Rough Set Theory Applied To Hyper
$BCK$-Algebra

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Abstract

The aim of this paper is to introduce the notions of lower and upper approximation of a subset of a hyper $BCK$-algebra with respect to a hyper $BCK$-ideal. We give the notion of rough hyper subalgebra and rough hyper $BCK$-ideal, too, and we investigate their properties.

Key words: rough set, rough (weak, strong) hyper $BCK$-ideal, rough hyper subalgebra, regular congruence relation

MSC2010: , .

1 Introduction

In 1966, Y. Imai and K. Iseki [2] introduced a new notion, called a $BCK$-algebra. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty [6] at the 8th Congress of Scandinavian Mathematicians. In [3], Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei applied the hyper structures to $BCK$-algebras and they introduced the notion of hyper $BCK$-algebra (resp. hyper $K$-algebra) which is a generalization of $BCK$-algebra (resp. hyper $BCK$-algebra). They also introduced the notion of hyper $BCK$-ideal, weak hyper $BCK$-ideal, hyper $K$-ideal and weak
hyper K-ideal and gave relations among them. In 1982, Pawlak introduced
the concept of rough set (see [7]). Recently Jun [5] applied rough set theory
to \textit{BCK}-algebras. In this paper, we apply the rough set theory to hyper
\textit{BCK}-algebras.

\section{Preliminaries}

Let \( U \) be a universal set. For an equivalence relation \( \Theta \) on \( U \), the set
of elements of \( U \) that are related to \( x \in U \), is called the \textit{equivalence class}
of \( x \) and is denoted by \([x]_\Theta\). Moreover, let \( U/\Theta \) denote the family of all
equivalence classes induced on \( U \) by \( \Theta \). For any \( X \subseteq U \), we write \( X^c \) to
denote the complement of \( X \) in \( U \), that is the set \( U \setminus X \). A pair \((U, \Theta)\) where
\( U \neq \emptyset \) and \( \Theta \) is an equivalence relation on \( U \) is called an \textit{approximation
space}.

The interpretation in rough set theory is that our knowledge of the objects
in \( U \) extends only up to membership in the class of \( \Theta \) and our knowledge
about a subset \( X \) of \( U \) is limited to the class of \( \Theta \) and their unions. This
leads to the following definition.

\textbf{Definition 2.1.} [7] For an approximation space \((U, \Theta)\), by a rough approxi-
mation in \((U, \Theta)\) we mean a mapping \( \text{Apr} : P(U) \longrightarrow P(U) \times P(U) \) defined
for every \( X \in P(U) \) by \( \text{Apr}(X) = (\text{Apr}^L(X), \text{Apr}^U(X)) \), where

\[
\text{Apr}^L(X) = \{ x \in U | [x]_\Theta \subseteq X \}, \\
\text{Apr}^U(X) = \{ x \in U | [x]_\Theta \cap X \neq \emptyset \}.
\]

\( \text{Apr}^L(X) \) is called a \textit{lower rough approximation} of \( X \) in \((U, \Theta)\), whereas \( \text{Apr}^U(X) \)
is called an \textit{upper rough approximation} of \( X \) in \((U, \Theta)\).

\textbf{Definition 2.2.} [7] Given an approximation space \((U, \Theta)\), a pair \((A, B) \in
P(U) \times P(U) \) is called a \textit{rough set} in \((U, \Theta)\) if and only if \((A, B) = \text{Apr}(X) \)
for some \( X \in P(U) \).

\textbf{Definition 2.3.} ([7]) Let \((U, \Theta)\) be an approximation space and \( X \) be a
non-empty subset of \( U \).

(i) If \( \text{Apr}(X) = \text{Apr}^U(X) \), then \( X \) is called \textit{definable}.

(ii) If \( \text{Apr}(X) = \emptyset \), then \( X \) is called \textit{empty interior}.
(iii) If $\overline{\text{Apr}}(X) = U$, then $X$ is called empty exterior.

Let $H$ be a non-empty set endowed with a hyper operation “$\circ$”, that is $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. For two subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 2.4. ([3]) By a hyper $BCK$-algebra we mean a non-empty set $H$ endowed with a hyper operation “$\circ$” and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call “$\ll$” the hyper order in $H$.

Theorem 2.5. ([3]) In any hyper $BCK$-algebra $H$, the following hold:

(a1) $0 \circ 0 = \{0\}$,

(a2) $0 \ll x$,

(a3) $x \ll x$,

(a4) $A \ll A$,

(a5) $A \ll 0$ implies $A = \{0\}$,

(a6) $A \subseteq B$ implies $A \ll B$,

(a7) $0 \circ x = \{0\}$,

(a8) $x \circ y \ll x$,

(a9) $x \circ 0 = \{x\}$,

(a10) $y \ll z$ implies $x \circ z \ll x \circ y$,

(a11) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,

(a12) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsets $A$ and $B$ of $H$. 

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Definition 2.6. ([3]) Let $H$ be a hyper $BCK$-algebra and let $S$ be a subset of $H$ containing 0. If $S$ be a hyper $BCK$-algebra with respect to the hyper operation “◦” on $H$, we say that $S$ is a hyper subalgebra of $H$.

