((a \circ \gamma \circ b)(r) = \bigvee_{t \in S} ((a \circ \gamma \circ t)(r) \wedge \mu(t)) \) and \((\mu \circ \gamma \circ a)(r) = \bigvee_{t \in S} (\mu(t) \wedge (t \circ \gamma \circ a)(r)) \) for all \( r \in S, \gamma \in \Gamma \).

If \( A \) is a nonempty subset of \( S \) and \( x \in S \), then for all \( r \in S, \gamma \in \Gamma \) we have:

\[
(x \circ \gamma \circ A)(r) = \bigvee_{a \in A} (x \circ \gamma \circ a)(r),
\]

and

\[
(A \circ \gamma \circ x)(r) = \bigvee_{a \in A} (a \circ \gamma \circ x)(r).
\]

A fuzzy \( \Gamma \)-hypersemigroup \((S, \circ)\) is called a fuzzy \( \Gamma \)-hypergroup if for all \( a \in S \) and \( \gamma \in \Gamma \), we have \( a \circ \gamma \circ S = S \circ \gamma \circ a \). We say that an element \( e \) of \((S, \circ)\) is identity (resp. scalar identity) if for all \( s, r \in S, \gamma \in \Gamma \), we have

\[
(e \circ \gamma \circ r)(r) > 0, \quad \text{and} \quad (r \circ \gamma \circ e)(r) > 0,
\]

\[
((e \circ \gamma \circ r)(s) > 0, \quad \text{and} \quad (r \circ \gamma \circ e)(s) > 0 \quad \text{if} \quad f(s, r = s).
\]

Let \((S, \circ)\) be a fuzzy hypergroup, endowed with at least an identity. An element \( a' \in S \) is called an inverse of \( a \in S \) if there is an identity \( e \in S \), such that

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\[(a \circ a')(e) > 0, \text{ and } (a' \circ a)(e) > 0.\]

**Definition 3.1.** A fuzzy hypergroup \(S\) is regular if it has at least one identity and each element has at least one inverse. A regular fuzzy hypergroup \((S, \circ)\) is called reversible if for any \(x, y, a \in S\), it satisfies the following conditions:

1. if \((a \circ x)(y) > 0\), then there exists an inverse \(a_1\) of \(a\), such that \((a_1 \circ y)(x) > 0\);
2. if \((x \circ a)(y) > 0\), then there exists an inverse \(a_2\) of \(a\), such that \((y \circ a_2)(x) > 0\).

**Definition 3.2.** We say that a fuzzy hypergroup \(S\) is a fuzzy canonical if

1. it is commutative;
2. it has an scalar identity;
3. every element has a unique inverse;
4. it is reversible.

Let \(\mu\) and \(\nu\) be two nonzero fuzzy subsets of a fuzzy \(\Gamma\)-hypergroupoid \((S, \circ)\). We define

\[(\mu \circ \gamma \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \land (p \circ \gamma \circ q)(t) \land \nu(q)), \forall t \in S, \gamma \in \Gamma.\]

In the following we introduce and study fuzzy gamma hyperrings.

**Definition 3.3.** Let \(R, \Gamma\) be two nonempty sets and \(\boxplus, \boxdot\) be two fuzzy hyperoperations on \(R\) and \(\otimes\) be a fuzzy hyperoperation on \(\Gamma\). Let \((R, \boxplus)\) and \((\Gamma, \otimes)\) be two canonical fuzzy hypergroups. \(R\) is called a fuzzy \(\Gamma\)-hyperring if there exists the mapping:

\[
\square : R \times \Gamma \times R \longrightarrow F^*(R)
\]

\[
(r, \gamma, s) \longmapsto r \boxdot \gamma \boxdot s,
\]

such that for all \(r, s, t \in R, \alpha, \beta \in \Gamma\), the following conditions are satisfied:

1. \(r \boxdot \alpha \boxdot (s \boxplus t) = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \alpha \boxdot t)\);
2. \(r \boxdot (\alpha \otimes \beta) \boxdot s = (r \boxdot \alpha \boxdot s) \otimes (r \boxdot \beta \boxdot s)\);
3. \((r \boxplus s) \boxdot \alpha \boxdot t = (r \boxdot \alpha \boxdot t) \boxplus (s \boxdot \alpha \boxdot t)\);
4. \(r \boxdot \alpha \boxdot (s \boxdot \beta \boxdot t) = (r \boxdot \alpha \boxdot s) \boxdot \beta \boxdot t\).

**Definition 3.4.** Let \((\Gamma, \otimes)\) be a fuzzy canonical hypergroups. Let \((R, \boxplus, \boxdot)\) be a fuzzy \(\Gamma\)-hyperring. A nonempty set \(M\), endowed with two fuzzy \(\Gamma\)-hyperoperation \(\oplus, \odot\) is called a left fuzzy \(\Gamma\)-hypermodule over \((R, \boxplus, \boxdot)\) if the following conditions hold:
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(1) \((M, \oplus)\) is a canonical fuzzy \(\Gamma\)-hypergroup;
(2) \(\odot : R \times \Gamma \times M \rightarrow F^*(M)\) is such that for all \(a, b \in M, r, s \in R\) and \(\alpha, \beta \in \Gamma\) we have

\[
\begin{align*}
(i) & \quad r \odot \alpha \odot (a \oplus b) = (r \odot \alpha \odot a) \oplus (r \odot \alpha \odot b); \\
(ii) & \quad (r \boxplus s) \odot \alpha \odot a = (r \odot \alpha \odot a) \oplus (s \odot \alpha \odot a); \\
(iii) & \quad r \odot (\alpha \otimes \beta) \odot a = (r \odot \alpha \odot a) \oplus (r \odot \beta \odot a); \\
(iv) & \quad r \odot (s \odot \alpha \odot a) = (r \cdot s) \odot \alpha \odot a.
\end{align*}
\]

If both \((R, \boxplus), (\Gamma, \otimes)\) and \((M, \oplus)\) have scaler identities, denoted by \(0_R, 0_\Gamma\) and \(0_M\), then the fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) also satisfies the condition:

\[
\begin{align*}
0_R \odot \gamma \odot a &= \chi_{0_M}, \\
r \odot 0_\Gamma \odot a &= \chi_{0_R}, \\
r \odot \gamma \odot 0_M &= \chi_{0_M},
\end{align*}
\]

for all \(a \in M\). Moreover, if \((R, \boxtimes)\) has an identity, say \(1\), then the fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) is called unitary if it satisfies the condition:

\[1 \odot \gamma \odot a = \chi_a\]

Clearly, any fuzzy \(\gamma\)-hyperring is a fuzzy \(\Gamma\)-hypermodule over itself.

