Size-dependent bistability of an electrostatically actuated arch NEMS based on strain gradient theory

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Abstract
This paper deals with the investigation of the size-dependent nature of nonlinear dynamics, in a doubly clamped shallow nano-arch actuated by spatially distributed electrostatic force. We employ strain gradient theory together with the Euler–Bernoulli and shallow arch assumptions in order to derive the nonlinear partial differential equation governing the transverse motion of the arch with mid-plane stretching effects. Using the Galerkin projection method, we derive the lumped single degree of freedom model which is then used for the study of the size effects on the nonlinear snap-through and pull-in instabilities of the arch nano-electro-mechanical-system (NEMS). Moreover, using strain gradient theory, the size-dependent bistability and fundamental frequencies of the nano-arch are scrutinized, revealing that, despite what is predicted by the classical theory, the bistability region in the parameter space of the nano-structure shrinks as the structure scales down. Also, we show that the minimum initial elevation, required for bistability, increases as the nano-arch scales down.

Keywords: NEMS, strain gradient theory, snap-through, pull-in, bistability, shallow arch

(Some figures may appear in colour only in the online journal)

1. Introduction

Micro- and nano-electro-mechanical-systems (MEMS/NEMS) have gained tremendous attention in the research community in the past decades. Along with the development of conventional MEMS–NEMS, growing attention has been directed toward the bistable family of these systems. Bistability, which refers to the capability of the system to operate in two separate stable configurations at the same value of the actuation parameters, motivates a series of applications including MEMS-based memories [1, 2], sensors [3], switches [4], microvalves [5], bandpass filters [6] and etc. Such a bistable MEMS is possible to be constructed using a doubly clamped initially curved microbeam, being curved either by buckling under the axial compressive force or by micromachining techniques [7, 8]. Bistability in these systems is verified by the existence of two convex and concave stable configurations, the transition between which occurs precipitately and is referred to as the snap-through. Various drive mechanisms for these systems are reported, such as the electro-magnetic [9], electro-thermal [10], light pressure [11] and electrostatic [8]. Among them the electrostatic actuation is more popular due to many advantages such as low cost, low power consumption, improved reliability and the possibility of integrating the MEMS with electronic circuits. Electrostatically actuated MEMS/NEMS are vulnerable to another instability, known as pull-in, which addresses the collapse of the structure...
and sudden adhesion of the electrodes due to the intrinsic softening effect arising from the electrostatic force [12].

Some researchers have addressed the nonlinear phenomena arising in the electrostatically actuated initially curved micro-beams. Krylov et al. [8, 13] have investigated bistable arch MEMS, both theoretically and experimentally. They presented theoretical studies using the reduced order model obtained by application of the Galerkin projection method to the non-linear partial differential equation governing transverse motion of the arch. Using this lumped model, they have investigated the requirements for bistability in the arch MEMS as well as the conditions for snap-through and pull-in instabilities. For instance, they have shown that, for the initially curved beam to present bistability, the initial elevation parameter of the arch should be larger than a minimum value [13]. Zhang et al. [14] investigated instabilities in the electrostatically actuated arch MEMS concluding that the snap-through motion can be used in the design of highly sensitive low-power sensors. Das and Batra [15] presented finite element and boundary element approaches for the study of this system. Some other research groups have proposed the arch MEMS as the forced oscillatory systems, namely resonators, under harmonic electrostatic actuation. Younis et al. [7] and Ouakad and Younis [16] have investigated the nonlinear dynamics of bistable MEMS resonator experimentally and theoretically, respectively. They also discussed the application of these bistable resonators as band-pass filters [6, 17]. The authors [18] have recently presented analytical discussions on the nonlinear dynamics of electrostatically actuated arch MEMS resonators. Furthermore, we have investigated the possibility of chaotic vibrations in such bistable resonators [19].

Due to possible advantages for applications, such as increased natural frequencies and faster responses, NEMS have received growing interest in recent years. However, many experimental investigations have shown that, as the thickness of the beam decreases below several micrometres, size-dependent behaviour appears; which is not accurately predictable by the conventional continuum theories. For instance, Namazu et al. [20] have shown that for a silicon beam with a thickness of 255 nm the bending stiffness is approximately 4 times larger than the bending stiffness of silicon beams in macro scale. Also, Fleck et al. [21] have shown that, as the diameter of a copper wire decreases from 170 to 12 μm, its relative torsion stiffness triples. For further experimental reports delineating size-dependency of mechanical characteristics in micron and sub-micron scales for various materials see [22–24].

In order to account for size-dependencies, various non-classical continuum theories have been introduced. Among these non-classical continuum theories, is the couple stress theory which brings two higher-order constants into the constitutive equations of the material, and originates back to the works of Mindlin and Tiersten [25] and Toupin [26]. Yang et al. [27] proposed a modified couple stress theory which imposes the implementation of only one higher-order constant for the material and is extensively implemented in the past years for the study of size-dependent statics and dynamics of micro-beams; see [28–36] for a few examples.

Using earlier ideas of Mindlin [37] on the elasticity gradient, Fleck and Hutchinson [38, 39] assumed that the strain energy density of the continuum is dependent on the strain as well as the first spatial derivative of the strain tensor. Based on this assumption, they proposed their strain gradient theory which brings five higher-order material constants into the constitutive equations of the continuum. Later, Lam et al. [22] proposed a modified strain gradient theory which depends on three independent higher-order material constants, and reduces to the modified couple stress theory when neglecting additional higher-order terms. It is worth noting that a modified version of Mindlin strain gradient theory, having only one higher-order constant, is also reported in the literature [40]. In the rest of this paper, the term ‘strain gradient theory’ stands for the theory proposed by Lam et al. [22]. Kong et al. [41, 42] proposed a linear strain gradient Euler–Bernoulli beam formulation for the study of micro-beams. Kahrabaiyan et al. [43] developed a large deflection strain gradient Euler–Bernoulli model capable of accounting for nonlinearities coming from the mid-plane stretching effects. These nonlinear effects arise in micro/nano-beams with fixed ends where their lateral deflection is comparable to their thickness. Many researchers have employed strain gradient theory for the study of static and dynamic behaviours of micro/nano-scaled beams; see [29, 36, 44–47] for a few examples.

The development of another non-classical continuum theory, known as the nonlocal elasticity theory, originates back to the earlier works of Eringen [48–50]. In nonlocal theory, stress at any point within the continuum is assumed to be a function of strain not only at the same point but also at all other points. This theory is also capable of accounting for size effects, and has been extended for the study of micro/nano-scaled beams [35, 45, 51–53].

