Analytical study on torsion of shape-memory-polymer prismatic bars with rectangular cross-sections

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Abstract
In this paper, the response of shape memory polymer (SMP) bars with rectangular cross-sections under torsional loadings is analytically studied. To this end, we first reduce the recently proposed small-strain 3D phenomenological constitutive model for SMPs to the shear case. Then, an analytical solution for torsional response of SMP rectangular bars in a full cycle of stress-free strain recovery is derived. We also implement the 3D constitutive equations in a finite element program and simulate a full cycle of stress-free strain recovery of a rectangular SMP bar. Analytical and numerical results are then compared showing that the analytical solution gives, besides the global load–deflection response, accurate stress distributions in the cross-section of the rectangular SMP bar. Some case studies are also presented to show the validity of the presented analytical method. Results are compared with the experimental data recently reported in the literature which showing an agreement between the predicted results and experiments. The analytical solution can also be used for analysis of helical springs in which both the curvature and pitch effects are negligible. This is the case for helical springs with large ratios of mean coil radius to the cross-sectional equivalent radius (spring index) and also small pitch angles. Using this solution simplifies the analysis of the helical springs to that of the torsion of a straight bar with rectangular cross-section.

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1. Introduction

Since the first observation of the shape memory effect in some polymers, research on SMPs has been an active field. Recently, manufacturing of SMP devices, has considerably increased, thanks to their unique ability in recovering large stored strains (Lendlein et al., 2009; Lendlein & Langer, 2002; Liang, Rogers, & Malafeew, 1997; Barot, Rao, & Rajagopal, 2008).

SMPs have been researched, developed, and utilized in a wide range of applications such as advanced technologies in the aerospace, medical, microelectromechanical systems (MEMS) and oil exploration industries (Díaz Lantada et al., 2010; Monkman, 2000; Ghosh, Reddy, & Srinivasa, 2012). Compared to other smart materials such as shape memory alloys, SMPs have ability of large elastic deformation, low energy consumption for shape programming, potential biocompatibility, low cost, low density, biodegradability and excellent manufacturability (Baghani, Naghdabadi, & Arghavani, 2012b, 2012a; Beloshenko, Varyukhin, & Voznyak, 2005; Cheng & Li, 2008; Jarali, Raja, & Upadhya, 2010; Sar, 2010; Ghosh & Srinivasa, 2011).

Despite of all advantages mentioned in the above, the much lower stiffness of un-reinforced SMPs prevents them from practical applications in cases where a large recovery stress is required (e.g., as actuators). To overcome such disadvantage, different reinforced-SMP composites have been developed and utilized (Li & Wang, 2011; Xu & Li, 2010).

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Up to now, the characterization of SMP behavior has been carried out with traditional uniaxial tension and compression tests (Liu, Gall, Dunn, Greenberg, & Diani, 2006; Xu & Li, 2010), however torsional thermomechanical tests can play an important role in the characterization of an SMP constitutive model. Besides, torsional test may be useful to verify and evaluate the validity of a 3D constitutive model as well as its numerical counterpart.

Recently, Diani et al. (2011) have performed a series of experiments on unconstrained recovery of rectangular bars made of SMPs. They also studied the effect of rate of applied temperature on the responses of SMPs. The necessity of obtaining an accurate analytical and numerical solution for the SMP devices, besides their new emerging applications, motivated the authors to seek analytical and numerical solutions for SMP rectangular bars. Although, analytical and numerical modeling of torsion of rectangular bars for some materials are presented in the literature, but to the knowledge of the authors, there are no relevant publications in this context for SMPs.

In this paper, employing the constitutive model recently presented by Baghani, Naghdabadi, Arghavani, and Sohrabpour (2012c), we present an analytical solution for SMP rectangular bars in a stress-free strain recovery process. In addition, to have a numerical tool for comparing the results of analytical solution with, we also develop a 3D finite element solution. Employing the finite element solution, we will verify the validity of assumptions used in the analytical solution as well as showing its accuracy. The solution is also validated by comparing the predicted results with experimental data reported in the literature. Moreover, it is observed that the solution time in the analytical method is much less than the computational time for the finite element simulations.