**Proposition 3.5.** Let \((M, +, \cdot)\) be a module over a ring \((R, \oplus, \circ)\) and \(\Gamma = R\). We define the following fuzzy \(\Gamma\)-hyperoperations:

for \(a, b \in M\), \(a \oplus b = \chi_{[a,b]}\),

for all \(a \in M\) and \(r \in R, \gamma \in \Gamma\), \(r \odot \gamma \odot a = \chi_{[r,\gamma,a]}\),

for all \(r, s \in R\), \(r \boxplus s = \chi_{[r,s]}\) and \(r \boxtimes \gamma \boxtimes s = \chi_{[r\gamma,\gamma s]}\).

Then \((M, \oplus, \odot)\) is a fuzzy \(\Gamma\)-hypermodule over the fuzzy \(\gamma\)-hyperring \((R, \boxplus, \boxtimes)\).

Note that the last theorem is satisfied, when \(M\) is a \(\Gamma\)-module over a \(\Gamma\)-ring \(R\), such that \(\Gamma \neq R\).

**Proposition 3.6.** Let \((R, \circ)\) and \((S, \bullet)\) be two fuzzy \(\Gamma\)-hyperrings. Let \((M, \oplus, \odot)\) be a left fuzzy \(\Gamma\)-hypermodule over \(R\) and a right fuzzy \(\Gamma\)-hypermodule over \(S\). Then

\[A = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M \right\}\]

is a fuzzy \(\Gamma\)-hyperring and fuzzy \(\Gamma\)-hypermodule over \(A\), under the mappings

\[
\star : A \times \Gamma \times A \rightarrow F^*(A)
\]

\[
\left( \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \gamma, \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \right) \mapsto
\begin{pmatrix} r \odot \gamma \odot r_1 & r \odot \gamma \odot m_1 \oplus m \odot \gamma \odot s_1 \\ 0 & s \bullet \gamma \bullet s_1 \end{pmatrix}.
\]
such that

\[
\begin{pmatrix}
  r \circ \gamma \circ r_1 & r \circ \gamma \circ m_1 \oplus m \circ \gamma \circ s_1 \\
  0 & s \cdot \gamma \cdot s_1
\end{pmatrix}
\begin{pmatrix}
  r_2 \\
  m_2
\end{pmatrix}
= \begin{pmatrix}
  (r \circ \gamma \circ r_1)(r_2) & (r \circ \gamma \circ m_1 \oplus m \circ \gamma \circ s_1)(m_2) \\
  0 & (s \cdot \gamma \cdot s_1)(m_2)
\end{pmatrix}
\begin{pmatrix}
  1, & r_2, m_2, s_2 \neq 0 \\
  0, & \text{otherwise}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1, & r_2, m_2, s_2 \neq 0 \\
  0, & \text{otherwise}
\end{pmatrix}
\]

Proof. Straightforward. □

Example 3.7. Let \( R \) be a \( \Gamma \)-ring and \((M, +, \cdot)\) a \( \Gamma \)-module. Consider the mapping \( \alpha : M \rightarrow R \). Then \( M \) is an fuzzy \( \Gamma \)-hypermodule over \( M \), under the following operations:

\[
m \oplus n = m + n \quad \text{and} \quad \circ : M \times \Gamma \times M \rightarrow F^*(M)(m, \gamma, n) \rightarrow m \circ \gamma \circ n = \chi_{\alpha(m), \gamma, n},
\]

for all \( m, n \in M, \gamma \in \Gamma \).

Proposition 3.8. Let \((M, +, \cdot)\) be a \( \Gamma \)-module over \( \Gamma \)-ring \( R \) and \( \nu \) be a nonzero fuzzy \( \Gamma \)-semigroup on \( M \). Let \( \mu \) and \( \rho \) be two nonzero fuzzy \( \Gamma \)-semigroups on \( R \). For \( r \in R, a, b \in M \) and \( \gamma \in \Gamma \), define a fuzzy \( \Gamma \)-hyperoperation \( \odot \) on \( M \) by

\[
(r \circ \gamma \circ a)(t) = \begin{cases}
  \mu(r) \wedge \rho(\gamma) \wedge \nu(a), & \text{if } t = r \cdot \gamma \cdot a \\
  0, & \text{otherwise.}
\end{cases}
\]

Also, \( a \oplus b = \chi_{(a+b)} \). It is easy to verify that \((M, \oplus, \odot)\) is a fuzzy \( \Gamma \)-hypermodule.

Let \( S, \Gamma \) be nonempty sets, and \( S \) endowed with a fuzzy \( \Gamma \)-hyperoperation \( \circ \).

For all \( a, b \in S, \gamma \in \Gamma \) and \( p \in [0,1] \) consider the \( p \)-cuts:

\[
(a \circ \gamma \circ b)_p = \{ t \in S : (a \circ \gamma \circ b)(t) \geq p \}
\]

of \( a \circ \gamma \circ b \), where \( p \in [0,1] \).