In an attempt to investigate size-dependencies, many researchers have focused on surface effects. In fact, reconstruction of surface atoms or other surface effects such as contaminations induce surface stresses, and cause the surface elastic parameters to differ from their bulk values [54–57]. As the size approaches tens of nanometres, the number of surface atoms becomes comparable to that of bulk atoms; thus, the surface effects become significant in these very small sizes. Many researchers have attempted to estimate size-dependent elastic parameters of various materials either by experiments [58] or by atomistic simulations [59, 60]. Continuum models incorporating surface stresses and surface elasticity have also been introduced to account for the size-dependent behaviours [61–63].

The main motivation of the present work arises from benefits of the combined advantages of nano-sized and initially curved beams. To the authors’ best knowledge, the size-dependent bistability and nonlinear dynamics of the electrostatically actuated arch NEMS has not thoroughly been addressed in literature. Although some researchers have reported investigations on the statics of shallow nano-arches [64], but neither a practical mathematical model nor sufficient investigations on the bistability and nonlinear dynamics has been proposed focusing on the electrostatically actuated arch NEMS.

In this paper, we employ strain gradient theory along with the Euler–Bernoulli beam assumptions to investigate the size-dependent nonlinear behaviours of the shallow arch NEMS. Toward this aim, primarily, we derive strain gradient partial
differential equations with mid-plane stretching effects which govern the transverse motion of the shallow arch under spatially distributed electrostatic actuation. Then, we apply the Galerkin’s decomposition method to the obtained dimensionless equation of motion, and derive a lumped single degree of freedom model which can be used for the investigations of nonlinear dynamics of the nano-arch. Throughout the paper, we compare the results of the strain gradient theory with those of the classical theory. Our investigations elucidate how the bistability region in the parameter space of the arch NEMS varies as the size of the structure scales down. We focus our study on the size-dependent bistability to nano-arches with thicknesses large enough for surface effects not to be significant, and to which the strain gradient theory is applicable for the description of size-dependencies. Moreover, as shown by Maani et al. [45] strain gradient theory accurately fits the experimental results for nano-beams with the smallest dimension, i.e. thickness, between several hundred nanometres and almost one micron, suggesting that the surface effects can be neglected for devices on this scale. Additionally, we show that the minimum required initial elevation parameter for bistability of the structure increases due to the size effects. Further, we discuss the influence of the size-dependent parameters on the snap-through and pull-in instabilities as well as on the fundamental frequency of the shallow nano-arch. At each stage we compare the results with those obtained by the classical theory, in order to provide a better insight into the size-dependent behaviours.

In section 2 we propose the employed procedure for the derivation of the equation of motion for the shallow nano-arch using the strain gradient formulation. In section 3, we present the dimensionless lumped equation obtained by the Galerkin projection method. Section 4 continues with the study of static equilibrium or fixed points and presents the size-dependent bifurcation diagrams and bistability requirements of the nano-arch. In section 5, the stability of the fixed points is studied and the size-dependent fundamental frequency of the nano-arch is proposed. The paper ends with the conclusions in section 6.

2. Mathematical modelling

An initially curved doubly clamped nano-beam of length \( L \), width \( b \) and thickness \( d \), as shown in figure 1, is investigated in this paper. We suppose the initial deflection of the arch to be given by \( w_0(x) \) in the direction shown in figure 1. Also, the transverse deflection of the arch in the positive \( z \) direction and measured from the initial resting condition is given by \( \tilde{w}(x,t) \); the absolute displacement is measured from the \( x \) axis and is represented by \( w(x,t) \), as shown in figure 1. The arch is actuated by the electrostatic transverse load distributed over its length, by applying a voltage \( V \) to the beam. We aim to derive equations of motion of the given nano-arch in the framework of the Euler-Bernoulli assumptions and strain gradient continuum theory. The coordinate system and the main parameters are shown in figure 1. We further assume that the slopes are always small after deformation (see appendix for the reason) and the initial rise of the arch is very small compared to its length (\( h_0 \ll L \)).

2.1. Basics

Based on strain gradient theory proposed by Lam et al. [22], the strain density energy \( U \) of a linear elastic continuum occupying region \( \Omega \) is given as a function of the symmetric strain tensor \( \varepsilon_{ij} \), the dilatation gradient vector \( \gamma \), the deviatoric stretch gradient tensor \( \eta_{ij}^1 \) and the symmetric rotation gradient tensor \( \chi_{ij} \), as follows:

\[
U = \frac{1}{2} \int_{\Omega} \left[ \sigma_{ij} \varepsilon_{ij} + p_{i} \gamma_{i} + \tau_{ijk} \eta_{jk}^1 + m_{ij} \chi_{ij} \right] \, dv. \tag{1}
\]

\( \sigma_{ij} \) is the classical stress component, while \( p_{i} \), \( \tau_{ijk} \) and \( m_{ij} \) are the higher-order stress components defined in the following, as:

\[
\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}),
\]

\[
\gamma_{i} = \varepsilon_{num,i},
\]

\[
\eta_{ij}^1 = \frac{1}{3}(\varepsilon_{ik,i} + \varepsilon_{ij,k} + \varepsilon_{ij,k}) - \frac{1}{15} \delta_{ij}(\varepsilon_{num,k} + 2\varepsilon_{nm,m}) - \frac{1}{15}[\delta_{ij}(\varepsilon_{num,i} + 2\varepsilon_{nm,m}) + \delta_{ii}(\varepsilon_{num,j} + 2\varepsilon_{mj,m})],
\]

\[
\chi_{ij} = \frac{1}{2}(\theta_{ij} + \theta_{ji}),
\]

\[
\theta_{ij} = \frac{1}{2}(\text{curl}(u))_{ij}, \tag{2}
\]

where components of the displacement vector \( u \), rotation vector \( \theta \) and the dilatation gradient vector \( \gamma \) are represented by \( u_{ij}, \theta_{ij} \) and \( \gamma_{i} \), respectively. \( \delta_{ij} \) represents the Kronecker’s delta. According to Lam et al. [22], the classical and higher-order stress components are calculated as below using the constitutive equations for linear elastic isotropic materials:

\[
\sigma_{ij} = \lambda \varepsilon_{ij} \delta_{ij} + 2G \varepsilon_{ij},
\]

\[
p_{i} = 2Gi_{i}^{1},
\]

\[
\tau_{ijk} = 2G\eta_{ijk}^1,
\]

\[
m_{ij} = 2G\chi_{ij}, \tag{3}
\]

where \( \lambda \) and \( G \) are called the Lame constants and are obtained as a function of the Poisson’s ratio \( \nu \) and Young’s modulus \( E \):

