On the dynamics of bistable micro/nano resonators: Analytical solution and nonlinear behavior

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A B S T R A C T

With the rapid development of micro/nano-electro-mechanical systems (MEMS/NEMS), arch shaped resonators are becoming increasingly attractive for different applications. Nevertheless, the dynamics of bistable resonators is poorly understood, and the conditions for their appropriate performance are not well known. In this paper, an initially curved arch shaped MEMS resonator under combined DC and AC distributed electrostatic actuation is investigated. A reduced order model obtained from first mode Galerkin's decomposition method is used for numerical and analytical investigations. We have used the Homotopy Analysis Method (HAM) in order to derive analytical solutions both for the amplitude and the temporal average of nonlinear vibrations. The obtained analytical expressions, validated by numerical simulations, are able to capture nonlinear behaviors including softening type vibrations and dynamic snap-through. We have used the derived analytical results in order to study the nonlinear vibrations of the bistable MEMS resonator. According to our results, in the bistability region the overall dynamic response of the system is described by means of a pair of softening type frequency responses merging together in a specific frequency band. The dynamic snap-through is then described by transitions between these two frequency responses, each of which corresponding to one of the stable configurations of the arch. This fresh insight to the problem can be used in the design and optimization of bistable resonators and determination of their sharp roll-off frequencies. A feature that can be implemented in the design of bandpass filters.

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1. Introduction

Micro-electro-mechanical systems (MEMS) resonators have gained tremendous attention due to their various applications such as accelerometers [1], load cells [2], gyroscopes [3], gas and chemical sensors [4] and biosensors [5]. A list of advantages including low power consumption, low cost and improved reliability makes MEMS resonators suitable for the aforementioned applications. So far, most of the presented MEMS resonators are comprised of a fixed electrode and an
electrostatically actuated movable electrode. Dynamic behaviors of these types of resonators are widely investigated in the literature [6–11]. These nonlinear behaviors include hardening or softening effects, dynamic pull-in or chaotic vibrations.

In recent years, bistable MEMS devices have gained lots of attention in the research community. These interests are motivated by applications of bistable MEMS devices including microrelays [12], microvalves [13], switches [14] and MEMS based memories [15]. Bistability refers to the ability of the system to operate in more than one stable configuration at certain values of geometrical and loading parameters. A MEMS device composed of an arch shaped microbeam is a typical representative of the family of bistable MEMS devices. These bistable microbeams are intentionally made in a curved (arch) configuration. This can be achieved by either micro machining or buckling a straight beam under compressive axial loads [16,17]. In addition to other nonlinear phenomena, these mechanically bistable MEMS undergo a nonlinear phenomenon, known as snap-through, which is responsible for the precipitate mechanical transitions of the micro arch between its convex and concave configurations. Various drive mechanisms including electromagnetic [18], mechanical touch [19], electrostatic [20], thermal [21] and optical [22] were reported as the starters of the so-called snap-through motion for bistable arch shaped MEMS devices.

Many researchers studied the static snap-through and pull-in response in electrostatically actuated bistable MEMS devices. Analytical and experimental studies on the static and dynamics of arch shaped MEMS are reported in the works of Krylov et al. [20,23–25]. Their reports include: (i) derivation of the governing equations of motion based on the Euler–Bernoulli beam theory and the Galerkin’s method, (ii) presentation of bistability conditions and (iii) investigation of the effects of various parameters. They also studied response of the system to DC electrostatic shocks using phase plane analysis. Zhang et al. [26] experimentally studied the bistability of an arch shaped micro-machined beam. They construed snap-through, due to its zero power consumption and precipitate motion, as an appropriate phenomenon in the development of highly sensitive sensors. In order to study the pull-in and snap-through instabilities, finite and boundary element methods were used by Das and Batra [27].

A few research groups have focused on the resonant behaviors of the bistable arch shaped microbeams. Casals-Terre et al. [28] demonstrated an initially curved double clamped micro beam under combined DC and AC electrostatic actuation. They studied, mainly due to snap-through triggered by mechanical resonance, the possibility of transitions between stable configurations. Ouakad and Younis [17] investigated an arch shaped MEMS resonator. They scrutinized the dynamic behavior of the resonator using Galerkin’s method. Also, they used a perturbation technique (the Multiple Scales Method) in order to examine the response of the resonator under small DC and AC actuations. The main limitation of the perturbation method is that it is only valid for small deflections of the resonator [29]. Thus, only problems with weak nonlinearities can be analyzed by this approach. This assumption is not valid for large displacements imposed by dynamic snap-through. In another work, Younis et al. [16] studied an arch shaped MEMS resonator; they showed, both numerically and experimentally, various interesting nonlinear phenomena including softening behavior, dynamic snap-through as well as dynamic pull-in. They also suggested application of these types of MEMS as bandpass filters or low-powered switches. Recently, they [30,31] have reported further discussions on the bistable-MEMS-based bandpass filters.