Theorem 2.7. ([3]) Let $S$ be a non-empty subset of hyper $BCK$-algebra $H$. Then $S$ is a hyper subalgebra of $H$ if and only if $x ◦ y ⊆ S$, for all $x, y ∈ S$.

Definition 2.8. ([3]) Let $I$ be a non-empty subset of hyper $BCK$-algebra $H$ and $0 ∈ I$.

(i) $I$ is said to be a hyper $BCK$-ideal of $H$ if $x ◦ y ≪ I$ and $y ∈ I$ imply $x ∈ I$ for all $x, y ∈ H$.

(ii) $I$ is said to be a weak hyper $BCK$-ideal of $H$ if $x ◦ y ⊆ I$ and $y ∈ I$ imply $x ∈ I$ for all $x, y ∈ H$.

(iii) $I$ is called a strong hyper $BCK$-ideal of $H$ if $(x ◦ y) ∩ I ≠ φ$ and $y ∈ I$ imply $x ∈ I$ for all $x, y ∈ H$.

Theorem 2.9. ([3]) If $H$ be a hyper $BCK$-algebra, then

(i) every hyper $BCK$-ideal of $H$ is a weak hyper $BCK$-ideal of $H$.

(ii) every strong hyper $BCK$-ideal of $H$ is a hyper $BCK$-ideal of $H$.

Definition 2.10. ([4]) Let $H$ be a hyper $BCK$-algebra. A hyper $BCK$-ideal $I$ of $H$ is called reflexive if $x ◦ x ⊆ I$ for all $x ∈ H$.

Definition 2.11. ([1]) Let $Θ$ be an equivalence relation on hyper $BCK$-algebra $H$ and $A, B ⊆ H$. Then,

(i) $AΘB$ means that, there exist $a ∈ A$ and $b ∈ B$ such that $aΘb$,

(ii) $AΘB$ means that, for all $a ∈ A$ there exists $b ∈ B$ such that $aΘb$ and for all $b ∈ B$ there exists $a ∈ A$ such that $aΘb$,

(iii) $Θ$ is called a congruence relation on $H$, if $xΘy$ and $x′Θy′$ imply $x ◦ x′Θy ◦ y′$ for all $x, y, x′, y′ ∈ H$.

(iv) $Θ$ is called a regular relation on $H$, if $x ◦ yΘ{0}$ and $y ◦ xΘ{0}$ imply $xΘy$ for all $x, y ∈ H$. 
Example 2.12. Let $H_1 = \{0, 1, 2\}$, $H_2 = \{0, a, b\}$ and hyper operations “$\circ_1$” and “$\circ_2$” on $H_1$ and $H_2$ are defined respectively, as follow:

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<tr>
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<tr>
<td>a</td>
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<td>{0, a}</td>
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<tr>
<td>b</td>
<td>{b}</td>
<td>{a, b}</td>
<td>{0, b}</td>
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</table>

Then $(H_1, \circ_1)$ and $(H_2, \circ_2)$ are hyper $BCK$-algebras. Define the equivalence relation $\Theta_1$ and $\Theta_2$ on $H_1$ and $H_2$, respectively, as

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 2), (2, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (0, a), (a, 0)\}.$$  

It is easily checked that $\Theta_1$ is a congruence relation on $H_1$. But $\Theta_2$ is not a congruence relation on $H_2$, since $b \Theta_2 b$ and $0 \Theta_2 a$ but $b \circ (b \Theta_2 b \circ a)$ is not true.

Example 2.13. Let $(H_1, \circ_1)$ be a hyper $BCK$-algebra as Example 2.12. Let $H_2 = \{0, a, b, c\}$ and define the hyper operation “$\circ_2$” on $H_2$ as follow:

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<th>$\circ_2$</th>
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Then $(H_2, \circ_2)$ is a hyper $BCK$-algebra. Define the congruence relation $\Theta_1$ and $\Theta_2$ on $H_1$ and $H_2$, respectively, by

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (c, c), (0, b), (b, 0)\}.$$  

It is easily checked that $\Theta_1$ is a regular congruence relation on $H_1$, but $\Theta_2$ is not a regular relation on $H_2$, since $a \circ b \Theta_2 \{0\}$ and $b \circ a \Theta_2 \{0\}$ but $(a, b) \notin \Theta_2$.

Theorem 2.14. ([1]) Let $\Theta$ be a regular congruence relation on hyper $BCK$-algebra $H$. Then $[0]_{\Theta}$ is a hyper $BCK$-ideal of $H$. 

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Theorem 2.15. ([1]) Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_{\Theta}$ and 
$I_x = \{ x : x \in H \}$, where $I_x = [x]_{\Theta}$ for all $x \in H$. Then $I_x$ with hyper 
operation “$\circ$” and hyper order “$<$” which is defined as follow, is a hyper $BCK$-
algebra which is called *quotient hyper $BCK$-algebra*,
\[I_x \circ I_y = \{ z : z \in x \circ y \},\]
and
\[I_x < I_y \iff I \in I_x \circ I_y.\]

Theorem 2.16. ([1]) Let $I$ be a reflexive hyper $BCK$-ideal of $H$ and relation $\Theta$ on $H$ be defined as follow:
\[x \Theta y \iff x \circ y \subseteq I \text{ and } y \circ x \subseteq I\]
for all $x, y \in H$. Then $\Theta$ is a regular congruence relation on $H$ and $I = [0]_{\Theta}$.