For all \( p \in [0,1] \), we define the following crisp \( \Gamma \)-hyperoperation on \( S \):

\[
a \circ_p \gamma \circ_p b = (a \circ \gamma \circ b)_p.
\]

Example 3.9. Let \( R = \Gamma = Z \) and \( M = Z_n \) for \( n \in N \). Define following fuzzy \( \Gamma \)-hyperoperations for all \( a, b \in M, \gamma \in \Gamma \):

\[
a \oplus b = \chi_{(a,b)}, \forall a \in M, \forall r \in R, \gamma \in \Gamma,
\]
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\[ r \odot \gamma \odot a = \chi_{\{r\gamma a\}}, \quad \forall r, s \in R, \forall \gamma \in \Gamma, \]

\[ r.\gamma.s = \chi_{\{r\gamma s\}} \quad \text{and} \quad r + s = \chi_{\{r,s\}}, \quad \text{for all} \ \alpha, \beta \in \Gamma, \]

and

\[ \alpha \boxplus \beta = \chi_{\{\alpha,\beta\}}, \]

such that \( \pi \) is denote a typical element in \( \mathbb{Z}_n \). Then it is easy to verify that \( (M, \oplus, \odot) \) is a fuzzy \( \Gamma \)-hypermodule over fuzzy \( \Gamma \)-hyperring \( R \) and canonical fuzzy hypergroup \( (\Gamma, \boxplus) \).

**Proposition 3.10.** Let \( (M, \circ) \) be a fuzzy \( \Gamma \)-hyperoperation. For all \( a, b, c, u \in M \) and \( \alpha, \beta \in \Gamma \) and for all \( p \in [0, 1] \) the following equivalence holds:

\[
(a \circ \alpha \circ (b \odot \beta \odot c)) \geq p \iff u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c).
\]

\[
((a \circ \alpha \circ b) \odot \beta \odot c) \geq p \iff u \in (a \circ_p \alpha \circ_p b) \circ_p \beta \circ_p c.
\]

**Proof.** Clearly,

\[
(a \circ \alpha \circ (b \odot \beta \odot c))(u) = \bigvee_{t \in M} (a \circ \alpha \circ t)(u) \land (b \odot \beta \odot c)(t) \geq p,
\]

if and only if there exists \( t_0 \in M \), such that \((a \circ \alpha \circ t_0)(u) \geq p\) and \((b \odot \beta \odot c)(t_0) \geq p\), which means that \( u \in a \circ_p \alpha \circ_p t_0 \odot_p b \odot_p \beta \odot_p c \). Therefore, \( u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c) \). \( \square \)

**Proposition 3.11.** Let \( (M, \oplus, \odot) \) be a fuzzy \( \Gamma \)-hypermodule over a fuzzy \( \Gamma \)-hyperring \( (R, \boxplus, \boxtimes) \). Then for all \( a \in M, r \in R, \gamma \in \Gamma \), conditions are equivalence:

(1) \( a \oplus M = \chi_M \iff \forall p \in [0, 1], \ a \oplus_p M = M \);

(2) \( r \odot \gamma \odot M = \chi_M \iff \forall p \in [0, 1], \ r \odot_p \gamma \odot_p M = M \).

**Proof.** We only proof (2). Let \( r \odot \gamma \odot M = \chi_M \). Then for all \( t \in M \) and \( p \in [0, 1] \), we have \( \bigvee_{u \in M} (r \odot \gamma \odot u)(t) = 1 \geq p \), whence there exists \( m \in M \), such that \((r \odot \gamma \odot m)(t) \geq p\), which means that \( t \in r \odot_p \gamma \odot_p m \). Hence, \( \forall p \in [0, 1], \ r \odot_p \gamma \odot_p M = M \). Conversely, for \( p = 1 \) we have \( r \odot_1 \gamma \odot_1 M = M \), whence for all \( t \in M \), there exists \( u \in M \), such that \( t \in r \odot_1 \gamma \odot_1 u \), which means that \((r \odot \gamma \odot u)(t) = 1 \). In other words, \( r \odot \gamma \odot M = \chi_M \). \( \square \)

**Proposition 3.12.** The structure \( (M, \oplus, \odot) \) is a fuzzy \( \Gamma \)-hypermodule over a fuzzy \( \Gamma \)-hyperring \( (R, \boxplus, \boxtimes) \) if and only if \( \forall p \in [0, 1], \ (M, \oplus_p, \odot_p) \) is a \( \Gamma \)-hypermodule over the hyperring \( (R, \boxplus_p, \boxtimes_p) \).

**Proof.** It is straightforward. \( \square \)
Consider \((M, \oplus, \ominus)\) as a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \boxminus)\) and canonical fuzzy hypergroup \((\Gamma, \odot)\). Now we follow [8], and define a new types of \(\Gamma\)-hyperoperations on \(M, R, \Gamma\), as follows:

\[
\forall a, b \in M, \quad a + b = \{x \in M | (a \oplus b)(x) > 0\}, \quad \forall r, s \in R,
\]

\[
r \uplus s = \{t \in R | (r \boxplus s)(t) > 0\}, \quad \text{forall} \alpha, \beta \in \Gamma,
\]

\[
\alpha * \beta = \{\gamma \in \Gamma | (\alpha * \beta)(\gamma) > 0\}, \quad \forall a \in M, \quad \forall r \in R, \forall \gamma \in \Gamma,
\]

\[
r.\gamma.a = \{b \in M | (r \odot (\gamma \circ a))(b) > 0\}, \quad \forall r, s \in R, \quad \forall \gamma \in \Gamma,
\]

\[
r \circ \gamma \circ s = \{t \in R | (r \circ \gamma \circ s)(t) > 0\}.
\]

**Proposition 3.13.** If \((M, \oplus, \ominus)\) is a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \boxminus)\) and canonical fuzzy hypergroup \((\Gamma, \odot)\), then \((M, +, .)\) is a \(\Gamma\)-hypermodule over the \(\Gamma\)-hyperring \((R, \uplus, \circ)\) and canonical hypergroup \((\Gamma, \ast)\).