This solution can also be used for helical springs (with rectangular cross-sections) under axial loadings in which both the curvature and pitch effects are negligible. This is the case for helical springs with large ratios of mean coil radius to the cross-sectional equivalent radius (spring index) and also small pitch angles (such a spring is depicted in Fig. 1). Using this solution simplifies the analysis of the helical springs to that of the torsion of a straight bar with a rectangular cross-section.

This paper is organized as follows. In Section 2, a 3D constitutive model for SMPs is briefly reviewed. In Section 3, the 3D constitutive relations are reduced to a constitutive equation for the cases in which only the shear strains and stresses exist. In the case of SMP rectangular bars, we also make use of the constitutive equations reduced to shear case and solve the torsion of rectangular bars analytically in a stress-free strain recovery cycle. In Section 4, finite element simulations are given and some case studies are reported for rectangular bars. In addition, a comparison is made between the proposed analytical solution, finite element simulations and experimental data available in the literature. Finally, we present a summary and draw conclusions in Section 5.

2. A 3D constitutive model for SMPs

In this section, a typical cycle of an SMP under thermomechanical loadings is described. A 3D constitutive model for SMPs is then briefly reviewed.

From a macroscopic point of view, shape memory effect can be characterized in a stress–strain-temperature diagram as illustrated in Fig. 2. The thermomechanical cycle starts at a strain- and stress-free state while temperature is high, $T_h$ (point ☐, permanent shape). At this point, a purely mechanical loading is applied to SMP and the material demonstrates a rubbery behavior up to point ☐. At point ☐, strain is held fixed (external loadings) and the temperature is decreased until the rubber-like polymer drastically turns into a glassy polymer at a low temperature $T_l$ (point ☒, fixed shape). In fact, in the neighborhood of the transition temperature $T_g$, SMP exhibits a combination of rubbery and glassy behaviors. Subsequently, the material is unloaded. Regarding the much higher stiffness of the glassy phase in comparison to the rubbery phase, after unloading, strains change slightly (point ☐). Finally, we increase the temperature up to $T_h$. It is seen that the strain will relax and the original permanent shape can be recovered (point ☐). This cycle is called a stress-free strain recovery in SMP applications. In practice, other types of recoveries may happen. If at point ☐, the strain is fixed and the temperature is increased, the fixed-strain stress recovery (point ☐) happens. Dotted line in Fig. 2, illustrates the mentioned behavior.

We now use an equivalent representative volume element (RVE) of the material composed of a glassy phase, a rubbery phase and a hard segment (Fig. 3) to derive a 3D SMP constitutive equation. It is assumed that the volume fraction of the
The hard phase is constant while the rubbery and glassy phases are able to be transformed to each other through external stimuli of heat. Assuming small strains, the mixture rule in the RVE is used and the total strain is decomposed as:

\[ e = \phi_p (\zeta_r e^r + \zeta_g e^g + e^s) + \phi_h e^h + e^T \]

where \( e^r \), \( e^g \) and \( e^h \) stand for the elastic strain in the rubbery and glassy phases and elastic strain in the hard segment, respectively, while \( e^s \) denotes the stored strain and \( e^T \) represents the thermal strain and is defined by

\[ R a_T (T) dT \]

where \( a_T \) is the effective thermal expansion coefficient. Also, \( n_r \) and \( n_g \) are volume fractions of the rubbery and glassy phases, respectively, with constraint \( n_r + n_g = 1 \). It is assumed that \( n_r \) and \( n_g \) are only functions of temperature while \( \phi_p \) and \( \phi_h \) (volume fraction of SMP segment and hard segment, respectively) are constant parameters, with constraint \( \phi_p + \phi_h = 1 \).

It is noted that a prescribed evolution equation is employed for \( e^s \). This equation is derived using the unconstrained strain recovery of the material as a function of temperature. Moreover, the evolution equation for stored strain \( e^s \), is defined as (Baghani et al., 2012c):

\[ \dot{e}^s = \left( k_1 \frac{e^s}{\zeta_g} + k_2 \frac{e^g}{\zeta_g} \right) \cdot \frac{\zeta_g}{\zeta_g} : \begin{cases} k_1 = 1, & k_2 = 0; \quad \dot{T} < 0 \\ k_1 = 0, & k_2 = 1; \quad \dot{T} > 0 \\ k_1 = 0, & k_2 = 0; \quad \dot{T} = 0 \end{cases} \]

where \( (\cdot) = \partial / \partial T \) and \( (\dot{\cdot}) = \partial / \partial t \) denote the derivative with respect to temperature and time, respectively.