3 Rough hyper $BCK$-ideals

Throughout this section $H$ is a hyper $BCK$-algebra. In this section first 
we define lower and upper approximation of the subset $A$ of $H$ with respect 
to hyper $BCK$-ideal of $H$ and prove some properties. Then we give the 
definition of (weak, strong) rough hyper $BCK$-ideals and investigate the 
relation between them and (weak, strong) hyper $BCK$-ideals of $H$.

Definition 3.1. Let $\Theta$ be a regular congruence relation on hyper $BCK$-
algebra $H$, $I = [0]_{\Theta}$, $I_x = [x]_{\Theta}$ and $A$ be a non-empty subset of $H$. Then the sets
\[\text{Apr}_I(A) = \{ x \in H | I_x \subseteq A \},\]
\[\overline{\text{Apr}}_I(A) = \{ x \in H | I_x \cap A \neq \emptyset \}.\]
are called *lower and upper approximation* of the set $A$ with respect to the 
hyper $BCK$-ideal $I$, respectively.

Proposition 3.2. For every approximation space $(H, \Theta)$ and every subsets 
$A, B \subseteq H$, we have:
\[1. \ \text{Apr}_I(A) \subseteq A \subseteq \overline{\text{Apr}}_I(A),\]
\[2. \ \text{Apr}_I(\emptyset) = \emptyset = \overline{\text{Apr}}_I(\emptyset),\]
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(3) $\text{Apr}_I(H) = H = \overline{\text{Apr}_I}(H)$,

(4) if $A \subseteq B$, then $\text{Apr}_I(A) \subseteq \text{Apr}_I(B)$ and $\overline{\text{Apr}_I}(A) \subseteq \overline{\text{Apr}_I}(B)$,

(5) $\text{Apr}_I(\text{Apr}_I(A)) = \text{Apr}_I(A)$,

(6) $\overline{\text{Apr}_I}(\text{Apr}_I(A)) = \overline{\text{Apr}_I}(A)$,

(7) $\overline{\text{Apr}_I}(\text{Apr}_I(A)) = \overline{\text{Apr}_I}(A)$,

(8) $\text{Apr}_I(\overline{\text{Apr}_I}(A)) = \text{Apr}_I(A)$,

(9) $\text{Apr}_I(A) = (\overline{\text{Apr}_I}(A^c))^c$,

(10) $\overline{\text{Apr}_I}(A) = (\overline{\text{Apr}_I}(A^c))^c$,

(11) $\text{Apr}_I(A \cap B) \subseteq \text{Apr}_I(A) \cap \text{Apr}_I(B)$,

(12) $\text{Apr}_I(A \cap B) = \text{Apr}_I(A) \cap \text{Apr}_I(B)$,

(13) $\overline{\text{Apr}_I}(A \cup B) = \overline{\text{Apr}_I}(A) \cup \overline{\text{Apr}_I}(B)$,

(14) $\text{Apr}_I(A \cup B) \supseteq \text{Apr}_I(A) \cup \text{Apr}_I(B)$,

(15) $\text{Apr}_I(I_x) = H = \overline{\text{Apr}_I}(I_x)$ for all $x \in H$.

Proof. (1), (2) and (3) are straightforward.

(4) For any $x \in \text{Apr}_I(A)$ we have $I_x \subseteq A \subseteq B$ and so $x \in \overline{\text{Apr}_I}(B)$. Now, suppose that $x \in \overline{\text{Apr}_I}(A)$. Then $I_x \cap A \neq \phi$ and so $I_x \cap B \neq \phi$. Hence $x \in \overline{\text{Apr}_I}(B)$.

(5) Since $\overline{\text{Apr}_I}(A) \subseteq A$, by (4) we have $\overline{\text{Apr}_I}(\text{Apr}_I(A)) \subseteq \text{Apr}_I(A)$. Now, let $x \in \overline{\text{Apr}_I}(A)$. Then $I_x \subseteq A$. Since for any $y \in I_x$, we have $I_x = I_y$, then $I_y \subseteq A$ and so $y \in \text{Apr}_I(A)$. Therefore, $I_x \subseteq \text{Apr}_I(A)$ and then we obtain $x \in \overline{\text{Apr}_I}(\text{Apr}_I(A))$.

(6) By (1) and (4), $\overline{\text{Apr}_I}(A) \subseteq \overline{\text{Apr}_I}(\overline{\text{Apr}_I}(A))$. On the other hand, we assume that $x \in \overline{\text{Apr}_I}(\overline{\text{Apr}_I}(A))$. Then we have $I_x \cap \overline{\text{Apr}_I}(A) \neq \phi$ and so there exist $a \in I_x$ and $a \in \text{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \cap A \neq \phi$ which imply $I_x \cap A \neq \phi$. Therefore, $x \in \overline{\text{Apr}_I}(A)$.
(7) By (1), we have $\text{Apr}_I(A) \subseteq \overline{\text{Apr}}_I(A)$. Now, let $x \in \overline{\text{Apr}}_I(A)$. Then $I_x \cap \text{Apr}_I(A) \neq \emptyset$ and so there exist $a \in I_x$ and $a \in \text{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \subseteq A$ which imply $I_x \subseteq A$. Therefore, $x \in \overline{\text{Apr}}_I(A)$.