\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad G = \frac{E}{2(1 + \nu)}, \tag{4}
\]

\( l_0, l_1 \) and \( l_2 \) are the independent material parameters associated with dilatation, deviatoric stretch and rotation gradients, respectively [22]. For an initially curved (arch) Euler–Bernoulli beam, the displacement is given as:

\[
u_1 = u(x,t) = \frac{dw(x,t)}{dx} - \frac{dw_0(x)}{dx},
u_2 = 0, \nu_3 = w(x,t) - w_0(x), \tag{5}
\]

where \( w(x,t) \) represents the transverse deflection of the arch, and \( w_0(x) \) denotes its initial curvature; \( w(x,t) \) stands for the axial displacement of the centroid of cross sections, i.e. the
neutral axis, of the beam. Hereafter subscripts 1, 2 and 3 stand for the components of vector quantities in $x$, $y$ and $z$ directions, respectively. Using the given displacement field, the ratio of elongation of a line element resting in the axial direction of the beam before deformation, i.e. the axial strain of the arch, is approximated by von-Karman strain:

$$
\varepsilon_{11} = \frac{du}{dx} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 - \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right). 
$$

(6)

For details of the derivation of (5) and (6) for an initially curved Euler–Bernoulli beam, see appendix and [65], where we show that assuming small slopes and small strains are necessary for derivation of (6). Also see [8, 33, 64] for implementation of this strain component for the study of shallow micro/nano-arches.

Denoting the axial strain component by (6) and substituting it together with (5) into (2), one obtains the non-zero components of $\gamma$, $\eta$ and $\chi$ as [43, 64]:

$$
\gamma_1 = \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2}.
$$

(7)

$$
\chi_{12} = \chi_{21} = -\frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right).
$$

(8)

$$
\begin{align*}
\eta_{111} &= \frac{2}{5} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{d w_0}{dx} \frac{d^2 w_0}{dx^2} - \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right], \\
\eta_{113} &= \eta_{311} = \eta_{313} = -\frac{4}{15} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right), \\
\eta_{122} &= \eta_{121} = \eta_{212} = \eta_{211} = \eta_{133} = \eta_{313} = -\frac{1}{5} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{d w_0}{dx} \frac{d^2 w_0}{dx^2} - \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right], \\
\eta_{223} &= \eta_{232} = \eta_{322} = \frac{1}{15} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right), \\
\eta_{333} &= \frac{1}{5} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right).
\end{align*}
$$

(9)

The components of classical and non-zero higher-order stresses are found by substituting (7)–(9) into (3):

$$
\sigma_{11} = E\varepsilon_{11} \\
= E \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 - \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right) \right).
$$

(10)

$$
P_1 = 2IG \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{d w_0}{dx} \frac{d^2 w_0}{dx^2} - \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right],
$$

(11)

$$
P_3 = -2IG \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right).
$$

(12)

$$
\begin{align*}
\tau_{111} &= \frac{4}{5} I_2 G \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{d w_0}{dx} \frac{d^2 w_0}{dx^2} - \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right], \\
\tau_{113} &= \tau_{311} = \tau_{313} = -\frac{8}{15} I_2 G \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right), \\
\tau_{122} &= \tau_{121} = \tau_{212} = \tau_{211} = \tau_{133} = \tau_{313} = -\frac{2}{15} I_2 G \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right), \\
\tau_{123} &= \tau_{132} = \tau_{213} = \tau_{231} = \tau_{232} = \tau_{321} = \tau_{322} = \frac{2}{15} I_2 G \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right).
\end{align*}
$$

(13)

Using (10)–(13) along with (1), the potential energy due to the classical and higher-order stress terms is calculated as:

$$
U = \frac{1}{2} \int_0^L \left[ \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right)^2 + K \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right)^2 \right] \, dx, \\
+ \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{d w_0}{dx} \frac{d^2 w_0}{dx^2} - \frac{\partial^2 w}{\partial x^2} - \frac{d^2 w_0}{dx^2} \right)^2 \, dx,
$$

(14)
\[ S = EI_x + GA \left( 2l_0^2 + \frac{8}{15} l^2 + l^2_z \right) \]  \hspace{1cm} (15) \\

\[ K = GL \left( 2l_0^2 + \frac{4}{5} l^2_0 \right) \]  \hspace{1cm} (16) \\

\[ P = 2l_0^2 + \frac{4}{5} l^2_0 \]  \hspace{1cm} (17) \\

Defining \( N_0 \) as the residual axial force corresponding to residual axial stress \( N_0 / A \) which is supposed to be uniformly distributed over the beam cross section, the strain potential energy due to the stretching effects induced by the residual initial axial stress is calculated as:

\[
U_0 = \int_0^L \frac{N_0}{A} \left( \frac{d\alpha}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) dx \\
- \int_0^L \left( \frac{d\alpha}{dx} - \frac{\partial w}{\partial x} - \frac{d^2 w}{d^2 x^2} \right) dx \\
= \int_0^L \left( \frac{d\alpha}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) dx.
\]  \hspace{1cm} (18)

The kinetic energy of the beam is:

\[
T = \frac{1}{2} \int_0^L \left\{ \rho A \left( \frac{dw}{dt} \right)^2 + \rho L \left( \frac{\partial^2 w}{\partial t^2} \right) + \rho A \left( \frac{\partial^2 u}{\partial t^2} \right) \right\} dx.
\]  \hspace{1cm} (19)

The virtual work done by the external loads is given by:

\[
\delta W = \int_0^L F(x, t) \delta w \, dx + \int_0^L G(x, t) \delta u \, dx \\
- C_d \int_0^L \frac{\partial^2 w}{\partial t \partial x} \delta w \, dx - C_v \int_0^L \frac{\partial w}{\partial t} \delta w \, dx,
\]  \hspace{1cm} (20)

where \( F(x, t) \) is the electrostatic external load applied in the transverse direction with \( w(x, t) \) as its work conjugate. \( G(x, t) \) represents external loads applied in the axial direction of the beam. Dissipative effects are represented by the structural and viscous damping with coefficients \( C_d \) and \( C_v \), respectively. Hamilton’s principle is written as:

\[
\int_0^L (\delta W + \delta T - \delta U) \, dt = 0.
\]  \hspace{1cm} (21)

