The SPAC Method for a Finite M-Station Circular Array
Using Horizontally Polarized Ambient Noise

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Abstract The spatial autocorrelation (SPAC) method is a useful tool for interpreting ambient noise, and several theoretical tools based on this method are available for data collection, processing, and analysis. The method uses the three components of seismic microtremors recorded simultaneously with arrays of different configurations and yields dispersion curves of Rayleigh and Love waves. Existing theories of the SPAC method are based on an idealized circular array with an infinite number of stations (the continuous model). Errors due to the use of a finite number of stations have been derived theoretically for vertically polarized ambient noise. This paper presents an extension of that error theory to horizontally polarized ambient noise so that it is valid for a circular array with M uniformly spaced stations. The quantities that are obtained as the output of the SPAC method, however, depend on the ratio of the power spectral densities (PSDs) of Rayleigh and Love waves, which are difficult to identify beforehand. Instead we introduce a quantity, $\kappa$, that does not depend on the PSD ratio and therefore is more convenient for separating the properties of Rayleigh and Love waves. We derive an expression of $\kappa$ for a finite, M-station circular array ($\kappa_M$) and compare it with what it should be in the continuous model ($\kappa_T$). We illustrate $\kappa_M$ and $\kappa_T$ using field data from a site in Tehran and discuss their discrepancies. Finally, we demonstrate how we have estimated Love-wave phase velocities using the calculated $\kappa_M$.

Introduction

In the domain of seismic hazard assessment, predictions of local site effects depend crucially on the elastic properties of the subsurface (Hartzell et al., 1996; Yamanaka, 1998). Both array-based and single-station measurements of ambient noise (also known as microtremors, ambient vibrations, or seismic noise) can be used to deduce such information (e.g., Milana et al., 1996; Bard, 1998; Ohmachi and Umezono, 1998). Ambient vibration methods are cost effective and easily realized so have become increasingly popular in recent years.

It is commonly assumed that surface waves dominate the ambient vibration field (Asten, 1976; Lachet and Bard, 1994; Cornou and Bard, 2003; Bonnefoy-Claudet et al., 2004). Surface wave dispersion curves can be determined using arrays of stations (Aki, 1957; Lacoss et al., 1969; Asten and Henstridge, 1984; Horike, 1985; Tokimatsu and Miyadera, 1992; Tokimatsu, 1997; Asten et al., 2004) and allow inversion of the one-dimensional velocity structure below the array (Malagnini et al., 1995; Herrmann, 2001; Scherbaum et al., 2003; Parolai et al., 2005; Wathelet et al., 2005). The horizontal components of ambient vibrations include a mixture of Rayleigh and Love waves, which cannot be separated easily. For this reason, the majority of array-based ambient vibration studies only make use of the vertical-component wave field, with the goal of estimating the Rayleigh-wave dispersion curve alone. However, one should be able to reduce the nonuniqueness of inverted shear-wave velocity structures if both Rayleigh-wave and Love-wave dispersion curves are available. Love-wave phase velocities are therefore of particular interest.

In recent times, the spatial autocorrelation (SPAC) method, introduced by Aki (1957) and reinterpreted by Henstridge (1979), has seen significant application in urban areas. The dataset for this passive method is simply the ambient noise recorded simultaneously by an array of seismometers. A typical arrangement consists of several seismometers distributed evenly over one or more concentric rings surrounding a central seismometer, their recordings linked using a sufficiently accurate common-time system. The optimal sensor characteristics and radius (aperture) of the array depend on the depth to be resolved and, more specifically, the wavelength of the employed waves.

The SPAC method is based on the work of Aki (1957), who showed how autocorrelation functions can be linked to phase velocities of surface waves. Under the assumption that Rayleigh waves only have a single mode (a single phase velocity) at every frequency, Aki (1957) demonstrated that, for an imaginary array of detectors around a circle of radius...
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r, the autocorrelation functions (after being normalized and averaged over all azimuths [SPAC coefficients]) should have the shape of zero order Bessel functions of the first kind \( (J_0(k^R)) \), where \( k^R \) is the wavenumber of the Rayleigh wave. Most existing SPAC prospecting applications are limited to determining the Rayleigh wave from observations of vertical microtremors. For the horizontal components, the SPAC coefficients are combinations of \( J_0(k^R), J_2(k^R), J_0(k^L), \) and \( J_2(k^L) \), where \( k^L \) is the wavenumber of the Love wave. The SPAC coefficient can therefore determine the dispersion characteristics of both Love and Rayleigh waves. No assumptions concerning the directions of wave propagation are required.

To test the SPAC method, Aki (1957) analyzed ambient noise in an urban area using three sensors to record horizontal and vertical ground motions. Papers by Aki (1957, 1965) have a shortcoming: the horizontal motions due to waves with Rayleigh-like polarization and Love-like polarization are described using separate formulas, which are not applicable to cases in which both types of waves are present. Thus, his method cannot be used to separate the two types of waves.

For the horizontal components, the SPAC coefficients are linear combinations of \( J_0(k^R) - J_2(k^R), J_0(k^R) + J_2(k^R), J_0(k^L) + J_2(k^L), \) and \( J_0(k^L) - J_2(k^L) \). Okada and Matsushima (1989) presented a new method, which is an improvement over the original formulation by Aki (1957) and uses three-component records at each station from a circular array with a central station. This method can work in situations where both Rayleigh and Love waves coexist.

Matsushima and Okada (1990) and Yamamoto (2000) later applied that method to microtremor data (see also Morikawa et al., 2004).