**Proof.** By [10], it is obtained that \((R, \uplus), (\Gamma, \ast)\) and \((M, +)\) are canonical hypergroups. It is sufficient to verify \((M, .)\) is a \(\Gamma\)-hypermodule. We consider the following cases:

**Case:** \((i)\)

\[
(r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a), \quad \text{for all} \quad r, s \in R, \gamma \in \Gamma, a \in M.
\]

Suppose that \(x \in (r \uplus s).\gamma.a = \bigcup_{y \in r \uplus s} y \odot (\gamma \circ a)\). Then \((y \odot (\gamma \circ a))(x) > 0\) and \((r \uplus s)(y) > 0\), for some \(y \in r \uplus s\), and hence \(\forall p \in M \quad ((r \uplus s)(p) \wedge (p \odot (\gamma \circ a))(x) > 0\). Thus \((p \odot (\gamma \circ a))(x) > 0\), which implies that \(((r \odot (\gamma \circ a)) \uplus (s \odot (\gamma \circ a)))(x) > 0\). Thus there exist \(z, t \in M\), such that \((z \odot t)(x) > 0\), \((r \circ (\gamma \circ a))(z) > 0\) and \((s \circ (\gamma \circ a))(t) > 0\) i.e., \(x \in z + t\), \(z \in r.\gamma.a\) and \(t \in s.\gamma.a\) and hence \(x \in (r.\gamma.a) + (s.\gamma.a)\). Therefore, \((r \uplus s).\gamma.a \subseteq (r.\gamma.a) + (s.\gamma.a)\). Similarly, we can show that \((r.\gamma.a) + (s.\gamma.a) \subseteq (r \uplus s).\gamma.a\). Therefore, \((r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a)\). The other conditions are verified similarly and omitted. \(\Box\)
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On the other hands, if \((M, +, \cdot)\) is a \(\Gamma\)-hypermodule over a \(\Gamma\)-hyperring \((R, \uplus, \circ)\), then we define the following fuzzy \(\Gamma\)-hyperoperations:

\[
a \oplus b = \chi_{\{a+b\}}, \forall a, b \in M, r \uplus s \\
= \chi_{\{r \uplus s\}}, \forall r, s \in R, \gamma \in \Gamma, r \odot \gamma \odot a \\
= \chi_{\{r \odot \gamma \odot a\}}, \forall a \in M, r \in R, r \sqcap \gamma \sqcap s \\
= \chi_{\{r \sqcap \gamma \sqcap s\}}, \forall r, s \in R, \forall \gamma \in \Gamma, \beta \\
= \chi_{\{a \oplus b\}} \forall \alpha, \beta \in \Gamma, \alpha \odot \beta.
\]

The next result is immediately follows from above discussion and [14].

**Proposition 3.14.** For every hypergroup \((M, +)\), there is an associated fuzzy hypergroup.

**Proposition 3.15.** Let \((M, +, \cdot)\) be a \(\Gamma\)-hypermodule over a \(\Gamma\)-hyperring. Let \((R, \uplus, \circ)\) be a canonical hypergroup \((\Gamma, \star)\). Then \((M, \oplus, \odot)\) is a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \uplus, \odot)\) and canonical fuzzy hypergroup \((\Gamma, \odot)\), where the fuzzy hyperoperations \(\oplus, \odot, \odot, \sqcap, \sqcup\) and \(\star\) are defined above.

**Proof.** By Proposition 3.14, \((M, \oplus)\) is a commutative fuzzy \(\Gamma\)-hypergroup. We show that \((M, \oplus)\) is canonical. Since \((M, +)\) is canonical \(\Gamma\)-hypergroup, then there exists \(e \in M, \forall a \in M, a = e + a = a + e \implies (e \oplus a)(a) = \chi_{\{e+a\}}(a) > 0, (a \oplus e)(a) = \chi_{\{e+a\}}(a) > 0\) and because for all \(a \in M\) there exists \(b \in M\), such that \(e \in a + b \cap b + a, b)\) is the inverse of \(a\) with respect to \(+\). Then

\[
(a \oplus b)(e) = \chi_{\{a+b\}}(e) = \chi_{\{b+a\}}(e) = (b \oplus a)(e) > 0.
\]

Let \((a \oplus x)(y) = \chi_{\{a+x\}}(y) > 0 \implies y \in a + x \implies \exists b \ (the \ inverse \ of \ a \ such \ that \ x \in b + y \implies (b \oplus y)(x) = \chi_{\{b+y\}}(x) > 0.\) The other cases is can be proved in a similar way and omitted. Then \((M, \oplus)\) is a canonical fuzzy \(\Gamma\)-hypergroup. Now, we show that \((M, \oplus, \odot)\) is a fuzzy \(\Gamma\)-hypermodule. We investigate only the condition \((iv)\) of Definition 3.4.

First, we show that for all \(r, s \in R, \alpha, \beta \in \Gamma, a \in M\), we have

\[
(r \odot \alpha \odot (s \odot \beta \odot a)) = (r \sqcap \alpha \sqcap s) \odot \beta \odot a, \ \forall t \in M.
\]

Then

\[
(r \odot \alpha \odot (s \odot \beta \odot a))(t) = \bigvee_{p \in M} \chi_{r \odot \alpha \odot (s \odot \beta \odot a)\odot (p)}(t)
\]

\[
[(r \odot \alpha \odot p) \land (s \odot \beta \odot a)(p)] = \bigvee_{p \in M} [\chi_{r \odot \alpha \odot p}(t) \land \chi_{s \odot \beta \odot a}(t)] =
\]

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\[ \begin{cases} 1, & t \in r.\alpha.(s,\beta.a) \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & t \in (r.\alpha.s).\beta.a \\ 0, & \text{otherwise} \end{cases} \]
\[(r \Box \alpha \Box s) \odot \beta \odot a)(t), \text{ for all } t \in M.\]

It is easy to verify that the other conditions of Definition 3.4 can be obtained in a similar way. □

**Proposition 3.16.** Let \( M \) an \( R_\Gamma \)-module and \( \mu \) be a fuzzy \( \Gamma \)-module of \( M \).

Then the set \( M \) will be a fuzzy \( \Gamma \)-hypermodule.

**Proof.** Let \((\Gamma, \ast)\) be an abelian group and \((M, +, .)\) be a \( \Gamma \)-module over \( \Gamma \)-ring \((R, \oplus, \odot)\).