Unlike perturbation techniques, the Homotopy Analysis Method (HAM), originally introduced and developed by Liao [32–34], does not depend on any small parameter assumption. This method provides a powerful straightforward tool for representation of the solution of highly nonlinear equations in the form of convergent series. Also, unlike other techniques targeting a series solution, the HAM can guarantee convergence of the series solution by introducing a so-called convergence-control parameter [35,36]. Because of its effectiveness and simplicity, HAM has been widely implemented in the literature for analysis of various highly nonlinear problems in recent years [37–40].

Though significant progresses in the investigation of the dynamic behaviors of the arch shape microbeam, there are still many issues under discussion which are beyond the capability of classical perturbation techniques. During analysis of curved microbeam behaviors, the highly nonlinear frequency responses elucidate the necessity of using a precise analytical solution. HAM method, with its strong capability, has a great potential to relate the microbeam response with the input voltages. This paper aims to study the nonlinear behavior of an electrostatically actuated bistable MEMS resonator using the HAM method. We assume that a double clamped micro arch under combined static DC and harmonic AC electrostatic actuation acts as a resonator near its first fundamental frequency. As a novel approach and with the aim of capturing dynamic snap-through instabilities, we have described the dynamic displacement of the arch by two unknowns, namely the amplitude and the temporal average of nonlinear vibrations. The latter directly describes the snap-through behavior of the arch. We present analytical expressions for these two unknowns providing appropriately precise tools for description of the nonlinear behaviors of the bistable MEMS resonator.

The rest of the paper is organized as follows. In Section 2, a first mode Galerkin’s decomposition method is used to convert the corresponding Euler–Bernoulli based governing partial differential equation to a second order nonlinear differential equation. In Section 3, we use HAM in order to obtain analytical solution for the nonlinear vibrations of the arch resonator. Section 4 continues with the verification of the obtained analytical solutions via numerical simulations. Also, we investigate in detail a typical case of nonlinear vibrations of the presented bistable resonator using both analytical solutions and numerical simulations. Then, we present a novel interesting description for the bistable vibrations and consequent snap-through instability of the arch resonator. We show that the nonlinear snap-through jumps can be described by merging of two frequency responses in the bistability region, each of which corresponding to one of the stable configurations of the arch. Further conclusions and discussions are presented in Section 5, where we conclude that this work is a forward step in the
literature of bistable MEMS resonators. Both analytical solutions and novel bistability descriptions presented in this paper would be useful in the design and implementation of the bistable MEMS resonators.

2. Mathematical modeling

2.1. Formulation

A double clamped arch shaped shallow microbeam having length \( \hat{L} \), width \( \hat{b} \) and thickness \( \hat{d} \) shown in Fig. 1 is studied in this paper. The beam undergoes combined AC and DC electrostatic actuation as a MEMS resonator. Hereafter, quantities having the physical dimension are denoted by \( \hat{\cdot} \). Supposing Young Modulus \( \hat{E} \) and Poisson’s ratio of \( \nu \) and due to wide beam assumptions (\( \hat{b} > 5\hat{d} \)), the effective modulus of elasticity is given as \( \hat{E} = \hat{E}_e/(1 - \nu^2) \) [41]. Using the given coordinates \( \hat{A} = \hat{b}\hat{d} \) is the cross sectional area, \( \hat{I}_y = \hat{b}\hat{d}^2/12 \) is area moment of inertia, \( \hat{\varepsilon}_0(\hat{x}) \) describes the initial shape of the arch and \( \hat{w}(\hat{x}) \) represents deflection of the arch in the positive \( \hat{Z} \) direction measured from its initial position. Supposing \( \hat{\rho}(\text{kg/m}^3) \) as the density of the arch and \( \hat{\varsigma} \) as viscous damping coefficient, the governing equation of motion based on the Euler–Bernoulli beam assumptions is given as following [20]:

\[
\rho\hat{A}\frac{\partial^2 \hat{w}}{\partial \hat{t}^2} + \hat{c}\frac{\partial \hat{w}}{\partial \hat{t}} + \hat{E}\hat{I}_y \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} - \frac{\hat{E}\hat{A}}{2\hat{L}} \int_0^1 \left( \frac{\partial \hat{w}}{\partial \hat{x}} \right)^2 + 2 \frac{\partial \hat{w}}{\partial \hat{x}} \frac{\partial \hat{w}}{\partial \hat{x}} \right) \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} \right) \frac{\partial^2 \hat{z}}{\partial \hat{x}^2} = \frac{\varepsilon_0 \hat{b} \hat{V}^2}{2(\hat{g}_0 - \hat{z}_0 - \hat{w})^2}
\]

(1)

Note that parallel plate assumptions are used in derivation of electrostatic force model and fringing field effects are neglected. Moreover, it is assumed that \( \hat{d} \ll \hat{L} \) and \( \hat{h} \ll \hat{L} \) and the deflections are small with respect to the beam length. In the given equation, \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \) is the air permittivity, \( \hat{g}_0 \) is the fixed gap between electrodes and \( \hat{V} \) represents the applied voltage. Boundary conditions of Eq. (1) are also given as follow.