The SPAC coefficients of horizontal motion, according to the Okada and Matsushima (1989) theory, depend on the power of microtremors and the power ratio of Rayleigh and Love waves in microtremors observed at the array. For example, Ferrazzini et al. (1991) observed three-component microtremors of volcanic origin near the Pu‘u O‘o volcano in Hawaii and estimated the subsurface structure that gives the best fit to both the Rayleigh-wave and Love-wave dispersion curves. Métaxian et al. (1997), Chouet et al. (1998), and Saccorotti et al. (2003) applied the SPAC method to signals of volcanic tremors. Their results also suggest a high proportion of Love waves, or at least a slight preponderance. Aki (1957) had also reported that Love waves dominated the horizontal components. However, in general, the partitioning between Rayleigh and Love waves is still under debate (Bonnefoy-Claudet et al., 2004).

Tada et al. (2006) came up with a different class of analytical method that uses only the horizontal components of tremors recorded on two concentric circular arrays. Also, Cho et al. (2006) reinterpreted Aki’s original equations of the horizontal SPAC method as a particular case of their generalized formulation. García-Jerez, Luzón, and Navarro (2008) derived the same result in a different mathematical framework. Recent papers by García-Jerez, Luzón, Navarro, and Pérez-Ruiz (2008) and García-Jerez et al. (2010) describe several alternative methods for dealing with the horizontal components of microtremors. Tada et al. (2009) have presented new formulas for the SPAC method that allow one to infer phase velocities of Love waves in a simple manner using two-component, horizontal-motion, circular-array records of microtremors.

Recently, Okada (2006) developed an efficient method of interpreting microtremor observations obtained on a circular array of M stations (\( M \geq 3 \)): M stations distributed uniformly around the circumference and one station placed at the center of the circle. By assuming that the vertical component consists only of surface (Rayleigh) waves, for which the fundamental mode is often dominant, he derived a two-part equation for the SPAC coefficient. Its first term is the Bessel function \( J_0(k^R) \), which is the SPAC coefficient derived for a continuous circular array (i.e., an imaginary array with a station at every possible angle). The error term, which depends on the number of stations \( M \) in the actual array, is expressed as a series of Bessel functions. Asten (2006) also argued that the finite nature of small seismic arrays, typically used for observing the microtremor wave field, causes predictable perturbations in the PSD.

In this paper, the Okada (2006) theory about the SPAC method for an M-station array is modified to include the horizontally polarized Rayleigh and Love waves.

As an alternative to the SPAC method, Okada (2003) proposed a new formula of ambient noise exploration by introducing a quantity, called \( \kappa_T \), which is a function of the angular frequency \( \omega \) and also the radius \( r \) of the circular array. Unlike the SPAC coefficients of horizontal motion, Okada’s \( \kappa_T \) coefficient does not depend on the relative powers of Rayleigh and Love waves. This presents an advantage over the SPAC method, although Okada’s method still does not allow one to infer Love wave velocities independently.

In the present paper, we introduce a quantity \( \kappa_M \), which is the value that \( \kappa_T \) is expected to take when there are only \( M \) stations around the circle. Our study does to the Okada (2003) kappa method what the Okada (2006) M-station theory did to the SPAC method.

Additionally \( \kappa_M \) is estimated using real microtremors recorded in a seven-station array in the south of Tehran. Rayleigh wavenumbers were estimated using the SPAC method applied on the vertical components. Then, the Love-wave phase velocities were estimated using Okada’s kappa method applied on the horizontal components.

Overview of the Theoretical SPAC Coefficient for Horizontally Polarized Waves

To interpret the horizontal components of microtremors, two types of waves propagating over the horizontal plane must be considered. The ambient noise can be polarized parallel and perpendicular to the direction of propagation. We consider the components of the waves parallel and perpendicular to the direction of \( \hat{r} \) in station S (Fig. 1),
The array in direction tangential components of plane waves propagating through face waves (both Rayleigh and Love waves). The radial and microtremors observed at a station with coordinate

\[ \phi \]

hereafter called radial and tangential components of waves, respectively.

The polarized waves are assumed to be generated by stationary stochastic processes (Okada, 2003). Let \( X_r(t, r, \theta) \) and \( X_\theta(t, r, \theta) \) be the radial and tangential components of the microtremors observed at a station with coordinate \((r, \theta)\) on the array (Fig. 1). These processes represent the complete surface waves (both Rayleigh and Love waves). The radial and tangential components of plane waves propagating through the array in direction \( \varphi \) (see Fig. 1) may be written as

\[ X_r(t, r, \theta) = X_r^R(t, r, \theta) + X_r^L(t, r, \theta) \]  

(1)

and

\[ X_\theta(t, r, \theta) = X_\theta^R(t, r, \theta) + X_\theta^L(t, r, \theta), \]  

(2)

where \( X_r^R(t, r, \theta) \) and \( X_r^L(t, r, \theta) \) are the horizontal components of Rayleigh and Love wave fields, respectively. It is not clear what contributions the Rayleigh and Love waves make to the total wave, but it is evident that their proportions change with frequency. Thus, Okada (2003) described this combination as a complex assemblage.

As illustrated in Figure 1, Rayleigh and Love components are written in polar coordinates as:

\[ X_r^R = X^R \cos(\theta - \varphi), \quad X_r^L = -X^L \sin(\theta - \varphi), \]

\[ X_\theta^R = X^L \sin(\theta - \varphi), \quad X_\theta^L = X^L \cos(\theta - \varphi). \]  

(3)

The Rayleigh and Love plane wave fields traveling in various directions \( \varphi \) with frequency \( \omega \) can be written in the following form (Okada, 2003):

\[
X_r^{(R/L)}(t, r, \theta) = \int_{\omega} \exp\{i\omega t + irk^{(R/L)}(\omega) \cos(\theta - \varphi)\}
\times d\eta^{(R/L)}(\omega, \varphi) ; \\
\omega \in (-\infty, \infty), \quad \varphi \in [0, 2\pi] 
\]