We define fuzzy \( \Gamma \)-hyperoperations on \( M \) as follows:

\[(a \oplus b)(t) = \chi_{[a+b]}(t), \quad (r \odot \gamma \odot a)(t) = \mu(r.\gamma.a - t), \]

\[(\alpha \odot \beta)(\gamma) = \chi_{[\alpha*\beta]}(\gamma) = \chi_{[\mu(\alpha)\mu(\beta)]}(\gamma),\]

\[\forall a, b, t \in M, r, s, z \in R, \alpha, \beta, \gamma \in \Gamma.\]

It is easy to verify that \((M, \oplus, \odot)\) is a canonical fuzzy hypergroup. Now, we show \((M, \oplus, \odot)\) is a fuzzy \( \Gamma \)-hypermodule with \( \mu(0) = 1. \)

\((i)\)

\[\begin{align*}
((r \Box s) \odot \gamma \odot a)(t) &= \bigvee_{p \in R}(r \Box s)(p) \land (p \odot \gamma \odot a)(t) \\
&= \bigvee_{p \in R} \chi_{[r \Box s]_\Gamma}(p) \land \mu(p.\gamma.a - t) \\
&= \mu((r \oplus s).\gamma.a - t) \quad \text{if } p = r \oplus s. \\
\end{align*}\]

Also, \((r \odot \gamma \odot a) \oplus (s \odot \gamma \odot a))(t) =

\[\begin{align*}
&= \bigvee_{p,q \in M}(r \odot \gamma \odot a)(p) \land (p \oplus q)(t) \land (s \odot \gamma \odot a)(q) \\
&= \bigvee_{p,q \in M} \mu(r.\gamma.a - p) \land \chi_{[p \oplus q]}(t) \land \mu(s.\gamma.a - q) \\
&= \bigvee_{p,q \in M,t = p+q} \mu(r.\gamma.a - p) \land \mu(s.\gamma.a - q) \\
&\leq \mu(r.\gamma.a - p + s.\gamma.a - q) \\
&= \mu((r \oplus s).\gamma.a - (p + q)). \\
\end{align*}\]

On the other hands, if \( q = s.\gamma.a, \ p = t - s.\gamma.a, \) then

\[\begin{align*}
\bigvee_{p,q \in M,t = p+q} \mu(r.\gamma.a - p) \land \mu(r.\gamma.a - q) &\geq \bigvee_{p \in M} \mu(r.\gamma.a - p) \\
&\geq \mu(r.\gamma.a - t + s.\gamma.a) \\
&= \mu((r \oplus s).\gamma.q - t). \\
\end{align*}\]

\((ii)\)

\[\begin{align*}
(r \odot (\alpha \odot \beta) \odot a)(t) &= \bigvee_{\gamma \in \Gamma} [(r \odot \gamma \odot a)(t) \land (\alpha \odot \beta)(\gamma)] \\
&= \bigvee \mu(r.\gamma.a - t) \land \chi_{[\alpha*\beta]}(\gamma) \\
&= \mu(r.(\alpha \ast \beta).a - t). \\
\end{align*}\]
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Also, \((r \circ \alpha \circ a) \oplus (r \circ \beta \circ a)) (t) = \)

\begin{align*}
&= \forall_{p,q \in M} [(r \circ \alpha \circ a)(p) \wedge (p \oplus q)(t) \wedge (r \circ \beta \circ a)(q)] \\
&= \forall_{p,q \in M} [\mu(r, \alpha.a - p) \wedge \chi_{(p+q)}(t) \wedge \mu(r, \beta.a - q)] \\
&= \forall_{t=p+q} \mu(r, \alpha.a - p) \wedge \mu(r, \beta.a - q) \\
&\leq \mu(r, \alpha.a - p + r, \beta.a - q) \\
&= \mu(r, (\alpha \ast \beta).a - (p + q)).
\end{align*}

On the other hands, suppose that \(q = r, \beta.a\), then for \(p = t - r, \beta.a\) we have

\begin{align*}
\forall_{t=p+q} \mu(r, \alpha.a - p) \wedge \mu(r, \beta.a - q) &= \forall_{p \in M} \mu(r, \alpha.a - p) \\
&\geq \mu(r, \alpha.a - (t - r, \beta.a)) \\
&= \mu(r, (\alpha \ast \beta).a - (p + q)),
\end{align*}

(iii)

\begin{align*}
r \circ \gamma \circ (a \oplus b) &= \forall_{p \in M} (r \circ \gamma \circ (a \oplus b))(t) \wedge (a \oplus b)(p) \\
&= \forall_{p, \gamma \in M} \mu(r, \gamma.p - t) \wedge \chi_{(a+b)}(p) \\
&= \mu(r, \gamma.(a + b) - t) \quad \text{and} \quad ((r \circ \gamma \circ a) \oplus (r \circ \gamma \circ b))(t) \\
&= \forall_{p, \gamma \in M} (r \circ \gamma \circ (a \oplus b))(p) \wedge (p \oplus q)(t) \wedge (r \circ \gamma \circ b)(q) \\
&= \forall_{p, \gamma \in M} \mu(r, \gamma.a - p) \wedge \chi_{(p+q)}(t) \wedge \mu(r, \gamma.b - q) \\
&= \forall_{p, \gamma \in M, t=p+q} \mu(r, \gamma.a - p) \wedge \mu(r, \gamma.b - q) \\
&\leq \mu(r, \gamma.a - p + r, \gamma.b - q) = \mu(r, \gamma.(a + b) - t).
\end{align*}

On the other hands, for \(q = r, \gamma.b, p = t - r, \gamma.b\) we have

\begin{align*}
\forall_{p, \gamma \in M, t=p+q} \mu(r, \gamma.a - p) \wedge \mu(r, \gamma.b - q) &\geq \forall_{p \in M} \mu(r, \gamma.a - p) \\
&\geq \mu(r, \gamma(a + b) - t).
\end{align*}