The free-energy density function is also defined in the form of:

\[ \Psi(e, T, \zeta_r, e^r, e^g, e^h) = \phi_p [(1 - \zeta_g) \Psi_r(e^r) + \zeta_g \Psi_g(e^g)] + \phi_h \Psi_h(e^h) + \Psi_T(T) \]

where elastic strain energies are assumed to be in the form of \( \Psi_\beta(e^\beta) = \frac{1}{2} e^\beta : \kappa_\beta : e^\beta, \beta = r, g, h \); in which \( \kappa_\beta \)'s are fourth-order elasticity tensors corresponding to each phase. Thermal energy is also denoted by \( \Psi_T \). Satisfying the second law of ther.mo-
Table 1
Time-continuous form of the 3D constitutive model (Baghani et al., 2012c).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Time-continuous form of the 3D constitutive model (Baghani et al., 2012c).</th>
</tr>
</thead>
<tbody>
<tr>
<td>External variables:</td>
<td>( \mathbf{e}, T )</td>
</tr>
<tr>
<td>Internal variables:</td>
<td>( \mathbf{e}_i, e )</td>
</tr>
<tr>
<td>Kinematics:</td>
<td>( \mathbf{e} = \phi_p(\xi \mathbf{e}^f + \zeta \mathbf{e}^g) + \phi_h \mathbf{e}^h + \mathbf{e}^T )</td>
</tr>
<tr>
<td>Stress:</td>
<td>( \mathbf{\sigma} = \frac{\partial W}{\partial \mathbf{e}} - \mathbf{h} \mathbf{E} )</td>
</tr>
<tr>
<td>Evolution equation:</td>
<td>( \dot{\mathbf{e}}^\beta = \frac{\partial W}{\partial \mathbf{e}} \left( k_1 \mathbf{e}^f + k_2 \mathbf{e}^g \right) ) with ( \begin{cases} k_1 = 1, k_2 = 0, &amp; T &lt; 0 \ k_1 = 0, k_2 = 1, &amp; T &gt; 0 \ k_1 = 0, k_2 = 0, &amp; T = 0 \end{cases} )</td>
</tr>
<tr>
<td>Free-energy density function:</td>
<td>( \Psi(\mathbf{e}, T, \mathbf{e}_i, \mathbf{e}^f, \mathbf{e}^g) = \phi_p(1 - \xi)\Psi_f(\mathbf{e}) + \phi_h \Psi_h(\mathbf{e}^h) + \Psi_T(T) )</td>
</tr>
</tbody>
</table>

where \( \mathbf{\sigma} \) and \( \mathbf{e} \) are the stress and strain, respectively, and subscripts \( xz, yz \) and \( xz, yz \) denote the components of tensors. For the sake of simplicity, we drop the subscripts \( xz, yz \) and \( xz, yz \) in the formulation. It is highlighted that where \( \tau, \gamma \) and \( \gamma^s \) are used without any subscripts, it means that it is valid for both components \( xz \) and \( yz \). We also drop the subscript \( g \) and show \( \mathbf{e}_i \) with \( \xi \) in the following. Substituting (4) into equations of Table 1 yields:

\[
\begin{align*}
\dot{\gamma} &= \gamma_p(1 - \xi)\mathbf{e}^f + \xi \gamma^g + \gamma^h \\
\tau &= \mathbf{G}_r \gamma^r = \mathbf{G}_h \gamma^h \\
\dot{\gamma}^s &= \frac{\partial W}{\partial \mathbf{g}} \left( k_1 \gamma^f + k_2 \gamma^g \right)
\end{align*}
\]

where \( \mathbf{G}_r, \mathbf{G}_h \) and \( \mathbf{G}_n \) are elastic shear modulus of the rubbery, glassy and hard phases, respectively. We now solve the set of Eqs. (5)–(7) in a stress-free strain recovery cycle (through path \( \alpha \rightarrow \beta \rightarrow \gamma \rightarrow \delta \rightarrow \epsilon \) shown in Fig. 2). During loading at \( T = T_h \) (path \( \alpha \rightarrow \beta \)), we apply the twist angle \( \theta \) until \( \theta = \theta_0 \). Whereas the temperature is higher than \( T_h \), no shape memory effect and strain storage occur. Hence, we have the elastic solution with rubbery-hard mixed phase material elastic modulus \( G_n \) (\( G_n = \phi_h G_h + \phi_r G_r \)). The classical elastic solution for torsion of rectangular bars presented by Prandtl (1903) and also reported in classical elasticity books (Timoshenko & Goodier, 1970; Boresi, Chong, & Lee, 2010) then can be used. The following relations describe the response of rectangular bar under torsion:

\[
\begin{align*}
\nabla^2 \Phi &= -2G_n \theta \\
\Phi &= 0 \quad \text{on} \quad S \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\end{align*}
\]

where \( \theta \) denotes the rotation of the bar per length and \( S \) stands for the lateral surface. \( \Phi \) is also Prandtl torsion function and is given by the following expression (Timoshenko & Goodier, 1970):

---

1 Although normal components of thermal strains are present but they do not contribute on the shear response of SMP rectangular bars.
\[\Phi = G_n \theta (h^2 - x^2) = \frac{32 G_n \theta h^2}{\pi^2} \sum_{n=1,3,5,\ldots} \left( \frac{(-1)^{(n-1)/2} \cos \left( \frac{\pi n x}{2h} \right) \cosh \left( \frac{\pi n y}{2h} \right)}{n^3 \cosh \left( \frac{\pi n b}{2h} \right)} \right)\] (9)

Moreover, shear stresses are related to Prandtl torsion function as:

\[
\begin{align*}
\tau_{xz} &= \Phi_x = \frac{16 G_n \theta h}{\pi^2} \sum_{n=1,3,5,\ldots} \left( \frac{(-1)^{(n-1)/2} \cos \left( \frac{\pi n x}{2h} \right) \sinh \left( \frac{\pi n y}{2h} \right)}{n^2 \cosh \left( \frac{\pi n b}{2h} \right)} \right) \\
\tau_{yz} &= -\Phi_y = 2 G_n \theta x - \frac{16 G_n \theta h}{\pi^2} \sum_{n=1,3,5,\ldots} \left( \frac{(-1)^{(n-1)/2} \sin \left( \frac{\pi n x}{2h} \right) \cosh \left( \frac{\pi n y}{2h} \right)}{n^2 \cosh \left( \frac{\pi n b}{2h} \right)} \right)
\end{align*}
\] (10)

In addition, applied twisting moment which is required to produce the twist angle \(\theta\) is obtained as:

\[M_t = 2 \int_A \Phi dA = G_n \theta \] (11)

where \(A\) is the cross-section area of the bar and \(J\) represents the polar moment of inertia of the cross-section. We now call the shear stresses at \(\theta = \theta_0\) as \(\tau_{xz}^0\) and \(\tau_{yz}^0\):

\[
\begin{align*}
\tau_{xz}^0 &= \tau_{xz}|_{\theta=\theta_0} \\
\tau_{yz}^0 &= \tau_{yz}|_{\theta=\theta_0}
\end{align*}
\] (12)

The temperature then will be decreased down to \(T = T_i\) (path \(\Box\sim\square\)). During this part of the cycle, the twist angle \(\theta\) is held fixed at \(\theta = \theta_0\). To solve Eqs. (5)–(7) during cooling \((k_1 = 1, k_2 = 0)\), taking derivative of (5) yields:

\[T < 0 \Rightarrow \dot{\gamma} = \dot{\tau} \left( \phi_p \left( 1 - \xi \right) \frac{1}{G_r} + \xi \frac{1}{G_g} \right) + \phi_n \frac{1}{G_n} \frac{\phi_p \tau_{xx}^0}{G_g} = 0\] (13)

Solving (13), we arrive at:

\[\tau = \tau_0 \left( \phi_p G_x G_r + \phi_p G_n (1 - \xi) G_g + \xi G_g \right) \frac{\phi_n}{G_r \left( \phi_p G_r + \phi_p G_n \right)} \] (14)