(8) By (1), we have $\overline{\text{Apr}}_I(\overline{\text{Apr}}_I(A)) \subseteq \overline{\text{Apr}}_I(A)$. Now, we assume that $x \in \overline{\text{Apr}}_I(A)$. Then $I_x \cap A \neq \emptyset$. For every $y \in I_x$, we have $I_y = I_x$ and so $I_y \cap A \neq \emptyset$. Hence $y \in \overline{\text{Apr}}_I(A)$ which implies $I_x \subseteq \overline{\text{Apr}}_I(A)$. Therefore, $x \in \overline{\text{Apr}}_I(\overline{\text{Apr}}_I(A))$.

(9) For any subset $A$ of $H$ we have:

$$(\overline{\text{Apr}}_I(A^c))^c = \{x \in H : x \notin \overline{\text{Apr}}_I(A^c)\}$$

$$= \{x \in H : I_x \cap A^c = \emptyset\}$$

$$= \{x \in H : I_x \subseteq A\}$$

$$= \{x \in H : x \in \text{Apr}_I(A)\}$$

$$= \text{Apr}_I(A).$$

(10) For any subset $A$ of $H$ we have:

$$(\overline{\text{Apr}}_I(A^c))^c = \{x \in H : x \notin \overline{\text{Apr}}_I(A^c)\}$$

$$= \{x \in H : I_x \nsubseteq A^c\}$$

$$= \{x \in H : I_x \cap A \neq \emptyset\}$$

$$= \{x \in H : x \in \overline{\text{Apr}}_I(A)\}$$

$$= \overline{\text{Apr}}_I(A).$$

(11) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by (4), $\overline{\text{Apr}}_I(A \cap B) \subseteq \overline{\text{Apr}}_I(A)$ and $\overline{\text{Apr}}_I(A \cap B) \subseteq \overline{\text{Apr}}_I(B)$. Hence $\overline{\text{Apr}}_I(A \cap B) \subseteq \overline{\text{Apr}}_I(A) \cap \overline{\text{Apr}}_I(B)$. 

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(12) For any subset $A$ and $B$ of $H$ we have:

$$x \in \text{Apr}_I(A \cap B) \iff I_x \subseteq A \cap B$$

$$\iff I_x \subseteq A \text{ and } I_x \subseteq B$$

$$\iff x \in \text{Apr}_I(A) \text{ and } x \in \text{Apr}_I(B)$$

$$\iff x \in \text{Apr}_I(A) \cap \text{Apr}_I(B).$$

(13) For any subset $A$ and $B$ of $H$ we have

$$x \in \text{Apr}_I(A \cup B) \iff I_x \cap (A \cup B) \neq \varphi$$

$$\iff (I_x \cap A) \cup (I_x \cap B) \neq \varphi$$

$$\iff I_x \cap A \neq \varphi \text{ or } I_x \cap B \neq \varphi$$

$$\iff x \in \overline{\text{Apr}}_I(A) \text{ or } x \in \overline{\text{Apr}}_I(B)$$

$$\iff x \in \overline{\text{Apr}}_I(A) \cup \overline{\text{Apr}}_I(B).$$

(14) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by (4), $\text{Apr}_I(A) \subseteq \text{Apr}_I(A \cup B)$ and $\text{Apr}_I(B) \subseteq \text{Apr}_I(A \cup B)$, which imply that $\text{Apr}_I(A) \cup \text{Apr}_I(B) \subseteq \text{Apr}_I(A \cup B)$.

(15) The proof is straightforward.

**Corollary 3.3.** Let $(H, \Theta)$ be an approximation space. Then

(i) for every $A \subseteq H$, $\text{Apr}_I(A)$ and $\overline{\text{Apr}}_I(A)$ are definable sets,

(ii) for every $x \in H$, $I_x$ is definable set.

**Proof.**

(i) By proposition 3.2 (5) and (7), we have $\text{Apr}_I(\text{Apr}_I(A)) = \text{Apr}_I(A) = \overline{\text{Apr}}_I(\overline{\text{Apr}}_I(A))$. Hence $\text{Apr}_I(A)$ is a definable set. On the other hand by proposition 3.2 (6) and (8), we have $\overline{\text{Apr}}_I(\overline{\text{Apr}}_I(A)) = \overline{\text{Apr}}_I(A) = \text{Apr}_I(\overline{\text{Apr}}_I(A))$. Therefore $\overline{\text{Apr}}_I(A)$ is a definable set.

(ii) By proposition 3.2 (15) the proof is clear.
Theorem 3.4. Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_{\Theta}$ be a hyper $BCK$-ideal of $H$ and $A,B$ are non-empty subsets of $H$. Then

(i) $\overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B) = \overline{\text{Apr}}_I(A \circ B)$,

(ii) $\overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B) \subseteq \overline{\text{Apr}}_I(A \circ B)$.