The variations of the potential and kinetic energies, using (14), (18) and (19), are substituted into Hamilton’s principle (21) together with (20). Integrating by parts and straightforwardly manipulating mathematical terms, we are left with the following equations and boundary conditions governing the transverse and axial displacements of the initially curved beam:

\[
- \frac{d}{dx} \left\{ \frac{\partial w}{dx} \left( N_0 + EA \left[ \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right] \right) \right\} \\
- G A \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right\} \\
+ S \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^4 w_0}{d^4 x^4} \right) - K \left( \frac{\partial^2 w}{dx^2} - \frac{d^4 w_0}{d^4 x^4} \right) + \rho A \frac{\partial^2 w}{\partial x^2} = 0,
\]  \hspace{1cm} (22)

\[
- \frac{\partial}{\partial x} \left\{ \frac{E A}{2} \left[ \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right] \right\} \\
- G A \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right\} = 0,
\]  \hspace{1cm} (23)

where \( w(0, t) = w_0(0) = w(L, t) = w_0(L) = 0 \), \( \frac{\partial w}{\partial x} \bigg|_{x=0} = \frac{\partial w}{\partial x} \bigg|_{x=L} = 0 \), \( \frac{\partial^2 w}{\partial x^2} \bigg|_{x=0} = \frac{\partial^2 w}{\partial x^2} \bigg|_{x=L} = 0 \), and \( u(0, t) = u(L, t) = 0 \).

Neglecting any longitudinal inertia \( \rho A \partial^2 u / \partial t^2 \) (as is usual in the literature [66]) and supposing that external forces do not have any component in the axial direction of the beam so that \( G(x, t) \) can be neglected, we can infer from (23) that:

\[
- \frac{d}{dx} \left\{ \frac{E A}{2} \left[ \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right] \right\} = 0.
\]  \hspace{1cm} (28)

On the other hand, we can calculate the axial force by integrating the axial stress over an arbitrary cross section [8, 43]:

\[
N = N_0 + \int_A E \alpha dA = N_0 + \int_A E \left( \frac{\partial u}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 \right) dA \\
- \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} - \frac{d^4 w_0}{d^4 x^4} \right) dA.
\]  \hspace{1cm} (29)

where \( \alpha \) is measured from the section centroid. Regarding (28) and (29), we can infer that the stretching axial force induced by the beam transverse deflection is uniform over the beam length. Thus, we can replace the induced stretching axial force.
by its average over the beam length and describe the overall axial force on the beam by:

\[
N = N_0 + \frac{E A}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 - \left( \frac{d w_0}{d x} \right)^2 \, dx. \tag{30}
\]

Using (28) and (30), and neglecting rotational kinetic energy terms due to negligibility of \( f_\nu \) compared to \( A \), we can re-write equation (22) in the following form:

\[
S \left( \frac{d^4 w}{d x^4} - \frac{d^4 w_0}{d x^4} \right) - K \left( \frac{d^2 w}{d x^2} - \frac{d^2 w_0}{d x^2} \right) \nonumber
- \frac{d^2 w}{d x^2} \left[ N_0 + \frac{E A}{2L} \int_0^L \left( \frac{d w_0}{d x} \right)^2 \, dx \right] + \rho A \frac{d^2 w}{d t^2} = F(x, t) + C_{ud} \frac{d^2 w}{d t^2} + C_s \frac{d w}{d t} = 0. \tag{31}
\]

Together with the boundary conditions (24)–(26). Now defining \( \vec{w}(x, t) \) as the deflection of the arch measured from its initial curved configuration, we have:

\[
w(x, t) = w_0(x) + \vec{w}(x, t). \tag{32}
\]

Using (32) we can convert (31) to the following final form:

\[
-K \frac{\partial^4 \vec{w}}{\partial x^4} + S \frac{\partial^4 \vec{w}}{\partial x^4} + \rho A \frac{\partial^2 \vec{w}}{\partial t^2} + C_v \frac{\partial \vec{w}}{\partial t} + C_{ud} \frac{\partial^2 \vec{w}}{\partial x \partial t} \nonumber
- \frac{\partial^2 \vec{w}}{\partial x^2} \left[ N_0 + \frac{E A}{2L} \int_0^L \left( \frac{d \vec{w}_0}{d x} \right)^2 + 2 \frac{d^2 \vec{w}_0}{d x \, d x} \, dx \right] \nonumber
= F(x, t). \tag{33}
\]

With the boundary conditions (24)–(26) for \( \vec{w}(x, t) \). It is clear that, if one neglects size-dependent parameters so that \( K = 0 \) and \( S = E I \), equation (33) is converted to the conventional shallow micro-arch equation \([8, 16, 67]\); while, assuming \( w_0(x) = 0 \) in equation (33) results in the strain gradient equation of a straight beam \([46]\).

3. Reduced order model

In this section we employ the Galerkin projection method to derive the reduced order model for the transverse motion of the arch. First, we proceed by introducing the electrostatic spatially distributed force as below:

\[
F(x, t) = - \frac{\varepsilon_0 E V^2}{2(\gamma_0 - \gamma_i - \vec{w})^2}, \tag{34}
\]

where \( \varepsilon_0 = 8.854 \times 10^{-12} \) F m\(^{-1}\) is the vacuum permittivity. Equation (34) denotes the electrostatic force per unit length of the parallel plates; which is also valid for the shallow arch with small deflections and small thickness to length ratio assumed here \( (h << L, d << L) \). Moreover, for wide shallow arches assumed here \( (b > 5d) \) the effective modulus of elasticity, \( E = E(1 - \nu^2) \), is used instead of \( E \) in the equations of motion.

As the first step, equations (33) and (34) with the corresponding boundary conditions are converted to the dimensionless form given by equation (35), using the non-dimensional parameters given in table 1, and assuming the residual axial stresses to be negligible.

\[
\theta_0 \frac{d^4 \theta}{d x^4} + \frac{d^2 \theta}{d x^2} + \frac{d^2 \theta}{d x^2} + \tilde{C}_v \frac{d \theta}{d t} + \tilde{C}_{ud} \frac{d^2 \theta}{d t d x} \nonumber
- \theta \left[ \frac{d^2 \theta}{d x^2} - h^2 \frac{d^2 \theta}{d x^2} \right] \int_0^L \left( \frac{d \theta}{d x} \right)^2 - 2h \frac{d \theta}{d x} \frac{d \theta}{d x} \, dx \right] \nonumber
= \frac{\beta}{(1 + h\gamma_0 - \gamma_i)^2}, \tag{35}
\]

with the following boundary conditions:

\[
\tilde{\theta} (0, \tilde{t}) = \tilde{\theta} (1, \tilde{t}) = 0, \tag{36}
\]

\[
\tilde{\theta} (\tilde{t}, \tilde{t}) = \tilde{\theta} (\tilde{t}, \tilde{t}) = 0, \tag{37}
\]

\[
\tilde{\theta}^{\infty} (0, \tilde{t}) = \tilde{\theta}^{\infty} (1, \tilde{t}) = 0, \tag{38}
\]

where \( \tilde{t} \) now represents differentiation with respect to \( \tilde{\xi} \). Note that, regarding the coordinate system shown in figure 1, the initial curvature function \( \theta_0(\tilde{\xi}) \) is replaced by \(-h_0\gamma_0(\tilde{\xi}) \), where \( \gamma_0(\tilde{\xi}) \) is a dimensionless function with the property \( \gamma_0(1/2) = 1 \).