(iv)

\begin{align*}
(r \circ \alpha \circ (s \circ \beta \circ a))(t) &= \forall_{p \in M} (r \circ \alpha \circ (s \circ \beta \circ a))(p) \\
&= \forall_{p, \alpha, \beta \in M} \mu((r \circ \alpha.p - t) \wedge \mu((s \circ \beta.a - p) \\
&= \mu(r, \alpha.(s \circ \beta.a) - t) \quad \text{and} \quad (r \circ \alpha \circ s) \circ \beta \circ a)(t) \\
&= \forall_{p \in R} (r \circ \alpha \circ s)(p) \wedge (p \circ \beta \circ a)(t) \\
&= \forall_{p \in R} \chi_{(r \circ \alpha \circ s \circ \beta \circ a)}(p) \wedge \mu(p, \beta.a - t) \\
&= \mu(r \circ \alpha \circ s \cdot (\beta \cdot a) - t) \quad \text{if} \quad p = r \circ \alpha \circ s.
\end{align*}
Remark. Let $\mathbb{H} = \langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra. Denote by $F^*(H)$ the set of the nonzero fuzzy subsets of $H$. Then $\mathbb{H}$ can be organized as a universal algebra under the following operations:

$$\beta_i(\mu_1, \ldots, \mu_n)(t) = \bigvee_{(x_1, \ldots, x_n) \in H^n} [(\mu_1(x_1) \wedge \ldots \wedge \mu_n(x_n)) \wedge \beta_i(x_1, \ldots, x_n)(t)],$$

for every $i \in I, \mu_1, \ldots, \mu_n \in F^*(H)$ and $t \in H$. We denote this algebra by $F^*(\mathbb{H})$.

Proposition 3.17. If $(M, \oplus, \odot)$ is a fuzzy $\Gamma$-hypermodule, then $(F^*(M), *, \odot)$ is a $\Gamma$-module.

Proof. We define operations $\ast, \odot$ on $F^*(M)$ by $\mu \ast \nu = \mu \oplus \nu$, and $r \odot \gamma \odot \mu = r \odot \gamma \odot \mu$ for all $\mu, \nu \in F^*(M), r \in R, \gamma \in \Gamma$. It is easy to see that $(F^*(M), \ast)$ is a group. Clearly, $(F^*(M), \oplus)$ is a semigroup.

(i) Identity: we must prove that there exists a $\nu \in F^*(M)$ such that $\mu \ast \nu = \mu$. We have

$$\mu \ast \nu(t) = (\mu \oplus \nu)(t) = \bigvee_{p, q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge \nu(q) = \bigvee_{p \in M} \mu(p) \wedge (p \odot e)(t) = \mu(t) \oplus \text{ if } q = e; \nu(q) = 1, p = t.$$

Thus it is enough we choose $\nu = \chi_e$.

(ii) Inverse: it must prove that for $\mu \in F^*(M)$, there exists a $\nu \in F^*(M)$, such that $\mu \ast \nu = \chi_e$. It is sufficient to consider $\nu = -\mu$, then we have

$$\mu \ast \nu(t) = (\mu \oplus \nu)(t) = \bigvee_{p, q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge (-\mu)(q) = \bigvee_{p \in M} \mu(p) \wedge (p \odot q)(t) \wedge \mu(-q) \leq \mu(p - (-q)) \wedge (p \odot q)(t) \leq (p \odot q)(t) = \chi_e(t) \text{ where, } p \text{ is inverse of } q.$$

On the other hands, we have

$$\bigvee_{p, q \in M} \mu(p) \wedge (p \odot q)(t) \wedge \mu(-q) \geq \bigvee_{p \in M} \mu(p) \wedge (p \odot -p)(t) \geq (p \odot -p)(t) = \chi_e(t).$$
Other cases are easy and omitted. □

**Definition 3.18.** Let \((M, \oplus, \circ)\) be a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \lhd, \rhd)\). A nonempty subset \(N\) of \(M\) is called a subfuzzy \(\Gamma\)-hypermodule if for all \(x, y \in N, r \in R \) and \(\gamma \in \Gamma\), the following conditions hold:

1. \((x \oplus y)(t) > 0 \Rightarrow t \in N\);
2. \(x \oplus N = \chi_N\);
3. \((r \circ \gamma \circ x)(t) > 0 \Rightarrow t \in N\).

**Proposition 3.19.** (i) If \((N, \oplus, \circ)\) is a subfuzzy \(\Gamma\)-hypermodule of \((M, \oplus, \circ)\) over a fuzzy \(\Gamma\)-hyperring \((R, \lhd, \rhd)\), then the associated \(\Gamma\)-hypermodule \((N, +, \cdot)\) is a \(\Gamma\)-hypersubmodule of \((M, +, \cdot)\) over \((R, \lhd, \rhd)\);

(ii) \((N, +, \cdot)\) is a \(\Gamma\)-hypersubmodule of \((M, +, \cdot)\) over \((R, \lhd, \rhd)\) if and only if \((N, \oplus, \circ)\) is a subfuzzy \(\Gamma\)-hypermodule of \((M, \oplus, \circ)\) over \((R, \lhd, \rhd)\).

### 4 Fundamental Relation of Fuzzy \(\Gamma\)-hypermodule

In [14], fuzzy regular relations are introduced in the context of fuzzy hypersemigroups. In this section we extend this notion to fuzzy \(\Gamma\)-hypermodules.

Let \(\rho\) be an equivalence relation on a fuzzy \(\Gamma\)-hypersemigroup \((M, \circ)\) and \(\mu, \nu\) be two fuzzy subsets on \(M\). We say that \(\mu \rho \nu\) if the following conditions hold:

1. if \(\mu(a) > 0\), then there exists \(b \in M\), such that \(\nu(b) > 0\) and \(a \rho b\) and;
2. if \(\nu(x) > 0\), then there exists \(y \in M\), such that \(\mu(y) > 0\) and \(x \rho y\).