Proof. (i) Let $z \in \overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B)$. Then there exist $a \in \overline{\text{Apr}}_I(A)$ and $b \in \overline{\text{Apr}}_I(B)$ such that $z \in a \circ b$. Hence $I_a \cap A \neq \phi$ and $I_b \cap B \neq \phi$ and so there exist $c \in I_a \cap A$ and $d \in I_b \cap B$ such that $a \Theta c$ and $b \Theta d$. Since $\Theta$ is a congruence relation on $H$, then we have $a \circ b \Theta c \circ d$ and because $z \in a \circ b$, then there exist $y \in c \circ d$ such that $z \Theta y$. Hence $y \in I_z$. On the other hand, $y \in c \circ d \subseteq A \circ B$ which implies $I_z \cap (A \circ B) \neq \phi$ and so $z \in \overline{\text{Apr}}_I(A \circ B)$. Therefore $\overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B) \subseteq \overline{\text{Apr}}_I(A \circ B)$.

Now, suppose that $x \in \overline{\text{Apr}}_I(A \circ B)$. Then $I_x \cap (A \circ B) \neq \phi$. Let $z \in I_x \cap (A \circ B)$, then there exist $a \in A$ and $b \in B$ such that $z \in a \circ b$ and $I_x = I_z$. Thus we have $I_z \subseteq I_a \cap I_b$ and so $I_x \subseteq I_a \circ I_b$. Hence $x \in a \circ b \subseteq A \circ B \subseteq \overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B)$. Therefore, $\overline{\text{Apr}}_I(A \circ B) \subseteq \overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B)$. □

(ii) Let $z \in \overline{\text{Apr}}_I(A) \circ \overline{\text{Apr}}_I(B)$. Then there exist $a \in \overline{\text{Apr}}_I(A)$ and $b \in \overline{\text{Apr}}_I(B)$ such that $z \in a \circ b$, $I_a \subseteq A$ and $I_b \subseteq B$. For every $y \in I_z$, we have $I_z = I_y \subseteq I_a \circ I_b$ and so $y \in a \circ b \subseteq A \circ B$. Then $y \in A \circ B$ and so $I_z \subseteq A \circ B$. Therefore $z \in \overline{\text{Apr}}_I(A \circ B)$. □

Example 3.5. Let $H = \{0,1,2\}$ and define the hyper operation “$\circ$” on $H$ as follow:

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Then $(H, \circ)$ is a hyper $BCK$-algebra. Define the equivalence relation $\Theta$ by

$\Theta = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$.

Then $\Theta$ is a regular congruence relation on $H$ and so we have:

$I = [0]_{\Theta} = \{0, 1\}, I_1 = [1]_{\Theta} = \{0, 1\}, I_2 = [2]_{\Theta} = \{2\}$.
Rough Set Theory Applied To Hyper BCK-Algebra

Now, if we let $A = \{1, 2\}$ and $B = \{0, 2\}$, then we have $A \cap B = \{0, 1, 2\}$ and so

\[
\begin{align*}
\text{Apr}_I(A) &= \{x \in H| I_x \subseteq A\} = \{2\}, \\
\text{Apr}_I(A) &= \{x \in H| I_x \cap A \neq \phi\} = \{0, 1, 2\}, \\
\text{Apr}_I(B) &= \{x \in H| I_x \subseteq B\} = \{2\}, \\
\text{Apr}_I(B) &= \{x \in H| I_x \cap B \neq \phi\} = \{0, 1, 2\}, \\
\text{Apr}_I(A \cap B) &= \{x \in H| I_x \subseteq A \cap B\} = \{0, 1, 2\}, \\
\text{Apr}_I(A \cap B) &= \{x \in H| (A \cap B) \neq \phi\} = \{0, 1, 2\}, \\
\text{Apr}_I(A) \cap \text{Apr}_I(B) &= \{0, 2\}.
\end{align*}
\]

Therefore, we see that $\text{Apr}_I(A) \cap \text{Apr}_I(B) \neq \text{Apr}_I(A \cap B)$ but $\overline{\text{Apr}_I(A)} \cap \overline{\text{Apr}_I(B)} = \overline{\text{Apr}_I(A \cap B)}$.

**Definition 3.6.** Let $\Theta$ be a regular congruence relation on $H$, $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$ and $A$ be a non-empty subset of $H$. If $\text{Apr}_I(A)$ and $\overline{\text{Apr}_I(A)}$ are hyper subalgebra of $H$, then $A$ is called a **rough hyper subalgebra** of $H$.

**Theorem 3.7.** If $I$ be a hyper $BCK$-ideal and $J$ be a hyper subalgebra of $H$, then

(i) $\overline{\text{Apr}_I(J)}$ is a hyper subalgebra of $H$.

(ii) If $I \subseteq J$, then $\text{Apr}_I(J)$ is a hyper subalgebra of $H$.

**Proof.** (i) Since $0 \in J \subseteq \overline{\text{Apr}_I(J)}$, then $\overline{\text{Apr}_I(J)} \neq \phi$. Now, we assume that $x, y \in \overline{\text{Apr}_I(J)}$. We must prove that $x \circ y \subseteq \overline{\text{Apr}_I(J)}$. Since $I_x \cap J \neq \phi$ and $I_y \cap J \neq \phi$, we can let $t \in I_x \cap J$, $s \in I_y \cap J$ and $z \in x \circ y$. Hence $I_z \subseteq I_x \circ I_y = I_t \circ I_s$ and so $z \in t \circ s \subseteq J$. Thus we have $z \in J$ and $z \in I_z$ and so $I_z \cap J \neq \phi$. Therefore, $z \in \overline{\text{Apr}_I(J)}$ and so $x \circ y \subseteq \overline{\text{Apr}_I(J)}$.