(4)

where \( i = \sqrt{-1} \), \( k^{(R/L)}(\omega) = \omega / c^{(R/L)}(\omega) \) is the wavenumber, \( c^{(R/L)}(\omega) \) is the phase velocity, and \( d\eta^{(R/L)}(\omega, \varphi) \) is the complex differential random amplitude of the wave. \( X_r^{(R/L)}(t, r, \theta) \) in the left-hand side of relation (4), which is the data sample recorded in a station, is a real quantity. The \( R(L) \) superscript identifies parameters that differ for Rayleigh (Love) waves. Assuming that no coupling takes place between Rayleigh and Love waves, or between waves with different frequencies and propagation directions, the orthogonality conditions imply that

\[
E[\eta^{(R)}(\omega, \varphi)\eta^{(L)}(\omega', \varphi')] = h^R(\omega, \varphi)\delta_{pq}b(\omega - \omega') \\
\times \delta(\varphi - \varphi')d\omega d\varphi d\omega' d\varphi' \\
\text{for } p, q = R, L. 
\]

(5)

Here, \( E[\cdot] \) denotes the expectation value, \( * \) is the conjugate symbol, and \( \delta_{pq} \) is the Kronecker delta, indicating that relation (5) is nonzero only when \( p = q \). The factor \( h^R(\omega, \varphi) \), for \( p = R \), is the directional PSD of the Rayleigh wave, considering \( d\eta^{(R)}(\omega, \varphi) = d\eta^{(R)}(\omega, \varphi') \) comes to a real quantity for \( X_r^{(R/L)}(t, r, \theta) \) in the left-hand side of relation (4). The SPAC functions between the signal received at the center of the array and the signal received at a single station on a circle of radius \( r \) are defined as follows with respect to radial and tangential wave components, respectively:

\[
S_r(r, \theta) \triangleq E[X^R_r(t, 0, \theta)X_r(t, r, \theta)] 
\]

(6)

and

\[
S_\theta(r, \theta) \triangleq E[X^L_\theta(t, 0, \theta)X_\theta(t, r, \theta)]. 
\]

(7)

Then, using relation (5), equations (6) and (7) reduce to the following forms:

\[
S_r(r, \theta) = E[X^R_r(t, 0, \theta)X^R_r(t, r, \theta) \\
+ X^L_r(t, 0, \theta)X^L_r(t, r, \theta)] 
\]

(8)

and

\[
S_\theta(r, \theta) = E[X^L_\theta(t, 0, \theta)X^L_\theta(t, r, \theta) \\
+ X^L_\theta(t, 0, \theta)X^L_\theta(t, r, \theta)]. 
\]

(9)

In equations (6)–(9), the quantities are real valued, and therefore their complex conjugates are equal to their original
values. Microtremors are assumed to be wide-sense stationary and ergodic; therefore, SPAC functions $S_s(r, \theta)$ and $S_\theta(r, \theta)$ are independent of time $t$. By substituting equation (3) into equation (8) and equation (9), using relation (4) and relation (5), according to $S_s(r, \theta) = \int_0^{2\pi} s_s(\omega, r, \theta) d\omega$ and $S_\theta(r, \theta) = \int_0^{2\pi} s_\theta(\omega, r, \theta) d\omega$, the SPAC functions at each angular frequency $\omega$ may be written as

$$s_s(\omega, r, \theta) = \int_0^{2\pi} \cos^2(\theta - \varphi) \exp\{irkr \cos(\theta - \varphi)\}$$

$$\times h^R(\omega, \varphi) d\varphi + \int_0^{2\pi} \sin^2(\theta - \varphi)$$

$$\times \exp\{irkr \cos(\theta - \varphi)\} h^L(\omega, \varphi) d\varphi,$$  

and

$$s_\theta(\omega, r, \theta) = \int_0^{2\pi} \sin^2(\theta - \varphi) \exp\{irkr \cos(\theta - \varphi)\}$$

$$\times h^R(\omega, \varphi) d\varphi + \int_0^{2\pi} \cos^2(\theta - \varphi)$$

$$\times \exp\{irkr \cos(\theta - \varphi)\} h^L(\omega, \varphi) d\varphi.$$  

The PSD function of the components is obtained by summing over all directions of propagation (Okada, 2003):

$$h^L(\omega) = \int_0^{2\pi} h^L(\omega, \varphi) d\varphi.$$  

The AV-SPAC functions are defined as the azimuthal average of the SPAC functions (Aki, 1957; Okada, 2003):

$$\bar{s}_r(\omega, r) \overset{\Delta}{=} \frac{1}{2\pi} \int_0^{2\pi} s_r(\omega, r, \theta) d\theta$$

$$= \frac{1}{2}[J_0(rkr^2) - J_2(rkr^2)]h_0^R(\omega)$$

$$+ \frac{1}{2}[J_0(rkr^2) + J_2(rkr^2)]h_0^L(\omega),$$  

and

$$\bar{s}_\theta(\omega, r) \overset{\Delta}{=} \frac{1}{2\pi} \int_0^{2\pi} s_\theta(\omega, r, \theta) d\theta$$

$$= \frac{1}{2}[J_0(rkr^2) + J_2(rkr^2)]h_0^R(\omega)$$

$$+ \frac{1}{2}[J_0(rkr^2) - J_2(rkr^2)]h_0^L(\omega),$$  

where $J_0$ and $J_2$ are zero-order and second-order Bessel functions of the first kind. The normalized AV-SPAC functions, or the SPAC coefficients, are defined as $\bar{s}_r(\omega, r)$ and $\bar{s}_\theta(\omega, r)$ divided by the maximum values they take at $r = 0$. The SPAC Coefficients of Horizontally Polarized Waves in the M-Station Model