An equivalence relation \(\rho\) on a fuzzy \(\Gamma\)-hypersemigroup \((M, \circ)\) is called a fuzzy regular relation (or fuzzy hypercongruence) on \((M, \circ)\) if, for all \(a, b, c \in M, \gamma \in \Gamma\), the following implication holds:

\[ a \rho b \implies (a \circ \gamma \circ c) \rho (b \circ \gamma \circ c) \text{ and } (c \circ \gamma \circ a) \rho (c \circ \gamma \circ b). \]

This condition is equivalent to

\[ a \rho a', b \rho b' \implies (a \circ \gamma \circ b) \rho (a' \circ \gamma \circ b'), \forall a, b, a', b' \in M, \gamma \in \Gamma. \]

**Definition 4.1.** An equivalence relation \(\rho\) on a fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \circ)\) over a fuzzy \(\Gamma\)-hyperring \((R, \lhd, \rhd)\) and a canonical fuzzy hypergroup \((\Gamma, \otimes)\) is called a fuzzy regular relation on \((M, \oplus, \circ)\) if it is a fuzzy regular relation on \((M, \oplus)\) and for all \(x, y \in M, r \in R, \gamma \in \Gamma\), the following implication holds:

\[ x \rho y \implies (r \circ \gamma \circ x) \rho (r \circ \gamma \circ y). \]
Let \((M, \oplus, \odot)\) be a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \boxdot)\) and a canonical fuzzy hypergroup \((\Gamma, \otimes)\). Suppose \((M, +, \cdot)\) is the associated \(\Gamma\)-hypermodule over the \(\Gamma\)-hyperring \((R, \oplus, \odot)\) and the canonical hypergroup \((\Gamma, \ast)\). Then we have the next result.

**Theorem 4.2.** An equivalence relation \(\rho\) is a fuzzy regular relation on \((M, \oplus, \odot)\) over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \boxdot)\) and canonical fuzzy hypergroup \((\Gamma, \otimes)\) if and only if \(\rho\) is a regular relation on \((M, +, \cdot)\) over the \(\Gamma\)-hyperring \((R, \oplus, \odot)\) and canonical hypergroup \((\Gamma, \ast)\).

**Proof.** Letting \(x\rho y\) and \(x'\rho y'\), where \(x, x', y, y' \in M\). We have \((x \oplus x')(u) > 0\) if and only if the following conditions hold:

\[
(x \oplus x')(u) > 0, \Rightarrow \exists v \in M : (y \oplus y')(v) > 0 \text{ and } uvv, \tag{1}
\]

and

\[
(y \oplus y')(t) > 0 \Rightarrow \exists w \in M : (x \oplus x')(w) > 0 \text{ and } atw. \tag{2}
\]

These are equivalent to:

- if \(u \in x + x'\), then there exists \(v \in y + y'\), such that \(uvv\);
- if \(t \in y + y'\), then there exists \(w \in x + x'\), such that \(twv\);

which mean that \((x + x')\rho(y + y')\). Hence \(\rho\) is fuzzy regular on \((M, \oplus)\) if and only if \(\rho\) is regular on \((M, +)\).

On the other hands, if \(xpy\) and \(x'py'\), where \(x, x', y, y' \in M\). We have \((x \oplus x')\rho(y \oplus y')\) if and only if the following conditions hold:

- if \((r \circ \gamma \circ x)(u) > 0\), then there exists \(v \in M\), such that \((r \circ \gamma \circ y)(v) > 0\) and \(u\rho v\);
- if \((r \circ \gamma \circ y)(t) > 0\), then there exists \(w \in M\), such that \((r \circ \gamma \circ x)(w) > 0\) and \(t\rho w\).

These are equivalent to:

- if \(u \in r.\gamma.x\), then there exists \(v \in r.\gamma.y\), such that \(uvv\);
- if \(t \in r.\gamma.y\), then there exists \(w \in r.\gamma.x\), such that \(twv\);

which means that \((r.\gamma.x)\rho(r.\gamma.y)\).

**Definition 4.3.** An equivalence relation \(\rho\) on a fuzzy \(\Gamma\)-hypersemigroup \((M, \odot)\) is called a fuzzy strongly regular relation on \((M, \odot)\) if, for all \(a, a', b, b'\) of \(M\) and for all \(\gamma \in \Gamma\), such that \(apb\) and \(a'pb'\), the following condition holds:

\[
(a \circ \gamma \circ c)(x) > 0, (b \circ \gamma \circ d)(y) > 0 \Rightarrow xpy,
\]

for all \(x, y \in M\). Note that if \(\rho\) is a fuzzy strongly relation on a fuzzy \(\Gamma\)-hypersemigroup \((M, \odot)\), then it is a fuzzy regular on \((M, \odot)\). An equivalence relation \(\rho\) on a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \boxdot)\) is called a fuzzy strongly regular
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relation on \((R, \boxplus, \ominus)\) if it is a fuzzy strongly regular relation both on \((R, \boxplus)\) and on \((R, \ominus)\).

**Definition 4.4.** Let \(\rho\) be a fuzzy strongly regular relation on a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \ominus)\) and \(\theta\) be a fuzzy strongly regular relation on a canonical fuzzy \(\Gamma\)-hypergroup \((\Gamma, \ast)\). An equivalence relation \(\delta\) on a fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) over a fuzzy \(\Gamma\)-hyperring \((\delta, \ast)\) and canonical fuzzy \(\Gamma\)-hypergroup \((\Gamma, \otimes)\) is called a fuzzy strongly regular relation on \((M, \oplus, \odot)\) if it is a fuzzy strongly regular relation on \((M, \oplus)\) and if \(x \delta y\), \(r \rho s\) and \(\alpha \theta \beta\), then the next condition holds:

\[\forall u \in M, \text{ such that } (r \circ \alpha \circ x)(u) > 0 \text{ and for all } v \in M, \text{ such that } (s \circ \beta \circ y)(v) > 0, \text{ we have } u \delta v.\]

**Theorem 4.5.** An equivalence relation \(\delta\) is a fuzzy strongly regular relation on \((M, \oplus, \odot)\) if and only if \(\delta\) is a strongly regular relation on \((M, +, .)\).