(ii) Since $I = I_0 \subseteq J$, we have $0 \in \overline{\text{Apr}_I(J)} \neq \phi$. Now, suppose that $a, b \in \overline{\text{Apr}_I(J)}$. Then $I_a \subseteq J$ and $I_b \subseteq J$. For every $z \in a \circ b$ and every $y \in I_z$, we have $I_z = I_y = I_a \circ I_b$ and so $y \in a \circ b \subseteq J$. Hence $I_z \subseteq J$, which implies that $z \in \overline{\text{Apr}_I(J)}$. Therefore, $a \circ b \subseteq \overline{\text{Apr}_I(J)}$. \qed

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Theorem 3.8. Let Θ and Φ be two regular congruence relations on $H$ and $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper $BCK$-ideals of $H$ such that $I \subseteq J$. Then for any nonempty subset $A$ of $H$, we have:

(i) $\text{Apr}_J(A) \subseteq \text{Apr}_I(A)$,

(ii) $\overline{\text{Apr}}_J(A) \subseteq \overline{\text{Apr}}_I(A)$.

Proof. (i) First we show that if $I \subseteq J$, then $I_x \subseteq J_x$. Let $y \in I_x$. Then $x \Theta y$. Since $\Theta$ is a congruence relation on $H$ and $x \Theta x$, then $x \circ x \overline{\Theta} x \circ y$. Since $0 \in x \circ x$, then there exist $t \in x \circ y$ such that $0 \Theta t$ and so $t \in [0]_\Theta = I \subseteq J = [0]_\Phi$. Thus by hypothesis, $t \in [0]_\Phi$ and so $x \circ y \Phi \{0\}$. By the similar way, we can show that $y \in [x]_\Phi = J_x$. Therefore, $I_x \subseteq J_x$. Now, let $x \in \text{Apr}_J(A)$. Then $J_x \subseteq A$ and so $I_x \subseteq A$ which implies $x \in \text{Apr}_I(A)$.

(ii) Assume that $x \in \overline{\text{Apr}}_I(A)$. Then $I_x \cap A \neq \emptyset$. Since $I_x \subseteq J_x$, we have $J_x \cap A \neq \emptyset$. Therefore, $x \in \overline{\text{Apr}}_J(A)$.

Corollary 3.9. Let Θ and Φ are two regular congruence relations on $H$, $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper $BCK$-ideals of hyper $BCK$-algebra $H$ and $A$ be a non-empty subset of $H$. Then

(i) $\text{Apr}_I(A) \cap \text{Apr}_J(A) \subseteq \text{Apr}_{I \cap J}(A)$,

(ii) $\overline{\text{Apr}}_{I \cap J}(A) \subseteq \overline{\text{Apr}}_I(A) \cap \overline{\text{Apr}}_J(A)$.

Proof. By theorem 3.8, the proof is clear.

Definition 3.10. Let Θ be a regular congruence relation on $H$, $I = [0]_\Theta$ be a hyper $BCK$-ideal of $H$, $A$ be a non-empty subset of $H$ and $\text{Apr}_I(A) = (\text{Apr}_J(A), \overline{\text{Apr}}_J(A))$ be a rough set in the approximation space $(H, \Theta)$. If $\text{Apr}_I(A)$ and $\overline{\text{Apr}}_I(A)$ are hyper $BCK$-ideals (resp, weak, strong) of $H$, then $A$ is called a rough hyper $BCK$-ideal (resp, weak, strong) of $H$. 

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Example 3.11. Let $H = \{0, 1, 2, 3\}$ and hyper operation “$\circ$” on $H$ is defined as follow:

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<th>3</th>
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<td>{2}</td>
<td>{0,1}</td>
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<tr>
<td>3</td>
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Then $(H, \circ, 0)$ is a hyper $BCK$-algebra. We define the regular congruence relation on $H$ as follow:

$\Theta = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0)\}$.

So we have:

$I = I_0 = I_1 = \{0,1\}, I_2 = \{2\}, I_3 = \{3\}$.

Now, let $A = \{0,1,3\}$ be a subset of $H$, then

$Apr_I(A) = \{x \in H | I_x \subseteq A\} = \{0,1,3\}$,

$\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\} = \{0,1,3\}$.

Easily we give that $Apr_I(A)$ and $\overline{Apr}_I(A)$ are hyper $BCK$-ideals. Therefore, $A$ is a rough hyper $BCK$-ideal of $H$.

Example 3.12. Let $H = \{0, a, b, c\}$. By the following table $(H, \circ)$ is a hyper $BCK$-algebra.

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<td>{0,a}</td>
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<td>c</td>
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<td>{0,c}</td>
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Now, let relation $\Theta$ on $H$ is defined as follow:

$\Theta = \{(0,0), (a,a), (b,b), (c,c), (0,b), (b,0), (0,a), (a,0), (a,b), (b,a)\}$.

Then,

$I_0 = I_a = I_b = \{0,a,b\}, I_c = \{c\}$.