Formally, the integrals with respect to $\theta$ in equation (13) and equation (14) are valid only when the circular array has an infinite number of stations. For a finite number of stations, the integrals should be approximated by sums. After some algebra (see Appendix), we arrive at an estimate of $\bar{s}_r(\omega, r)$ for a circular array with $M$ stations placed along the circumference at equal intervals, which we call $\bar{s}_{r,M}(\omega, r)$:

$$\bar{s}_{r,M}(\omega, r) = \frac{1}{2} \left\{ [J_0(rkr^2) - J_2(rkr^2)]h_0^R(\omega) \right. $$

$$+ [J_0(rkr^2) + J_2(rkr^2)]h_0^L(\omega) \right\} + \frac{1}{\pi \sum_{n=1}^{\infty} a_n(rkr^2)}$$

$$\times \sum_{j=1}^{M} \Delta \theta_j \int_0^{2\pi} \cos[n(\theta_j - \varphi)] h^R(\omega, \varphi) d\varphi$$

$$+ \frac{1}{\pi \sum_{n=1}^{\infty} b_n(rkr^2)} \sum_{j=1}^{M} \Delta \theta_j \int_0^{2\pi} \cos[n(\theta_j - \varphi)]$$

$$\times h^L(\omega, \varphi) d\varphi,$$  

where

$$a_n(x) = \frac{n(n-1)}{x^2} J_n(x) - \frac{1}{x} J_{n+1}(x),$$

$$b_n(x) = \frac{n(n-1)}{x^2} J_n(x) + \frac{1}{x} J_{n+1}(x).$$  

In the same way, using equation (11), one can find the tangential AV-SPAC function:
\[ s_{\theta M}(\omega, r) = \frac{1}{2} \left\{ [J_0(\maxL) + J_2(\maxL)]h_0^R(\omega) \\
+ [J_0(\maxL) - J_2(\maxL)]h_0^L(\omega) \right\} \\
+ \frac{1}{\pi} \sum_{n=1}^{\infty} b_n(\maxL) \sum_{j=1}^{\maxR} \Delta \theta_j \int_0^{2\pi} \cos[n(\theta_j - \varphi)] \\
\times h^R(\omega, \varphi) d\varphi \]  
(20)

The geometry of the proposed array results in \( \Delta \theta_j = \Delta \theta = \frac{\pi}{M} \), so the angles are \( \theta_j = 0, \theta_j = (j - 1) \frac{\pi}{M} \). Based on the assumptions just described, one can simply show that \( \sum_{j=1}^{\maxR} \cos[n(\theta_j - \varphi)] = M \cos(n\varphi) \delta_{n,1} \) for \( l = 1, 2, 3, \ldots \). In other words, the second and third lines of equation (18) and equation (20) can be nonzero only when \( n = \nu M \). Therefore, \( \tilde{s}_{r,\nu M}(\omega, r) \) and \( \tilde{s}_{\theta,\nu M}(\omega, r) \) in equations (18)-(20) are simplified to

\[ \tilde{s}_{r,\nu M}(\omega, r) = \frac{1}{2} \left\{ [J_0(\maxL) - J_2(\maxL)]h_0^R(\omega) \\
+ [J_0(\maxL) + J_2(\maxL)]h_0^L(\omega) \right\} \\
+ \frac{2}{\pi} \sum_{l=1}^{\maxR} a_{\nu M}(\maxL) \int_0^{2\pi} \cos(\nu M \varphi) h^R(\omega, \varphi) d\varphi \\
+ \frac{2}{\pi} \sum_{l=1}^{\maxR} b_{\nu M}(\maxL) \int_0^{2\pi} \cos(\nu M \varphi) h^L(\omega, \varphi) d\varphi \]  
(21)

and

\[ \tilde{s}_{\theta,\nu M}(\omega, r) = \frac{1}{2} \left\{ [J_0(\maxL) + J_2(\maxL)]h_0^R(\omega) \\
+ [J_0(\maxL) - J_2(\maxL)]h_0^L(\omega) \right\} \\
+ \frac{2}{\pi} \sum_{l=1}^{\maxR} a_{\nu M}(\maxL) \int_0^{2\pi} \cos(\nu M \varphi) h^R(\omega, \varphi) d\varphi \\
+ \frac{2}{\pi} \sum_{l=1}^{\maxR} b_{\nu M}(\maxL) \int_0^{2\pi} \cos(\nu M \varphi) h^L(\omega, \varphi) d\varphi \]  
(22)

The AV-SPAC functions are real functions because they are deduced from real signals \( X^{R(L)}(t, r, \varphi) \). According to the definitions given in equation (19), \( a_n(x) \) and \( b_n(x) \) are real only when \( n \) is even. Therefore, the second and third lines of equation (21) and equation (22) should be considered only when \( \maxR \) is even. We can use a unified notation and say that the only terms we have to consider are those with \( n = \nu \maxR \) for \( l = 1, 2, 3, \ldots \) where \( \nu = 1 \) for even \( \maxR \) and \( \nu = 2 \) for odd \( \maxR \). Therefore, equation (21) and equation (22) can be simplified to

\[ \tilde{s}_{r,\nu M}(\omega, r) = \frac{1}{2} \left\{ [J_0(\maxL) - J_2(\maxL)]h_0^R(\omega) \\
+ 4 \sum_{l=1}^{\infty} a_{\nu M}(\maxL)c_{\nu M}(\omega) h^R_0(\omega) \right\} \\
+ \frac{1}{2} \left( [J_0(\maxL) + J_2(\maxL)]h^L_0(\omega) \right) \\
+ 4 \sum_{l=1}^{\infty} b_{\nu M}(\maxL)c_{\nu M}(\omega) h^L_0(\omega) \]  
(23)

and

\[ \tilde{s}_{\theta,\nu M}(\omega, r) = \frac{1}{2} \left\{ [J_0(\maxL) + J_2(\maxL)]h_0^R(\omega) \\
+ 4 \sum_{l=1}^{\infty} a_{\nu M}(\maxL)c_{\nu M}(\omega) h^R_0(\omega) \right\} \\
+ \frac{1}{2} \left( [J_0(\maxL) - J_2(\maxL)]h^L_0(\omega) \right) \\
+ 4 \sum_{l=1}^{\infty} b_{\nu M}(\maxL)c_{\nu M}(\omega) h^L_0(\omega) \]  
(24)

where

\[ c_{\nu M}(\omega) = \frac{\int_0^{2\pi} \cos(\nu M \varphi) h^R(\omega, \varphi) d\varphi}{\int_0^{2\pi} h^R(\omega, \varphi) d\varphi}. \]  
(25)