Now, we apply the Galerkin decomposition method to the dimensionless equation (35) with the associated boundary conditions, in order to derive the reduced order equation of motion. To this aim, we employ an assumed mode model to express the deflection of the arch:

\[
\tilde{\theta}(\tilde{t}, \tilde{t}) = \sum_{n=1}^{\infty} \theta_n(\tilde{t}) \phi_n(\tilde{\xi}), \tag{39}
\]

where \( \phi_n(\tilde{\xi}) \) represent the non-classical linear undamped mode shapes of the shallow arch. The linear mode shapes for a straight gradient nano-beam are given as a solution to the following differential equation with \( \phi_n(\tilde{\xi}) \) satisfying the boundary conditions (36)–(38)

\[
\theta_0 \frac{d^4 \phi_n}{d x^4} + \frac{d^2 \phi_n}{d x^2} = k_n \phi_n(\tilde{\xi}) = 0, \tag{40}
\]

where \( k_n \) represents the \( n \)th eigenvalue. Maani et al. \([44]\) discussed in detail the solution of equation (40) and the derivation of the corresponding mode shapes and natural frequencies. They have shown that, for very small values of \( \theta_0 \), the solution converges to the mode shapes and natural frequencies of a classical doubly clamped beam. Under these conditions, although the classical mode shapes do not satisfy the non-classical boundary condition (38), the difference between the non-classical mode shapes, satisfying all boundary conditions, and the classical mode shapes are negligible. Thus, provided that \( \theta_0 \ll 1 \), we can use the classical mode shapes instead of non-classical ones, without loss of accuracy.

Moreover, Ouakad et al. \([68]\) have shown that using exact shallow arch mode shapes, proposed in \([69]\), instead of the straight beam mode shapes does not produce significant errors in the application of the Galerkin method in the analysis of a classical arch. Also, most researchers \([7, 13, 16]\) have used the
mode shapes of the straight beam for analysis of the shallow arch. Thus, for the sake of simplicity and recalling that the first mode shape is dominant, we use the first mode shape of a classical doubly clamped beam in equation (39) which is normalized as:

\[ \phi_0(\hat{x}) = \frac{\phi_0(\hat{x})}{\max |\phi_0(\hat{x})|} \quad \text{for} \quad \hat{x} \in [0, 1] \]  

(41)

Further, assuming that the initial curvature of the arch is similar to its first mode shape, we can substitute \( \hat{\omega}_0(\hat{x}) = \phi_0(\hat{x}) \) together with \( \hat{\omega}(\hat{x}, \hat{t}) = q(\hat{t})\phi_0(\hat{x}) \) in equation (35), multiply both sides by \( \phi_0(\hat{x}) \) and integrate over the arch length. This procedure yields:

\[ m_1 \frac{d^2 q}{d\hat{t}^2} + m_1 \hat{C}_v \frac{dq}{d\hat{t}} + \hat{C}_{ad} \alpha_1 \frac{dq}{d\hat{t}} + [\alpha_{10} + 2hq^2 \theta \alpha_1] q(\hat{t}) \]

\[ - 3h \theta \alpha_1 q^3(\hat{t}) + \theta \alpha_1 q^3(\hat{t}) = \int_0^1 \frac{\beta \rho \phi(\hat{x})}{(1 + (h - q(\hat{t}))\phi(\hat{x}))^2} \, d\hat{x}, \]  

(42)

where we have defined:

\[ m_1 = \int_0^1 q^2(\hat{x}) \, d\hat{x}, \]  

(43)

\[ \alpha_{10} = \int_0^1 \left[ \frac{\theta q(x)}{d\hat{x}^2} + \frac{d^2 q}{d\hat{x}^2} \right] \phi(\hat{x}) \, d\hat{x}, \]  

(44)

\[ \alpha_{11} = -\int_0^1 \left[ \frac{\phi(\hat{x})}{d\hat{x}^2} \right] d\hat{x}, \]  

(45)

\[ \alpha_{13} = -\int_0^1 \frac{q(x)}{d\hat{x}^2} \frac{q(x)}{d\hat{x}} \int_0^1 \left( \frac{dq(\hat{x})}{d\hat{x}} \right)^2 \, d\hat{x}, \]  

(46)

\[ \alpha_{14} = \int_0^1 \left( \frac{\phi(\hat{x})}{d\hat{x}} \right)^2 \, d\hat{x}. \]  

(47)

Integration by parts of (44)–(46) and using the classical and non-classical boundary conditions results in the following definitions:

\[ \alpha_{10} = \int_0^1 \left[ \left( \frac{d^2 q}{d\hat{x}^2} \right)^2 - \theta_1 \left( \frac{d^2 q}{d\hat{x}^2} \right)^2 \right] d\hat{x}, \]  

(48)

\[ \alpha_{11} = \int_0^1 \left( \frac{dq}{d\hat{x}} \right)^2 d\hat{x}, \]  

(49)

\[ \alpha_{13} = \alpha_{11}^2, \]  

(50)

With further change of variables \( \tau = t \sqrt{\alpha_{10}/m_1} \), we are left with the following ordinary differential equation from which the amplitude \( q \) of the fundamental mode is to be calculated:

\[ \frac{d^2 q}{d\tau^2} + \frac{\mu}{\tau} \frac{dq}{d\tau} + F(q) = 0, \]

\[ F(q) = (1 + 2h^2)q - 3h\alpha q^2 + \alpha^2 q^3 - \beta f_c(q), \]  

(51)

with \( \mu = \hat{C}_v \sqrt{m_1/\alpha_{10}} + \hat{C}_{ad} \alpha_1 \sqrt{m_1/\alpha_{10}}, \)

\[ \beta = h \alpha_{10}, \]

\[ \alpha = \alpha_{10}\alpha_1 \] and \( f_c(q) \) accounting for the modal electrostatic force, given by the following formula:

\[ f_c(q) = \int_0^1 \frac{q(\hat{x})}{(1 + (h - q(\hat{t}))\phi(\hat{x}))^2} d\hat{x}. \]  

(52)

An analytical solution to the integral given in (52) is not available using the classical mode shape implemented here. However, one can obtain an analytical approximation for the modal electrostatic force, provided that \( \phi(\hat{x}) \) is replaced by the approximation \( 1 - \cos(2\pi\hat{x})/2 \), which yields [13]:

\[ f_c(q) = \frac{1}{2\sqrt{(1 + h - q)^3}}. \]  

(53)

Equations (51) and (53) provide the reduced order strain gradient model which will be used in the rest of the paper for the investigation of the size-dependent behaviour of the electrostatically actuated nano-arch. Note that neglecting all of the material length scale parameters, i.e. for \( l_0 = l_1 = l_2 = 0 \), coefficients of equation (51) reduce to their classical form, and we are left with the reduced-order classical model which will also be used in the following sections for obtaining the results of the classical theory.