Let $J_1 = \{0,c\}, J_2 = \{0,b\}$ and $J_3 = \{c\}$. Then,

$Apr_I(J_1) = \{c\}, \overline{Apr}_I(J_1) = \{0,a,b,c\}$,

$Apr_I(J_2) = \{\}, \overline{Apr}_I(J_2) = \{0,a,b\}$,

$Apr_I(J_3) = \{c\}, \overline{Apr}_I(J_3) = \{c\}$.
Hence we can see that $J_1$ is a hyper $BCK$-ideal of $H$ but $\text{Apr}_I(J_1)$ is not a hyper $BCK$-ideal. Moreover $J_2$ is not a hyper $BCK$-ideal but $\overline{\text{Apr}}_I(J_2)$ is a hyper $BCK$-ideal of $H$. In follows, $J_3$ is not a hyper $BCK$-ideal and neither $\text{Apr}_I(J_3)$ nor $\overline{\text{Apr}}_I(J_3)$ is a hyper $BCK$-ideal of $H$.

**Theorem 3.13.** Let $\Theta$ be a regular congruence relation on $H$ and $I = [0]_{\Theta}$ be a hyper $BCK$-ideal of $H$. Then

(i) If $J$ be a weak hyper $BCK$-ideal of $H$ containing $I$, then $\overline{\text{Apr}}_I(J)$ is a weak hyper $BCK$-ideal of $H$,

(ii) If $J$ be a hyper $BCK$-ideal of $H$ containing $I$, then $\text{Apr}_I(J)$ is a hyper $BCK$-ideal of $H$,

(iii) If $J$ be a strong hyper $BCK$-ideal of $H$ containing $I$, then $\overline{\text{Apr}}_I(J)$ is a strong hyper $BCK$-ideal of $H$.

**Proof.** (i) Since $I = I_0 \subseteq J$, then $0 \in \overline{\text{Apr}}_I(J)$. Now, Let $x, y \in H$ be such that $x \circ y \subseteq \overline{\text{Apr}}_I(J)$ and $y \in \overline{\text{Apr}}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a \Theta x$ and $b \Theta y$. Since $\Theta$ is a congruence relation on $H$, we have $a \circ b \Theta b$ and so for every $z \in a \circ b$, there exist $t \in x \circ y$ such that $z \Theta t$. Since $x \circ y \subseteq \overline{\text{Apr}}_I(J)$, we have $t \in \overline{\text{Apr}}_I(J)$ and so $I_t = I_z \subseteq J$ which implies $z \in J$. Thus $a \circ b \subseteq J$. On the other hand, $b \in I_y \subseteq J$. Since $J$ is a weak hyper $BCK$-ideal, we have $a \in J$ and so $I_x \subseteq J$. Hence $x \in \overline{\text{Apr}}_I(J)$. Therefore, $\overline{\text{Apr}}_I(J)$ is a weak hyper $BCK$-ideal of $H$.

(ii) Let $x, y \in H$ be such that $x \circ y \subseteq \overline{\text{Apr}}_I(J)$ and $y \in \overline{\text{Apr}}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a \Theta x$ and $b \Theta y$. Since $\Theta$ is a congruence relation on $H$, we have $a \circ b \Theta x \circ y$ and so for every $z \in a \circ b$, there exist $z' \in x \circ y$ such that $z \Theta z'$. Since $z' \in x \circ y \subseteq \overline{\text{Apr}}_I(J)$, then there exists $t \in \overline{\text{Apr}}_I(J) \subseteq J$ such that $z' \ll t$ and so from $z \Theta z'$, we have $I_0 \in I_{z'} \circ I_t = I_z \circ I_t$. Hence $0 \in z \circ t$ and then $z \ll t$. Thus we have proved that for every $z \in a \circ b$, there exist $t \in J$ such that $z \ll t$ which means that $a \circ b \subseteq J$. On the other hand we have $b \in I_y \subseteq J$. Since $J$ is a hyper $BCK$-ideal of $H$, we
have \( a \in J \). Thus \( I_x \subseteq J \) which implies that \( x \in \overline{\text{Apr}}_I(J) \). Therefore, \( \overline{\text{Apr}}_I(J) \) is a hyper \( BCK \)-ideal of \( H \).

(iii) Suppose that \( x, y \in H \) be such that \((x \circ y) \cap \overline{\text{Apr}}_I(J) \neq \emptyset \) and \( y \in \overline{\text{Apr}}_I(J) \). Let \( a \in I_x \) and \( b \in I_y \). Then \( a \Theta x \) and \( b \Theta y \). Since \( \Theta \) is a congruence relation on \( H \), we have \( a \circ b \Theta x \circ y \). Since \((x \circ y) \cap \overline{\text{Apr}}_I(J) \neq \emptyset \), then there exist \( t \in H \) such that \( t \in x \circ y \) and \( t \in \overline{\text{Apr}}_I(J) \). Now, \( t \in x \circ y \Theta a \circ b \) implies that there exist \( z \in a \circ b \) such that \( z \Theta t \) and so \( I_t = I_z \subseteq J \). Hence \( z \in J \) and so \((a \circ b) \cap J \neq \emptyset \). On the other hand, we have \( b \in I_y \subseteq J \). Since \( J \) is a strong hyper \( BCK \)-ideal of \( H \), then we have \( a \in J \) which implies \( I_x \subseteq J \) that means \( x \in \overline{\text{Apr}}_I(J) \). Therefore, \( \overline{\text{Apr}}_I(J) \) is a strong hyper \( BCK \)-ideal of \( H \). \( \square \)