Now, the SPAC coefficients \( \rho_{r,\nu M}(\omega, r) \) and \( \rho_{\theta,\nu M}(\omega, r) \) are defined by normalizing \( \tilde{s}_{r,\nu M}(\omega, r) \) and \( \tilde{s}_{\theta,\nu M}(\omega, r) \) such that their maxima are equal to one. The maximum values of \( \tilde{s}_{r,\nu M}(\omega, r) \) and \( \tilde{s}_{\theta,\nu M}(\omega, r) \) occur at \( r = 0 \). One can readily show that the terms \( a_{\nu M}(x) \) and \( b_{\nu M}(x) \) for \( \maxR > 2 \) approach zero as \( x \) approaches zero. Therefore, the maximum values of \( \tilde{s}_{r,\nu M}(\omega, r) \) and \( \tilde{s}_{\theta,\nu M}(\omega, r) \) are \( 1/2[h_0^R(\omega) + h_0^L(\omega)] \). Hence, the SPAC coefficients are

\[ \rho_{r,\nu M}(\omega, r) = \left( J_0(\maxL) - J_2(\maxL) \right) \\
+ 4 \sum_{l=1}^{\infty} a_{\nu M}(\maxL)c_{\nu M}(\omega) h^R_0(\omega) \\
+ \left( J_0(\maxL) + J_2(\maxL) \right) \\
+ 4 \sum_{l=1}^{\infty} b_{\nu M}(\maxL)c_{\nu M}(\omega) h^L_0(\omega) \]  
(26)
\[ \rho_{\nu M}(\omega, r) = \left[ J_0(r^L) + J_2(r^R) + 4 \sum_{l=1}^{\infty} b_{\nu M}(r^L) c_{\nu M}(r^R) \right] H_0^2(\omega) \]

\[ + \left[ J_0(r^L) - J_2(r^L) + 4 \sum_{l=1}^{\infty} a_{\nu M}(r^L) c_{\nu M}^L(\omega) \right] H_0^L(\omega), \]

where \( \nu = 1 \) for even \( M \) and \( \nu = 2 \) for odd \( M \).

If \( M_1 = 2M_2 \) and \( M_2 \) is odd, the radial and tangential SPAC coefficients are identical for the two cases of \( M_1 \) stations and \( M_2 \) stations.

**Okada’s Kappa Method**

When calculating the SPAC coefficients for an idealized circular array with sensors at each angle, given in equation (15) and equation (16), it is necessary to discriminate between Rayleigh and Love waves. Because the PSD ratio of the two waves is not known beforehand, equation (15) and equation (16) cannot be applied directly. To circumvent this problem, Okada (2003) instead defined a coefficient \( \kappa_T(\omega, r) \) as

\[ \kappa_T(\omega, r) \triangleq \frac{\tilde{s}_r(\omega, r) + \tilde{s}_\theta(\omega, r) - J_0(r^R) H_0(\omega)}{\tilde{s}_s(\omega, r) - \tilde{s}_\theta(\omega, r) + J_2(r^R) H_0(\omega)}, \]

where \( H_0(\omega) = h_0^R(\omega) + h_0^L(\omega) \) is the total power of horizontal motion. \( H_0, \tilde{s}_r, \) and \( \tilde{s}_\theta \) can be calculated using observed records of horizontal-motion microtremors. \( J_0(r^R) \) and \( J_2(r^R) \) can be found by SPAC analysis of vertical-motion microtremors; it has been assumed that the wavenumbers of the vertical and horizontal components of Rayleigh waves are the same. Substituting equation (13) and equation (14) into equation (28) yields the theoretical representation for \( \kappa_T \):

\[ \kappa_T(\omega, r) = \frac{J_0(r^L) - J_2(r^R)}{J_2(r^L) + J_2(r^R)}. \]

The Love wave velocity \( r^L \) can be estimated by inverting equation (29) using observed \( \kappa_T(\omega, r) \) (equation 28). Finally, by substituting the estimated \( r^L \) and \( r^R \) back into equation (13) and equation (14), the PSDs of Rayleigh and Love waves can be obtained.

In practice, when there are only a finite number of \( M \) stations around the circle, \( \kappa_T \) has to be estimated by using finite-sum SPAC coefficients. Let us define \( \kappa_M \) as the value of \( \kappa_T \) thus estimated using observed array data:

\[ \kappa_M(\omega, r) \triangleq \frac{\tilde{s}_{r M}(\omega, r) + \tilde{s}_{\theta M}(\omega, r) - [J_0(r^R) + \Phi_M(r^R)] H_0(\omega)}{\tilde{s}_{s M}(\omega, r) - \tilde{s}_{\theta M}(\omega, r) + [J_2(r^R) + \Psi_M(r^R)] H_0(\omega)}, \]

where

\[ \Phi_M(r^R) \triangleq 2 \sum_{l=1}^{\infty} (-1)^{l/2M} J_{\nu M}(r^R) c_{\nu M}^L(\omega), \]

\[ (Okada, 2006), \]

\[ \Psi_M(r^R) \triangleq 2 \sum_{l=1}^{\infty} (-1)^{l/2M} \left[ \frac{2r^R \nu M (\nu M - 1)}{(r^R)^2} - 1 \right] J_{\nu M}(r^R) + \frac{2}{r^R} J_{\nu M+1}(r^R) c_{\nu M}^L(\omega), \]

where \( \nu = 1 \) for even \( M \) and \( \nu = 2 \) for odd \( M \). The \( r^R \) value, estimated by SPAC analysis of vertical-motion records, should be substituted into equation (31) and equation (32) to obtain \( \Phi_M(r^R) \) and \( \Psi_M(r^R) \).