4. Fixed points analysis

As the first step toward analysis of the nonlinear dynamics of the proposed nano-arch, we calculate the fixed points, or static equilibrium states, of the equation of motion (51). Supposing the geometrical and material parameters to be known, one can obtain the fixed point \( q \) versus the non-dimensional voltage parameter \( \beta \) by solving equation \( F(q) = 0 \) given by (51). This leads to the derivation of a bifurcation diagram of fixed points. Figure 2(a) displays a typical bistable bifurcation diagram with schematic representations of the arch configuration at various stages. As shown in this figure, the arch, which initially rests in its lower equilibrium configuration for \( q = 0 \), deflects upward under the applied electrostatic load. Once the electrostatic loads

---

**Table 1. Dimensionless parameters.**

<table>
<thead>
<tr>
<th>( \hat{x} = \frac{x}{L} )</th>
<th>( h = \frac{h_0}{\delta_0} )</th>
<th>( \dot{t} = t \sqrt{\frac{S}{\rho b d^4}} )</th>
<th>( \dot{w} = \frac{\pi}{\delta_0} )</th>
<th>( \dot{N}_0 = \frac{N_0 L^2}{S} )</th>
<th>( \theta_1 = \frac{E b d \gamma_0^2}{2S} )</th>
<th>( \beta = \frac{E_b d L^4 V^2}{2\eta_0^2 S} )</th>
<th>( \ddot{d} = \frac{d}{\delta_0} )</th>
<th>( C_c = \frac{C_L^2}{\sqrt{S} \delta_0} )</th>
<th>( C_{ad} = \frac{C_{ad}}{\sqrt{S} \delta_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>( m_1 \hat{C}<em>v \frac{dq}{d\hat{t}} + \hat{C}</em>{ad} \alpha_1 \frac{dq}{d\hat{t}} + [\alpha_{10} + 2hq^2 \theta \alpha_1] q(\hat{t}) )</td>
<td>( 3h \theta \alpha_1 q^3(\hat{t}) )</td>
<td>( \theta \alpha_1 q^3(\hat{t}) )</td>
<td>( \int_0^1 \frac{\beta \rho \phi(\hat{x})}{(1 + (h - q(\hat{t}))\phi(\hat{x}))^2} , d\hat{x} )</td>
<td>( \alpha_{10} = -\int_0^1 \left[ \frac{\phi(\hat{x})}{d\hat{x}^2} \right] d\hat{x}, )</td>
<td>( \alpha_{11} = -\int_0^1 \left[ \frac{\phi(\hat{x})}{d\hat{x}^2} \right] d\hat{x}, )</td>
<td>( \alpha_{13} = -\int_0^1 \frac{q(x)}{d\hat{x}^2} \frac{q(x)}{d\hat{x}} \int_0^1 \left( \frac{dq(\hat{x})}{d\hat{x}} \right)^2 , d\hat{x}, )</td>
<td>( \alpha_{14} = \int_0^1 \left[ \frac{\phi(\hat{x})}{d\hat{x}} \right]^2 , d\hat{x}. )</td>
<td></td>
</tr>
</tbody>
</table>
reaches a critical value $\beta_{ST}$, the arch undergoes snap-through and precipitately moves to its upper equilibrium position. Provided that the load keeps increasing, the arch reaches the pull-in critical point $\beta_{PI}$ and the two electrodes stick together suddenly. Decreasing the electrostatic load, while the arch is still in its upper configuration, directs the arch toward the snap-back (or release) critical point $\beta_{SB}$ where the arch switches to its initial lower equilibrium position. The formation of a hysteresis loop in the bifurcation diagram of fixed points is obvious from figure 2(a). Note that only solid curves in figure 2(a) are stable and can be achieved experimentally [8].

For a typical arch with the material and geometrical parameters given in table 2, one can obtain the bifurcation diagrams shown in figure 2(b) which depicts the equilibrium states versus dc voltage parameter for various values of the initial rise parameter $h$. Note that figure 2(b) is obtained for a classical arch with the material size parameters set to zero.

As shown in figure 2(b), for the arch with zero initial rise parameter resembling a straight beam, two fixed points exists for each value of the dc voltage, the larger of which will be proved in the next section to be unstable. Only one instability point associated with the pull-in $\beta_{PI}$ is present in figure 2(b) for $h = 0$. As the initial rise parameter exceeds a critical value $h_0$, the arch exhibits the possibility of bistability by undergoing snap-through and snap-back at voltages $\beta_{ST}$ and $\beta_{SB}$[8, 13]. Corresponding critical bifurcation points are shown in figure 2(b). Thus for the arch with the initial elevation $h > h_0$, bistability is expected at specific values of the dc voltage parameter. These specifications of the arch are already discussed in the literature [8, 13]. Also, one can obtain the condition of bistability by finding the minimum value of the initial rise $h_0$ required for bistability. This can be achieved by the fact that for the bistable arch the equation $q d \beta / dq = 0$ has three real-valued roots each of which corresponds to one of the critical bifurcation points [8].

Here we aim to study the size-dependent nature of the bifurcation diagrams depicted in figure 2(b). To this aim, we scale the dimensions of the typical arch given in table 2 by several scaling factors; then for each scaled arch we calculate the minimum initial rise parameter $h_0$ required for bistability either with the classical theory or with the strain gradient theory. We assume three equal material length scale parameters $l_0 = h_0 = 0.1$ = CI in the implementation of the strain gradient theory [43, 46]. Figure 3 depicts the critical value $h_0$ versus the $d / CI$ ratio with $CI = 100$ nm; thus, the ratio $d / CI = 10$ corresponds to a beam of $500 \times 30 \times 1 \mu \text{m}^3$. Note that as the thickness of the arch is varied, the ratios $L / d$, $b / d$ and $\beta_0 / d$ are kept constant. As shown in figures 2(b) and 3, the classical theory (that is for $CI = 0$) the plateau in the $q - \beta$ curve indicating the transition to instability, happens at $h_0 = 0.194$. When $d / CI$ decreases to 2, the critical value increases to $h_0 = 0.358$, as predicted by the strain gradient theory. This means that based on the non-classical theory, considerably deeper nano-arches become bistable.