Moreover, during cooling \((k_1 = 1, k_2 = 0)\), Eq. (7) gives the following relation for stored shear strain:

\[\gamma^s = \frac{\tau_0}{\phi_p G_n G_g G_r} \left\{ \phi_p G_x G_r + \phi_p G_n (G_g (1 - \xi) + \xi G_g) \right\} \frac{\tau_{xx}^0}{G_r} \times \left\{ G_r (\phi_p G_r + \phi_p G_n) \right\} \frac{\phi_n}{\phi_p G_n G_r} = 0\] (15)
In last part of the cycle (after unloading at $T_l$), in order to release the stored strains and recover the original configuration of the SMP rectangular bar, the temperature will be increased up to $T_g$. First of all, the Prandtl torsion function should be defined in such a way that satisfy the equilibrium equation automatically. The only equilibrium equation that should be satisfied is:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = 0$$  \hspace{1cm} (16)

Eq. (16) expresses a necessary and sufficient condition for the existence of a stress function $\Phi(x,y)$ such that:

$$\begin{align*}
\tau_{xx} &= \Phi_y \\
\tau_{yz} &= -\Phi_x
\end{align*}$$  \hspace{1cm} (17)

We now define the equivalent shear modulus, $G$ as:

$$G^{-1} = \phi_p \left(1 - \xi \right) G_s^{-1} + \phi_n G_n^{-1}$$  \hspace{1cm} (18)

This definition enables us to find a similar relation to elastic solution for shear stresses during the heating process in the following form:

$$\tau = G(\gamma - \phi_p \gamma^p)$$  \hspace{1cm} (19)

Combining (17) and (19) yields:

$$\begin{align*}
\gamma_{xz} &= G^{-1} \tau_{xz} + \phi_p \gamma^p_{xz} = G^{-1} \Phi_y + \phi_p \gamma^p_{xz} \\
\gamma_{yz} &= G^{-1} \tau_{yz} + \phi_p \gamma^p_{yz} = -G^{-1} \Phi_x + \phi_p \gamma^p_{yz}
\end{align*}$$  \hspace{1cm} (20)

In addition, the compatibility equation gives the following relation:

$$\frac{\partial^2 \gamma_{yz}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial y^2} \Rightarrow \Phi_{xx} + \Phi_{yy} + G \phi_p (\gamma^p_{yz} - \gamma^p_{xy}) = C$$  \hspace{1cm} (21)

where $C$ is a constant parameter. To determine the twist angle per unit length of the bar, we recall that the rotation $\omega_z$, of a volume element relative to the $z$ axis is:

$$\frac{\partial \omega_z}{\partial z} = -\frac{1}{2} \left( \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} \right) = \theta$$  \hspace{1cm} (22)

Substituting relations (20) into (22) gives the following expression:

$$\Phi_{xx} + \Phi_{yy} + G \phi_p (\gamma^p_{yz} - \gamma^p_{xy}) = -2G\theta$$  \hspace{1cm} (23)

Moreover, during heating ($k_1 = 0$, $k_2 = 1$), Eq. (7) arrives at the following relation for stored shear strain:

$$\gamma^s = \frac{\tau_0}{G_{rh}} f(T)$$  \hspace{1cm} (24)

where $f(T)$ is:

$$f(T) = \frac{\xi G_{rh}}{\phi_p G_n G_s G_{tr}} \left( (G_r (\phi_n G_s + \phi_p G_n))^{\frac{\phi}{\phi_n}} (G_s (\phi_n G_r + \phi_p G_n))^{\frac{\phi}{\phi_s}} - 1 \right)$$  \hspace{1cm} (25)

Combining relations (10), (12), (24) and (25), we expand the expression for stored shear strain components as:

$$\begin{align*}
\gamma^s_{xz} &= -f(T) \left[ \frac{16\phi h}{\pi^2} \sum_{n=1,3.5...} \left( \frac{-1}{n} \right)^{1/2} \frac{\cos \left( \frac{\pi n}{2} \right) \cosh \left( \frac{\pi z}{2} \right)}{n^2 \cosh \left( \frac{\pi h}{2} \right)} \right] \\
\gamma^s_{yz} &= f(T) \left[ 2\theta_0 - \frac{16\phi h}{\pi^2} \sum_{n=1,3.5...} \left( \frac{-1}{n} \right)^{1/2} \frac{\cos \left( \frac{\pi n}{2} \right) \cosh \left( \frac{\pi z}{2} \right)}{n^2 \cosh \left( \frac{\pi h}{2} \right)} \right]
\end{align*}$$  \hspace{1cm} (26)