By substituting equation (23) and equation (24) into equation (30), we obtain the theoretical value of \( \kappa_M(\omega, r) \):

\[ \kappa_M(\omega, r) = \frac{J_0(r^L) - J_0(r^R) + \Phi_M(r^L) - \Phi_M(r^R)}{J_2(r^L) + J_2(r^R) + \Psi_M(r^L) + \Psi_M(r^R)}. \]

In fact, the quantities \( \Phi_M(r^R) \) and \( \Psi_M(r^R) \) have been introduced in the M-station case because including them in equation (30) makes the final equation (33) formulaically simple (all terms with \( h_0^R(\omega) \) vanish both in the numerator and the denominator). The Love-wave velocity \( r^L \) can be estimated by inverting equation (33) using observed \( \kappa_M \) (equation 32).

We will also need an estimate of \( c_{\nu M}^L(\omega) \) to find \( \kappa_T(\omega, r) \) and \( \kappa_M(\omega, r) \). We cannot calculate \( c_{\nu M}^L(\omega) \) from equation (25) because \( h_{L M}^R(\omega, \phi) \) is unknown. Therefore, approximate values of \( c_{\nu M}^L(\omega) \) should be sought. Some special cases are predictable:

- It is obvious from equation (25) that \( 0 \leq |c_{\nu M}^L(\omega)| \leq 1 \), where \( | \cdot | \) denotes the absolute value of the function. In the limit \( c_{\nu M}^L(\omega) = 0 \) for uniformly distributed \( h_{L M}^R(\omega, \phi) \) with respect to \( \phi \), our M-station array becomes equivalent to the theoretical continuous model. In contrast, choosing \( c_{\nu M}^L(\omega) = 1 \) maximizes the difference between our M-station model and the continuous model.

On the other hand, for locally distributed (with special direction) \( h_{L M}^R(\omega, \phi) \), \( c_{\nu M}^L(\omega) = \cos(\nu M \phi) \). Note that approximating \( c_{\nu M}^L(\omega) \) will not have the same effect in \( \kappa_M(\omega, r) \) as in the actual SPAC coefficients.
Real Data Results

We conducted ambient noise measurements using a uniform circular array of seven stations with one station in the center (Fig. 2a). The array was composed of Guralp CMG–6TD seismic stations, and its aperture was near 20 m. Ambient noise measurements were performed in 2007 at a site in the south of Tehran; more detailed information can be found in Shabani et al. (2010). Figure 2b illustrates the data with a sample of simultaneous three-component microtremor records taken at the central station.

The Geopsy software package (see Data and Resources section) was used to compute the SPAC coefficient and phase velocity of Rayleigh waves, \( c_R(\omega) \), using the vertical components of the recorded microtremors (Fig. 3a,b). Then, the wavenumber of the Rayleigh wave \( k_R \) was determined by \( k_R = \frac{\omega}{c_R(\omega)} \). Also, radial and tangential AV-SPAC functions, \( \tilde{s}_{r,M}(\omega, r) \) and \( \tilde{s}_{\theta,M}(\omega, r) \), were computed for all pairs of stations; the results are shown in Figure 4a. In this regard, the horizontal seismogram components (N, E) were rotated with respect to the orientation of the station pair considered. The AV-SPAC functions at the array center were found by inserting \( r = 0 \) into equation (23) and equation (24):

\[
\tilde{s}_{r,M}(\omega, 0) = \tilde{s}_{\theta,M}(\omega, 0) = \frac{1}{2}[h_0^R(\omega) + h_0^R(\omega)] = \frac{1}{2}H_0(\omega).
\]

(34)

The value of \( H_0(\omega) \) was computed by adding \( \tilde{s}_{r,M}(\omega, 0) \) and \( \tilde{s}_{\theta,M}(\omega, 0) \), rather than by doubling \( \tilde{s}_{r,M}(\omega, 0) \) or doubling \( \tilde{s}_{\theta,M}(\omega, 0) \), because we preferred to make use of both the radial and tangential component records. Then \( \kappa_M \) was obtained using computed \( H_0(\omega) \), \( \tilde{s}_{r,M}(\omega, r) \), and \( \tilde{s}_{\theta,M}(\omega, r) \) in equation (30) for the known \( k_R \) deduced from vertical components of the recorded microtremors.

Finally, using the obtained value of \( \kappa_M \) in equation (33), a Love-wave dispersion curve is derived. Figure 5 presents the estimated Love-wave phase velocity in comparison with the theoretical Love wave dispersion curve obtained from geological borehole data. The array measurements were recorded at a site in southern Tehran, located near a 150-m-deep borehole named N13 (Japan International Cooperation Agency and Centre for Earthquake and Environmental Studies of Tehran, 2000). The distance between the microtremor measurement site and the borehole site is near 1 km.

The effects of the finite number of sensors on \( \kappa_T \) (or \( |\kappa_T - \kappa_M| \)) can only be evaluated theoretically because it is impossible to know the true value of \( \kappa_T \) from real data when there are only a finite number of M stations in the array.
Instead, we compare the case of considering the effects of taking into account terms $\Phi M/0.0133$ and $\Psi M/0.0133$ in Okada's kappa method, or $\kappa_M$, with the case of neglecting these terms (we call it theoretical $\kappa_M$, or $\kappa_T M$). Figure 6 presents an example showing these two cases applied to real data. When there are only a finite number of M stations in the array, for $M/0.0136 \geq 7$, the difference between the two cases is greater than 0.01 for frequencies higher than 6.5 Hz.

**Error Terms**

Equation (33) can be written in the form $\kappa_M = \kappa_T (1 - \epsilon_M)$, where $\kappa_T$ is defined in equation (29) for the continuous circular array and $\epsilon_M$ (or $\epsilon_L/\epsilon_R$) is the relative error. If $\kappa_T$ goes to zero, the relative error will be very large. Thus, in the following discussion, the absolute value of error $|\kappa_T - \kappa_M|$ will be used.