We can obtain the critical bifurcation values of the voltage for the pull-in $\beta_{PI}$, snap-through $\beta_{ST}$ and snap-back $\beta_{SB}$ instabilities by solving the equation $q d \beta / dq = 0$ for $\beta$. Figure 4 shows the obtained values from the non-classical theory, scaled by that from the classical theory. It’s clear that when the arch thickness is much larger than the material length scale parameter, then the results of the classical theory and the strain gradient theory converge. However, as the nano-arch structure scales down and the thickness of the nano-arch $d$ approaches the material length scale parameter $CI$, the critical voltages vary significantly and consequently the bistability region shrinks for small $d / CI$ ratios.

Based on the observations in figures 3 and 4, we can infer that as the structure scales down the size-dependent nature of the bifurcation diagram emerge, as predicted by the strain gradient theory. Based on these effects the bistability region shrinks at the smaller sizes. This is clear in figure 5 which illustrates the bifurcation diagram for various scales of the typical arch parameters given in table 2 with a fixed initial rise parameter $h = 0.32$ and using the strain gradient theory as well as the classical theory.

For better understanding of the size effects on the variation of the bistability region of the nano-arch, we can plot the critical values of $\beta_{PI}$, $\beta_{ST}$ and $\beta_{SB}$ obtained at various values...
5. Stability of the fixed points

We can study the stability nature of the fixed points, obtained in the previous section, by linearizing the undamped single degree of freedom equation of motion (51) around an equilibrium configuration, obtaining the Jacobian matrix of the linearized system:

\[
J = \begin{bmatrix}
0 & 1 \\
-\lambda^2 & 0
\end{bmatrix}
\]

where due to equation (51), we have:

\[
\lambda^2 = \frac{\partial^2 E}{\partial q^2} \bigg|_{q=q_0}.
\]  

Eigenvalues of the linearized system are given by \( \lambda_{1,2} = \pm \alpha \lambda \). Thus, for a stable equilibrium position, \( \lambda \) would be real positive leaving a couple of complex conjugate eigenvalues. This happens for the fixed points located on the branches with positive slope in the bifurcation diagrams, figure 2 or 5. On the branches with negative slope, \( \lambda \) is complex and the system possesses a positive real-valued eigenvalue, confirming the unstable nature of the associated equilibrium condition.

Based on the given descriptions, \( \lambda \) represents the fundamental non-dimensional frequency of the system in the vicinity of a stable configuration. Therefore, for each stable branch of the bifurcation diagram, a positive real-valued branch for \( \lambda \), and hence the dimensional fundamental frequency is expected. Figure 7 depicts the arch’s fundamental frequency in Hz versus the applied dc voltage for the nano-arch parameters given in table 2 for the two cases of \( h = 0 \) and \( h = 0.32 \) with \( d/Cl = 7 \) obtained from the classical and non-classical theory. It is worth summarizing here that for any prescribed positive value of the non-dimensional parameter \( q < 1 + h_0/g_0 \), the dc voltage (V in Volts) and the corresponding natural frequency (\( \omega \) in Hz) are obtained, as a function of dimensional quantities, from the following relations:

\[
V^2 = \frac{2gh_0^3}{bLSE}[2Sa_{10} + bdE\alpha_1^2(2h_0^2 - 3g_0h_0q + g_0^2q^2)]
\]

\[
\times \sqrt{1 + \frac{h_0}{g_0} - q^3},
\]

where \( \omega = \frac{1}{\rho bd m L^2} \left[ Sa_{10} + Ebd\alpha_1^2(h_0^2 - 3g_0h_0q + 1.5g_0^2q^2) \right] \)

\[
\omega^2 = \frac{3q(2Sa_{10} + bdE\alpha_1^2(2h_0^2 - 3g_0h_0q + g_0^2q^2))}{4\left(1 + \frac{h_0}{g_0} - q^3\right)},
\]
with $\alpha_0 = 198.463 - 9820\theta_0$, $\alpha_1 = 4.8777$ and $m_1 = 0.3965$ as calculated before, and with the other parameters described in table 3 and equation (15). Note that in our studies $\theta_0$ falls in the range $10^{-7}$ to $10^{-8}$; thus, we can use a fixed value of $\alpha_0 = 198.463$ in our calculations.

Figure 7(a) suggests that the strain gradient theory predicts an increase of in the fundamental frequency of the straight beam at small dc voltages. Thus, hardening of the beam is expected due to the size effects. However, the pull-in voltage predicted by the strain gradient theory is slightly larger than that predicted by the classical theory. In the case of the initially curved beam in figure 7(b), the fundamental frequency at small dc voltages is approximately two times larger than that of the straight beam. This is due to the hardening effect of the mid-plane stretching nonlinearity imposed by the initial curvature [13]. Also, due to the size effects, the fundamental frequency increases at small dc voltages. Moreover, due to the initial curvature of the beam, the pull-in voltage decreases in comparison with the case of straight beam.

One can see from figure 7 that, as the structure approaches the nano scale, the fundamental frequency tends to the MHz range which suddenly drops to zero at the snap-through and snap-back points. Considering that the system is still functional after these instabilities, because of the bistable nature, the large range of variation of the natural frequency versus the small changes in the applied dc voltage can open many applications such as the design of highly sensitive sensors. For a detailed discussion of this see [13].

Figure 7(b) also shows that, on the stable branch of equilibrium points which is located between the snap-back and pull-in critical points, the non-classical theory predicts a reduction of the fundamental frequency of the arch, compared to what is predicted by the classical theory. Also, the shrinkage of the bistability region due to the size effects is observed in figure 7(b).

In order to study the effects of the size parameters on the dimensionless frequencies of the nano-arch, we can compare the results for various scales of the system. Figure 8 depicts the values of $\lambda$ for various scales of the arch’s structure using strain gradient theory, showing that the size-dependent nature of the natural frequency is more pronounced for smaller sizes.