Substituting (26) into (23), we obtain:

$$\nabla^2 \Phi = -2G(\theta - \theta_0f(T))$$  \hspace{1cm} (27)

Comparing (27) and (8), it is concluded that the elastic solution derived in previous section can be used here only if in Eq. (8) the term $G_{rh}\theta$ be replaced by $G(\theta - \theta_0f(T))$. Therefore, in a similar method, the torsion in the heating process is obtained as:

$$M_t = 2 \int_A \Phi dA = G(\theta - \theta_0f(T))$$  \hspace{1cm} (28)
Whereas, in a stress-free strain recovery, after unloading, the torsional moment along bar axis \((z\text{-axis})\) vanishes, Eq. (28) arrives at:
\[ \theta = \theta_0 f(T) \]  
(29)

Remark. In order to give a better approximation of temperature, transient heat-transfer analysis of the studied structure should be performed. Considering only the time-dependency of the temperature, the lumped heat-transfer equation is expressed as:
\[ \frac{\partial T(t)}{\partial t} = -\kappa(T(t) - T_\infty(t)), \quad T(t = 0) = T_0, \quad \kappa = \frac{h' A^\prime}{\rho C_p V} \]  
(30)

where \(\rho\) denotes density, \(C_p\) is specific heat capacity and \(h'\) represents the heat convection coefficient of air. Also, \(A^\prime\) and \(V^\prime\) stand for total area and volume of the structure, while the ambient and initial temperature are represented by \(T_\infty\) and \(T_0\).
Solving (30) leads to the following relation for time-dependent temperature field:
\[ T(t) = e^{-\kappa t} \left[ T_0 + \kappa \int_0^t e^{\kappa \xi} T_\infty(\xi) d\xi \right] \]  
(31)

Although the constitutive model in Section 2 does not involve time-dependent behavior of the material but the temperature-rate dependent response of the material can be considered through the heat-transfer analysis.

4. Results

In order to have a 3D finite element solution, we use a numerical integration scheme in an implicit form and solve the nonlinear system of equations in Table 1. For discussions on the numerical solution, we refer to Baghani (2012).

For analysis of SMP rectangular bars via the finite element method, the 3D constitutive equations of Section 2, are implemented within a user-defined subroutine (UMAT) in the commercial non-linear finite element software ABAQUS/Standard. Details of implementing the constitutive relations in a displacement-based finite element formulation is given in Baghani et al. (2012c).

Moreover, the domain of field variable temperature will be solved using finite element method by software ABAQUS/Standard. Therefore, the heat-transfer and structural analysis are performed simultaneously.

In this section, analytical and numerical results of torsion of rectangular bars are compared with experiments recently reported by Diani et al. (2011). Initial configuration of the rectangular bar has been shown in Fig. 6. The rectangular bar has a length of 100 mm, width of 10 mm and thickness of 1 mm.

Material parameters applied in the simulations are adopted in Baghani, Naghdabadi, Arghavani, and Sohrabpour (2012d) for the experiments recently performed by Diani et al. (2011). The parameters together with their values are tabulated in Table 2. In Fig. 7, temperature–time response of a node at upper central free end of the rectangular bar with applied twist angle \(\theta_0 = 360^\circ\) is shown at different temperature rates (T.R.) of 0.9 °C/min and 2.5 °C/min. It is observed that, at the rate of 0.9 °C/min the structure experiences an almost thermal equilibrium state in the whole steps of the process. Hence, the difference between \(T_\infty\) and \(T\) is insignificant, while at the rate of 2.5 °C/min the bar does not get a chance to reach \(T_\infty\) quickly, thus, a gap appears between \(T_\infty\) and \(T\).