Figure 7 compares the variation of $|\kappa_T - \kappa_M|$, according to equation (29) and equation (33) with respect to $r_kR$ and $r_kL$ for $c_R/0.0133 L/0.0134 \nu L/0.0136 1$ where $M/0.0136 \geq 3$, 5, and 7, respectively. This comparison reveals that, in regular arrays with odd number of stations, the absolute value of the difference between $\kappa_T$ and $\kappa_M$ decreases as the number of stations is increased.

To present the shapes of $\kappa_T$ and $\kappa_M$ more simply, it is also possible to provide some two-dimensional plots for the fixed values of $r_kR$. Figure 8, Figure 9, and Figure 10 illustrate a comparison of $\kappa_T$ and $\kappa_M$ with respect to $r_kL$ for $c_R/0.0133 L/0.0134 \nu L/0.0136 1$, 0.7, 0.4, and 0.1. As Figure 4.

(a) AV-SPAC functions obtained for radial and tangential components of real data microtremor records in a seven-station array ($M = 7$). (b) Computed power spectrum $H_0(\omega)$, defined in equation (34) using radial and tangential components of microtremor records of real data in a seven-station array. The color version of this figure is available only in the electronic edition.

**Figure 4.**

**Figure 5.**

The estimated Love-wave dispersion curve ($\kappa_M$; squares) in comparison to the theoretical Love-wave dispersion curve ($\kappa_T M$; triangles) calculated from the geological data (nearest borehole log). The color version of this figure is available only in the electronic edition.

To present the shapes of $\kappa_T$ and $\kappa_M$ more simply, it is also possible to provide some two-dimensional plots for the fixed values of $r_kR$. Figure 8, Figure 9, and Figure 10 illustrate a comparison of $\kappa_T$ and $\kappa_M$ with respect to $r_kL$ for $r_kR = 1.0$ when $M = 3, 5$, and 7, respectively. These figures also show the ranges of Love wavenumbers, $r_{kL} < r_k < r_{kL2}$ within which the error $|\kappa_T - \kappa_M|$ is reasonably small (i.e., <0.01). The limit of 0.01 on the absolute value of the difference between $\kappa_T$ and $\kappa_M$ is an arbitrary criterion; one can choose any other value; for example, 0.001. The proper range increases as the number of stations is increased. Therefore, the proper range of $r_kL$ is $0.7 < r_kL < 1.2$ for $M = 3$, $r_kL < 3.75$ for $M = 5$, and $r_kL < 5.18$ for $M = 7$.

If $r_kR$ is interchanged with $r_kL$ on the right-hand side of equation (29), an identical function value is obtained, with the simple difference of a reversed plus/minus sign; the same is true for equation (33). Thus, one may easily find $\kappa_T$ and $\kappa_M$ for an arbitrary $r_kL = a$ by reversing the values of $\kappa_T$ and $\kappa_M$ obtained for $r_kR = a$.

To show the effects of $c_{R/L}$, Figure 11 compares $\kappa_T$ and $\kappa_M$ ($M = 3, r_kR = 1.0$) for $c_{R/L} = 1, 0.7, 0.4$, and 0.1.
expected, the maximum difference of $\kappa_T$ and $\kappa_M$ was found for $c_{R(L)}^{(M)} = 1$, and the difference vanishes for $c_{R(L)}^{(M)} = 0$.

Moreover, one can find a relationship between the deviation of Rayleigh (Love)-wave phase velocities and the partial derivative of $\kappa_M$ with respect to wavenumber:

$$\delta c^j = \frac{\omega}{k^2} \left( \frac{1}{\partial \kappa_M / \partial k^j} \right) \delta \kappa, \quad j = R, L, \quad (35)$$

where $\delta c^j$ is the absolute value of the variation of Rayleigh-wave or Love-wave phase velocity due to the absolute value of variation of $\kappa_M$, named $\delta \kappa$. The term $\partial \kappa_M / \partial k^j$ expresses the magnitude of the partial derivative of $\kappa_M$ with respect to wavenumber, which can be deduced from equation (33). The determination of $\delta \kappa$ is shown in Figures 8–10.

Conclusion

This paper initially presents the derivation of mathematical expressions for the horizontal-motion SPAC coefficients estimated with a finite number of $M$ circumferential stations, inspired by the work of Okada (2006) that deals with a similar problem for the vertical-motion SPAC method. The resultant horizontal SPAC coefficients depend on the PSD

\[ \delta c^j = \frac{\omega}{k^2} \left( \frac{1}{\partial \kappa_M / \partial k^j} \right) \delta \kappa, \quad j = R, L, \quad (35) \]

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The ratio of Rayleigh and Love waves, $H^R_0(\omega)$ and $H^L_0(\omega)$, which are not known beforehand. Therefore, phase velocities of Rayleigh and Love waves cannot be calculated from the two horizontal SPAC coefficients only. Hence, as the second step, Okada’s kappa method was used to introduce the quantity $\kappa_M$ as a replacement for the SPAC coefficients, for which $\kappa_M$ is based on both the averaged radial and tangential SPAC functions but is independent of the PSD ratio of Rayleigh and Love waves.

**Figure 8.** (a) Comparing $\kappa_T$ and $\kappa_M$ for $M = 3$, $c^R_{iM} = 1$, and $rk^L = 1.0$. The error $|\kappa_T - \kappa_M|$ is less than 0.01 only in a limited range, between $rk_{d1} = 0.7$ and $rk_{d2} = 1.2$. (b) Absolute value of the difference between $\kappa_T$ and $\kappa_M$ at $rk^L = 1.0$ for $M = 3$. The color version of this figure is available only in the electronic edition.

**Figure 9.** (a) Comparing $\kappa_T$ and $\kappa_M$ for $M = 5$, $c^R_{iM} = 1$, and $rk^L = 1.0$. The error $|\kappa_T - \kappa_M|$ is less than 0.01 in the range $rk^L < 3.75$. (b) Absolute value of the difference between $\kappa_T$ and $\kappa_M$ at $rk^L = 1.0$ for $M = 5$. The color version of this figure is available only in the electronic edition.
Love waves, $\kappa_M$ is obtained only if the Rayleigh-wave dispersion curve is estimated from the vertical-motion components.