**Theorem 3.14.** Suppose that \( I \) be a hyper \( BCK \)-ideal of \( H \) and \( \Theta \) be a regular congruence relation on \( H \) which is defined as follow:

\[
x \Theta y \iff x \circ y \subseteq I \quad \text{and} \quad y \circ x \subseteq I.
\]

(i) If \( J \) be a weak hyper \( BCK \)-ideal of \( H \) containing \( I \), then \( \overline{\text{Apr}}_I(J) \) is a weak hyper \( BCK \)-ideal of \( H \),

(ii) If \( J \) be a hyper \( BCK \)-ideal of \( H \) containing \( I \), then \( \overline{\text{Apr}}_I(J) \) is a hyper \( BCK \)-ideal of \( H \),

(iii) If \( J \) be a strong hyper \( BCK \)-ideal of \( H \) containing \( I \), then \( \overline{\text{Apr}}_I(J) \) is a strong hyper \( BCK \)-ideal of \( H \).

**Proof.** (i) Since \( I \subseteq J \subseteq \overline{\text{Apr}}_I(J) \), then we have \( 0 \in \overline{\text{Apr}}_I(J) \). Let \( x, y \in H \) be such that \( x \circ y \subseteq \overline{\text{Apr}}_I(J) \) and \( y \in \overline{\text{Apr}}_I(J) \). Then \( I_y \cap J \neq \emptyset \) and for every \( z \in x \circ y \), we have \( z \in \overline{\text{Apr}}_I(J) \) which means \( I_z \cap J \neq \emptyset \). Thus there exist \( a, b \in H \) such that \( a \in I_y \cap J \) and \( b \in I_z \cap J \) which imply that \( a \Theta y, b \Theta z \) and \( a, b \in J \). Thus \( y \circ a \subseteq I \subseteq J \) and \( z \circ b \subseteq I \subseteq J \) and so we get \( y, z \in J \), since \( J \) is a weak hyper \( BCK \)-ideal. Thus we have proved that for any \( z \in x \circ y \), we have \( z \in J \) and so \( x \circ y \subseteq J \). Since \( J \) is a weak hyper \( BCK \)-ideal and \( y \in J \), obviously we have \( x \in J \). Since \( x \in I_x \), then \( I_x \cap J \neq \emptyset \). Therefore \( x \in \overline{\text{Apr}}_I(J) \) and so \( \overline{\text{Apr}}_I(J) \) is a weak hyper \( BCK \)-ideal of \( H \).
(ii) Let \( x, y \in H \) be such that \( x \circ y \ll \overrightarrow{Apr}_I(J) \) and \( y \in \overrightarrow{Apr}_I(J) \). Then \( I_y \cap J \neq \phi \) and for every \( z \in x \circ y \), there exist \( t \in \overrightarrow{Apr}_I(J) \) such that \( z \ll t \) and \( I_t \cap J \neq \phi \). Thus, there exist \( c, d \in H \) such that \( c \in I_t \cap J \) and \( d \in I_y \cap J \) and so \( c \Theta t, d \Theta y \) and \( c, d \in J \). Hence \( t \circ c \subseteq I \subseteq J \) and \( y \circ d \subseteq I \subseteq J \). Since \( J \) is a hyper \( BCK \)-ideal and \( c, d \in J \), we have \( y, t \in J \). Thus, we have proved that for every \( z \in x \circ y \), there exist \( t \in J \) such that \( z \ll t \) which means that \( x \circ y \ll J \) and so from \( y \in J \) we get \( x \in J \). Consequently, \( I_x \cap J \neq \phi \) and so \( x \in \overrightarrow{Apr}_I(J) \). Therefore, \( \overrightarrow{Apr}_I(J) \) is a hyper \( BCK \)-ideal.

(iii) Let \( x, y \in H \) be such that \( (x \circ y) \cap \overrightarrow{Apr}_I(J) \neq \phi \) and \( y \in \overrightarrow{Apr}_I(J) \). Then \( I_y \cap J \neq \phi \) and so there exist \( z \in H \) such that \( z \in x \circ y \) and \( z \in \overrightarrow{Apr}_I(J) \). Hence \( I_z \cap J \neq \phi \) and so there exist \( c, d \in H \) such that \( c \in I_z \cap J \) and \( d \in I_y \cap J \). Hence \( c \Theta z \) and \( d \Theta y \) where \( c, d \in J \). Thus we have \( z \circ c \subseteq I \subseteq J \) and \( y \circ d \subseteq I \subseteq J \). Since \( J \) is a strong hyper \( BCK \)-ideal and \( c, d \in J \), we have \( z \in J \) and \( y \in J \). Thus we have proved that \( (x \circ y) \cap J \neq \phi \) and \( y \in J \). Since \( J \) is a strong hyper \( BCK \)-ideal, we have \( x \in J \) and so \( I_x \cap J \neq \phi \) which means that \( \overrightarrow{Apr}_I(J) \) is a strong hyper \( BCK \)-ideal of \( H \).

4 Conclusion

This paper is intend to built up connection between rough sets and hyper \( BCK \)-algebras. We have presented a definition of the lower and upper approximation of a subset of a hyper \( BCK \)-algebra with respect to a hyper \( BCK \)-ideal. This definition and main results are easily extended to other algebraic structures such as hyper \( K \)-algebra, hyper \( I \)-algebra, etc.

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References