In Fig. 8, von-Mises stress and maximum logarithmic principal strain in temporary shape of the rectangular bar (corresponding to point 4 in Fig. 2) are illustrated while the applied twist angle at \(T_h\) is \(\theta_0 = 360^\circ\) and the temperature rate is 0.9 °C/min. As observed from Fig. 8b, the maximum strain value is about 5%. This value shows that the employment of a small-strain constitutive model is appropriate and in this case, there is no need to use a large-strain SMP constitutive model.
Twisting moment-temperature-angle diagram of an SMP in a stress-free strain recovery cycle is depicted in Fig. 9. The effect of temperature rate and applied twist angle on the response of SMP is also shown. As observed from Fig. 9, increasing the applied twist angle \( h_0 \), increases the amount of the required twisting moment \( (Mt) \). On the other hand, as already mentioned, increasing the temperature rate signifies the importance of heat transfer analysis. The effect of volume fraction of the hard segment \( \phi_h \) and also the width of the bar \( b \) is investigated in Fig. 10. As we expected, increasing \( \phi_h \) fortifies the structure and increases the stresses as well as the required twisting moment to produce the same deformation. Clearly, using a thicker bar increases the amount of twisting moment significantly. Moreover, in the same temperature rate (0.9 °C/min), the bar with larger width has a more temperature rate dependent response than its thinner counterpart.

Analytical solution for the shear stress \( \tau = \sqrt{\tau_{xx}^2 + \tau_{yy}^2} \) in the cross-section of rectangular bar is plotted in Fig. 11 while \( \theta_0 = 180^\circ \), width \( b = 4 \text{ mm} \) and T.R. = 0.9 °C/min (as shown in Fig. 7). Figs. 11a and 11b are corresponding to points \( \text{①} \) and \( \text{②} \) in Fig. 2, respectively with \( \phi_h = 0 \). In Figs. 11c and 11d, the effect of \( \phi_h \) is investigated and similar to Figs. 11a and 11b, shear stress \( \tau \) for \( \phi_h = 0.4 \) is depicted.
In Fig. 12, the analytical and numerical angle-temperature response of the bar in different applied twist angles $\theta_0$ are illustrated and the results are compared with the experimental data reported by Diani et al. (2011). As shown, there is a good qualitative agreement between the numerical results and the experimental data.
The effect of temperature rate on a stress-free strain recovery of the rectangular bar is investigated in Fig. 13 via both analytical solution and its finite element counterpart. According to Fig. 13, the experimental results show a 4°C shift in temperature dependent response of twist angle recovery. The same effect is observed in traditional 1D compression-tension experiments reported by Volk, Lagoudas, Chen, and Whitley (2010), Volk, Lagoudas, and Chen (2010) and Li and Xu (2011). As depicted in Fig. 13, although we took glassy temperature as a constant material parameter, but numerical results are capable of capturing the mentioned shift in twist angle-temperature response of the bar.

In Fig. 14, angular velocity of the free end of the bar as a function of temperature during stress-free heating at T.R. = 0.9°C/min is depicted. Analytical and numerical results are also compared with the

![Fig. 12. Angle recovery of the rectangular bar for different $\theta_0$s at T.R. = 0.9 °C/min (F.E. stands for finite element solution while A.S. stands for analytical solution). Experimental data are reported from Diani et al. (2011).](image1)

![Fig. 13. Angle recovery of the rectangular bar at different T.R.s (F.E. stands for finite element solution while A.S. stands for analytical solution). Experiments reported by Diani et al. (2011).](image2)

![Fig. 14. Analytical and finite element solution for angular velocity-temperature response of the free end of the bar with $\theta_0 = 360°$ during stress-free heating at T.R. = 0.9 °C/min. Experiments for two shape memory epoxy samples reported by Diani et al. (2011).](image3)
experimental data reported by Diani et al. (2011) for two shape memory epoxy samples. Fig. 14, indicates that there is a good agreement between the simulation results and the experimental data.

5. Summary and conclusions

In this paper, an analytical solution is presented for the torsion of SMP prismatic bars with rectangular cross-sections. We reduce a 3D phenomenological macroscopic constitutive model for SMPs to shear case and receive at an explicit expression for the shear stress in rectangular SMP bars under torsional loadings.

SMP rectangular bars are also analyzed using a 3D finite element method. The results of presented analytical and 3D finite element solutions are compared to validate the assumptions made in the analytical solution. Moreover, the analytical and numerical solutions are validated by comparing the predicted results with the experimental data reported in the literature.

The analytical solution can also be used for the helical springs in which both the curvature and pitch effects are negligible. Moreover, it simplifies the analysis of the helical springs to that of the torsion of a straight bar with rectangular cross-section.

It is observed that the solution time for analytical method is much less than the computational time in finite element simulations (about 0.5%). The presented analytical solution can be used to study the effect of material or geometrical parameters on the rectangular SMP bar response as well as for their design and optimization which involve a large number of simulations.

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References