The term $\kappa_M$ is compared theoretically with $\kappa_T$ (Okada’s kappa method for an idealized circular array) for different numbers of stations in an array. This comparison reveals that the absolute value of the difference between $\kappa_T$ and $\kappa_M$ decreases with an increasing number of stations.

Moreover, $\kappa_M$ was calculated with real data from the field. It is impossible to know the true value of $\kappa_T$ from real data when there are a finite number of $M$ stations in the array. Instead, we compared the case of considering the effects of taking into account terms $\Phi_M(rk^R)$ and $\Psi_M(rk^R)$ on Okada’s kappa method ($\kappa_M$) with the case of neglecting these terms (theoretical $\kappa_M$, or $\kappa_T$). The results show that the absolute value of the difference between $\kappa_T$ and $\kappa_M$ is very small in long wavelengths but increases in small wavelengths. In the preceding calculations, an approximate value should be used for $cR/L/\omega$, the ratio of power spectral density function, which is unknown beforehand.

A value of $\kappa_M$ calculated with real field data was then used in equation (33) to find the Love-wave dispersion curve, whereas the Rayleigh-wave dispersion curve is known from the observations of the vertical components of real data. The estimated Love-wave dispersion curve from real field measurement data in a site in Tehran, in comparison with the theoretical Love wave dispersion curve calculated from the nearest borehole log, is in good agreement for frequencies higher than 5 Hz (Fig. 5).

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**Figure 10.** (a) Comparing $\kappa_T$ and $\kappa_M$ for $M = 7$, $c_{R/L} = 1$, and $rk^R = 1.0$. The error $|\kappa_T - \kappa_M|$ is less than 0.01 in $rk^L < 5.18$. (b) Absolute value of the difference between $\kappa_T$ and $\kappa_M$ at $rk^L = 1.0$ for $M = 7$. The color version of this figure is available only in the electronic edition.

**Figure 11.** (a) Comparing $\kappa_T$ with $\kappa_M$ for $M = 3$, $rk^R = 1.0$, and $c_{R/L} = 1, 0.7, 0.4, 0.1$. (b) Absolute value of the difference between $\kappa_T$ and $\kappa_M$ at $rk^R = 1.0$ for $M = 3$ and $c_{R/M} = 1, 0.7, 0.4, 0.1$. 
The data used in this study cannot be released to the public. They were borrowed from the International Institute of Earthquake Engineering and Seismology (IIIES).

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References


Appendix

Equation (10) can be rewritten as

\[
\begin{align*}
\tilde{s}_r(M, \omega, r, \theta) & = \frac{1}{2\pi} \sum_{j=1}^{M} s_r(\omega, r, \theta_j) \Delta \theta_j, \\ & \quad \text{(A4)}
\end{align*}
\]

where \( \Delta \theta_j \) is the central angle between the \( j \)-th and \( (j+1) \)-th stations. Therefore, using equation (A1) and equation (A2), the azimuthal average of equation (A4) is

\[
\tilde{s}_r(M, \omega, r) = \frac{1}{2\pi} h^R(\omega) \left( \frac{d^2}{d(rk^L)^2} \right) \left( \frac{d^2}{d(rk^L)^2} \right) \sum_{j=1}^{M} \Delta \theta_j
\]

To simplify this expression, one may use the following recurrence relation (Arfken and Weber, 2001):

\[
J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x). 
\]

From equation (A6), the following recurrence relation is readily inferred:

\[
J'_n(x) = \frac{n(n-1)}{x^2} J_n(x) - \frac{2n+1}{x} J_{n+1}(x) + J_{n+2}(x). 
\]

In equation (A6) and equation (A7), the prime and double-prime represent the first and the second derivatives with respect to \( x \). On the other hand, the recurrence relation (Arfken and Weber, 2001)

\[
J_{n+2}(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_n(x) 
\]

is also true and, with equation (A7), can be changed to
\( J'_n(x) = \left( \frac{n(n-1)}{x^2} - 1 \right) J_n(x) + \frac{1}{x} J_{n+1}(x). \quad \text{(A9)} \)

Finding \( J_1(x) \) from equation (A8) and substituting it into equation (A7) for \( n = 0 \), one finds \( J'_0(x) = \frac{1}{x} (J_2(x) - J_0(x)) \).

Using the deduced value of \( J'_0(x) \) and the relation \( \sum_{j=1}^{M} \Delta \theta_j = 2\pi \), equation (A5) can be written as:

\[
\tilde{S}_{r,M}(\omega, r) = \frac{1}{2} \left[ (J_0(rk^R) - J_2(rk^R)) h^R_0(\omega) \right. \\
+ \left. (J_0(rk^L) + J_2(rk^L)) h^L_0(\omega) \right] \\
+ \frac{1}{\pi} \sum_{n=1}^{\infty} a_n(rk^R) \sum_{j=1}^{M} \Delta \theta_j \int_0^{2\pi} \cos[n(\theta_j - \varphi)] \\
\times h^R(\omega, \varphi) d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} b_n(rk^L) \\
\times \sum_{j=1}^{M} \Delta \theta_j \int_0^{2\pi} \cos[n(\theta_j - \varphi)] \\
\times h^L(\omega, \varphi) d\varphi. \quad \text{(A10)}
\]

where

\( a_n(x) \triangleq \int \left[ \left( 1 - \frac{n(n-1)}{x^2} \right) J_n(x) - \frac{1}{x} J_{n+1}(x) \right] \frac{1}{x^n} dx \),

\( b_n(x) \triangleq \int \left[ \frac{n(n-1)}{x^2} J_n(x) + \frac{1}{x} J_{n+1}(x) \right] \frac{1}{x^n} dx. \quad \text{(A11)} \)