6. Conclusions

We aimed in this paper to study the size-dependent nature of nonlinear behaviours in a bistable NEMS comprised of a doubly clamped initially curved shallow nano-beam under dc electrostatic actuation. Toward this aim, we presented the details of the procedure for derivation of the mathematical model which employs strain gradient theory together with the Euler–Bernoulli shallow arch assumptions. We first non-dimensionalized the obtained partial differential equation and then converted it to a nonlinear single degree of freedom equation, using the first mode Galerkin projection method.
Figure 7. Fundamental frequency versus the applied dc voltage for $d/Cl = 7$ with (a) $h = 0$ and (b) $h = 0.32$ obtained by the classical theory and the strain gradient theory. Dashed arrows on the classical curves in part (b) show the loading path and clarify formation of a bistability loop. The same path is followed on the strain gradient curves.

Table 3. Nomenclature.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>Width</td>
<td>m</td>
<td>$G$</td>
<td>Shear modulus</td>
<td>Pa</td>
</tr>
<tr>
<td>$g_0$</td>
<td>Fixed gap between electrodes</td>
<td>m</td>
<td>$l_0$</td>
<td>Material parameter for dilatation gradients</td>
<td>m</td>
</tr>
<tr>
<td>$h_0$</td>
<td>Initial elevation</td>
<td>m</td>
<td>$l_1$</td>
<td>Material parameter for deviatoric stretch gradients</td>
<td>m</td>
</tr>
<tr>
<td>$d$</td>
<td>Thickness</td>
<td>m</td>
<td>$l_2$</td>
<td>Material parameter for rotation gradient</td>
<td>m</td>
</tr>
<tr>
<td>$A$</td>
<td>Cross sectional area</td>
<td>m$^2$</td>
<td>$\nu$</td>
<td>Poisson’s ratio</td>
<td>—</td>
</tr>
<tr>
<td>$I_y$</td>
<td>Cross sectional moment of inertia</td>
<td>m$^4$</td>
<td>$V$</td>
<td>Electrostatic voltage</td>
<td>V</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mass density</td>
<td>kg m$^{-3}$</td>
<td>$C_v$</td>
<td>Viscous damping coefficient</td>
<td>N s m$^{-2}$</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus</td>
<td>Pa</td>
<td>$C_{sd}$</td>
<td>Structural damping coefficient</td>
<td>N s m$^{-1}$</td>
</tr>
<tr>
<td>$\bar{E}$</td>
<td>Effective Young’s modulus</td>
<td>Pa</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 8. Non-dimensional natural frequency of vibrations for various scales of the typical nano-arch with $h = 0.32$ versus the non-dimensional voltage parameter (a), and the equilibrium deflection (b) using strain gradient theory.
Figure A1. Schematics of the shallow arch.

Through calculation of the fixed points of the lumped non-linear equation, bifurcation diagrams of fixed points were obtained. We discussed in detail the effect of size-dependent parameters on the critical pull-in, snap-through and snap-back values, using the bifurcation diagram of fixed points. For instance, we showed that, contrary to the classical theory which predicts a size-independent minimum value of the non-dimensional initial elevation required for bistability, this minimum value increases as the nano-structure scales down predicted by the strain gradient theory. Moreover, the non-dimensional pull-in and snap-through voltages at various scales, which are size-independent from the classical point of view, decrease as the structure scales down, while the non-dimensional snap-back voltage increases when reducing the nano-arch size. All of these effects reduce the bistability region in the parameter space of the NEMS.

The stability nature of the fixed points has been studied by calculation of the non-dimensional fundamental frequency of the nano-arch at each fixed point. Results show that, for a bistable nano-arch, due to the hardening effects, the fundamental frequency increases at lower values of the applied dc voltages. But when the arch snaps-through to the higher stable configuration, the fundamental frequency predicted by the non-classical theory reduces, as well as the pull-in voltage. Also, while the non-dimensional fundamental frequency is independent of the size from the classical point of view, due to the non-classical theory, a large reduction of the frequency is found depending on the value of the scaling parameter.

Results of this work will be helpful in the analysis and design of the bistable arch NEMS which will have natural frequencies in the scale of several MHz. The proposed mathematical model can be used for further investigations of the nonlinear dynamics of this family of NEMS, such as the forced vibrations of the system. Overall, this work elucidates the size-dependent bistability and nonlinear dynamics of the arch NEMS. Experimental investigations are needed for definite validation of the obtained results. This is left as future work.

Appendix. Axial strain derivation

We first assume a shallow arch with small initial curvature described by the function \( w_0(x) \). Suppose a material point located on the mid-line (neutral axis) to be described by \((x, w_0(x))\) initially. After deformation, location of this material point on the deformed mid-line is given by \((x + u(x, t), w(x, t))\), as shown in figure A1. Supposing \( ds \) and \( ds^* \) as the length of line elements on the undeformed and deformed mid-lines, the engineering definition of the extensional strain at the midplane is given by:

\[
epsilon_E = \frac{ds^*-ds}{ds}.
\]  

(58)

In the strain density energy \( U \) given by equation (1), the appropriate strain is the Green strain, defined as:

\[
epsilon_g = \frac{1}{2} \left( \frac{ds^*}{ds} \right)^2 - \frac{ds^*}{ds^2}.
\]  

(59)

Combining (58) and (59), one obtains the following relation:

\[
epsilon_g = \epsilon_E + \frac{1}{2} \epsilon_E^2.
\]  

(60)

which confirms that both definitions yield the same result at small strains. However, \( ds \) and \( ds^* \) are calculated as below:

\[
ds^2 = dx^2 + dw_0^2,
\]  

(61)

\[
ds^2* = (dx + du)^2 + dw^2.
\]  

(62)

Substituting (61) and (62) into (59) yields:

\[
epsilon_g = \frac{1}{2} \left( \frac{dx^2 + 2 dx du + du^2 + dw^2 - dx^2 - dw_0^2}{dx^2 + dw_0^2} \right).
\]  

(63)

Supposing \( dw_0/dx \ll 1 \) and \( |w(x, t) - w_0(x)| \ll u(x, t) \), the axial strain of mid-line is obtained as:

\[
epsilon_g = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 = \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2.
\]  

(64)

Additionally, the expression for the change in curvature due to the lateral deformation is obtained assuming that \((dw/dx)^2 \ll 1\) and \(dw_0/dx \ll 1\):

\[
\kappa = -\left( \frac{d^2 w}{dx^2} - \frac{d^2 w_0}{dx^2} \right).
\]  

(65)
Thus, for any material point having a distance $z$ from the mid-plane, the extensional strain component $e_{11}$ is calculated using (64) and (65):

$$e_{11} = \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} = \left( \frac{\partial^2 w}{\partial x^2} \right)^2.$$

which is used in the derivation of equations of motion in this paper. For more information on the above assumptions, see [65].

References