**Proof.** Set \(x \delta y\) and \(x' \delta y'\), where \(x, x', y, y' \in M\) and set \(r \rho s\), where \(r, s \in R\) and \(\alpha \theta \beta\), where \(\alpha, \beta \in \Gamma\). The relation \(\delta\) is strongly regular on \((M, \oplus, \odot)\) if and only if the following conditions are satisfied:

\[\forall u \in M, \text{ such that } (x \oplus x')(u) > 0 \text{ and } \forall v \in M, \text{ such that } (y \oplus y')(v) > 0, \text{ we have } u \delta v;\]

\[\forall t \in M, \text{ such that } (r \circ \alpha \circ x)(t) > 0 \text{ and } \forall w \in M, \text{ such that } (s \circ \beta \circ y)(w) > 0, \text{ we have } t \delta w.\]

These conditions are equivalent to the following ones:

\[\forall u \in M, \text{ such that } u \in x + x'\text{ and } \forall v \in M, \text{ such that } v \in y + y', \text{ we have } u \delta v;\]

\[\forall t \in M, \text{ such that } t \in r.\alpha.x \text{ and } \forall w \in M, \text{ such that } w \in s.\beta.y, \text{ we have } t \delta w,\]

which mean that \((x + x')\tilde{\delta}(y + y')\) and \((r.\alpha.x)\tilde{\delta}(s.\beta.y)\). Hence \(\delta\) is strongly regular on \((M, \oplus, \odot)\) if and only if \(\delta\) is strongly regular on \((M, +, .)\).

Now, Let \(\delta\) be a fuzzy regular relation on a fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \ominus)\) and canonical fuzzy \(\Gamma\)-hypergroup \((\Gamma, \otimes)\) and \(\rho, \theta\) be fuzzy strongly regular relations on the \(\Gamma\)-hyperring \((R, \boxplus, \ominus)\) and canonical fuzzy \(\Gamma\)-hypergroup \((\Gamma, \otimes)\).

We consider the following \(\Gamma\)-hyperoperations on the quotient set \(M/\tilde{\delta}\):

\[\bar{x} \ast \bar{y} = \{\bar{z} | z \in x + y\} = \{\bar{z} | (x \oplus y)(z) > 0\},\]

\[\bar{r} \circ \bar{a} \odot \bar{t} = \{\bar{z} | z \in r.\alpha.x\} = \{\bar{z} | (r \circ \alpha \circ x)(z) > 0\}.\]

**Theorem 4.6.** Let \((M, \oplus, \odot)\) be a fuzzy \(\Gamma\)-hypermodule over a fuzzy \(\Gamma\)-hyperring \((R, \boxplus, \ominus)\) and canonical fuzzy hypergroup \((\Gamma, \ast)\). Let \((M, +, .)\) be the associated \(\Gamma\)-hypermodule over the corresponding \(\Gamma\)-hypergroup \((R, \boxplus, \ominus)\) and canonical hypergroup \((\Gamma, \ast)\). Then we have:

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(i) The relation \( \delta \) is a fuzzy regular relation on \((M, \oplus, \odot)\) if and only if \((M/\delta, \star, \odot)\) is a \(\Gamma\)-hypermodule over \((R, \oplus, \odot)\) and \((\Gamma, \star)\).

(ii) The relation \( \delta \) is a fuzzy strongly regular relation on \((M, \oplus, \odot)\) over \((R, \oplus, \odot)\) and \((\Gamma, \star)\) if and only if \((M/\delta, \star, \odot)\) is a \(\Gamma\)-module over \(R/\rho\) and \(\Gamma/\theta\).

If we denote by \(\Upsilon\) the set of all expressions consisting of finite fuzzy \(\Gamma\)-hyperoperations either on \(R, \Gamma\) and \(M\) or the external fuzzy \(\Gamma\)-hyperoperations applied on finite sets of elements of \(R, \Gamma\) and \(M\), then we have

\[ x \epsilon y \iff \exists u \in \Upsilon : \{x, y\} \subseteq u. \]

Now, we introduced \textit{fundamental relation} on fuzzy \(\Gamma\)-hypemodules.

\textbf{Definition 4.7.} An equivalence relation \(\epsilon^*\) is called \textit{fundamental relation} on a fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) if \(\epsilon^*\) is fundamental relation on the associated \(\Gamma\)-hypermodule \((M, +, .)\).

Hence, \(\epsilon^*\) is fundamental relation on a fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\) if and only if \(\epsilon^*\) is the smallest fuzzy strongly equivalence relation on \((M, \oplus, \odot)\).

Denote by \(\text{UFS}\) the set of all expressions consisting of finite fuzzy \(\Gamma\)-hyperoperations either on \(R, \Gamma\) and \(M\) or the external fuzzy \(\Gamma\)-hyperoperation applied on finite sets of elements of \(R, \Gamma\) and \(M\). We obtain

\[ x \epsilon y \iff \exists \mu_f \in \text{UFS} : \{x, y\} \subseteq \mu_f, \iff \mu_{f,1}(x) > 0 \text{ and } \mu_{f,2}(y) > 0. \]

The relation \(\epsilon^*\) is the transitive closure of \(\epsilon\).

Denote by \(\sum_{\oplus}^*\) any finite fuzzy hypersum and by \(\prod_{\odot}^*\) any finite fuzzy \(\Gamma\)-hyperproduct of the fuzzy \(\Gamma\)-hypemodule \((M, \oplus, \odot)\). As above, we obtain that

\[ (\sum_{i \in I}^* \prod_{j \in J}^* a_{ji})(p) > 0 \text{ if and only if } p \in \sum_{i \in I}^* \prod_{j \in J}^* a_{ji}. \]

Hence, \(\{x, y\} \subseteq \sum_{i \in I}^* \prod_{j \in J}^* a_{ji}\) if and only if \((\sum_{i \in I}^* \prod_{j \in J}^* a_{ji})(x) > 0 \) and \((\sum_{i \in I}^* \prod_{j \in J}^* a_{ji})(y) > 0\). Therefore, we obtain \(x \epsilon y \iff \exists \mu_f \in \text{UFS}\) such that \(\mu_{f,1}(x) > 0 \) and \(\mu_{f,2}(y) > 0\).

So, in order to obtain a fuzzy \(\Gamma\)-module starting from a fuzzy \(\Gamma\)-hypermodule, we consider first the relation \(\epsilon\), then the transitive closure \(\epsilon^*\) of \(\epsilon\) and finally the quotient structure \((M/\epsilon^*, \star, \odot)\) of the fuzzy \(\Gamma\)-hypermodule \((M, \oplus, \odot)\).

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