\[
\hat{w}(0, \hat{t}) = \hat{w}(\hat{L}, \hat{t}) = 0, \quad \frac{\partial \hat{w}}{\partial \hat{x}} \bigg|_{\hat{x} = 0} = \frac{\partial \hat{w}}{\partial \hat{x}} \bigg|_{\hat{x} = \hat{L}} = 0.
\]

(2)

2.2. Reduced order model

Using a normalization technique [42], Eq. (1) and its boundary conditions (2) are first converted to nondimensional form using the nondimensional parameters of Table 1. Then, implementing the single mode Galerkin method, with the assumption \( w(x, \tau) = q(\tau) \phi(x) \), eventuates in the following reduced order ordinary differential equations [20].

\[
\frac{d^2 q}{d\tau^2} + \mu q + F(q) = 0
\]

(3)

\[
F(q) = (1 + 2h^2\alpha_1)q - 3\alpha_1 h q^2 + \alpha_1 q^3 - \frac{\beta_1}{\sqrt{(1 + h - q)^3}}
\]

Fig. 1. Schematic of the arch shaped resonator.

### Table 1

<table>
<thead>
<tr>
<th>Nondimensional parameter</th>
<th>Description</th>
<th>Nondimensional parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{w} = \frac{\hat{w}}{\hat{L}} )</td>
<td>Deflection</td>
<td>( \hat{h} = \frac{\hat{h}}{\hat{L}} )</td>
<td>Initial rise</td>
</tr>
<tr>
<td>( \hat{x} = \frac{\hat{x}}{\hat{L}} )</td>
<td>Length coordinate</td>
<td>( \hat{c} = \frac{\hat{c}}{\sqrt{\frac{\hat{m}_u}{\hat{M}_u \hat{\varsigma}_u}}} )</td>
<td>Viscosity parameter</td>
</tr>
<tr>
<td>( \hat{\tau} = \frac{\hat{t}}{\sqrt{\frac{\hat{b}_L \hat{d}_L}{\hat{m}_u \hat{\varsigma}_u}}} )</td>
<td>Time</td>
<td>( \alpha_1 = \frac{\hat{\alpha}<em>1}{\hat{\alpha}</em>{10}} )</td>
<td>Stretching parameter</td>
</tr>
<tr>
<td>( \hat{b} = \frac{\hat{b}}{\hat{L}} )</td>
<td>Width</td>
<td>( \beta_1 = \frac{\hat{\beta}<em>1}{\hat{\alpha}</em>{10} \hat{\varsigma}_u} )</td>
<td>Voltage parameter</td>
</tr>
<tr>
<td>( \hat{d} = \frac{\hat{d}}{\hat{L}} )</td>
<td>Thickness</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Note in Table 1 that \( m_{11} = \int_0^1 (\phi_1(x))^2 dx \), \( s_{11} = \int_0^1 (\phi_1''(x))^2 dx \), \( b_{11} = \int_0^1 (\phi_1''(x))^2 dx \), with \( \phi_1(x) \) being equal to the normalized first mode shape of straight double clamped beam \[20\], and \( \beta = 2b_{11}/s_{11} \). Potential energy function related to the dynamic equation (3) is calculated by integrating \( F(q) \):

\[
U(q) = \int F(q) dq \tag{4}
\]

Shape of this potential function depends on the number of fixed points of Eq. (3) as its extremums. Effect of this potential function on the overall behavior of the resonator is discussed later in this paper.

2.3. Bistability analysis

Neglecting dynamic terms in Eq. (3) results in the static equilibrium or fixed points of the system. This is thoroughly studied in the literature \[20\]. The main result of this type of analysis is that the micro arch shows bistable static behavior for values of initial rise parameter \( h \) larger than a specific value \( h^* \), and also for the static voltage parameter \( \beta_1 \) being in a specific interval. Bounds of static snap-through and static pull-in are obtained via bifurcation diagrams related to these types of analysis \[20,25\].

For the parameter values out of the bistability region, the potential function given by Eq. (4) consists of one local minimum relating to a center point in the corresponding phase plane diagram (Fig. 2a), and a local maximum relating to the pull-in unstable saddle point. On the other hand, once the parameter values are in the bistability region, two local minima appear in the potential function relating to two center points in the phase plane (Fig. 2b). Moreover, in this case, two local maxima related to the unstable saddle points of snap-through and pull-in exist. Fig. 2 shows typical potential functions for these two cases.

2.4. Dynamic mathematical model

In this section, we consider the combined effects of the static DC and harmonic AC voltages on the dynamics of the micro-arch. Supposing \( \Omega_0 \) to be the dimensional frequency of the applied harmonic AC load, we define its respective normalized frequency \( \omega_0 \) to be consistent with the formulation of non-dimensional single degree of freedom model given in Eq. (3):

\[
\omega_0 = 2\pi \sqrt{\frac{m_{11}\beta AL^4}{b_{11}EI_y}} \Omega_0 \tag{5}
\]

Using the electrostatic actuation and also supposing the applied voltage to be equal to the sum of a static DC voltage and a harmonic AC load, we rewrite equation (3) to obtain the governing dynamic equation:

\[
\frac{d^2q}{d\tau^2} + \mu \frac{dq}{d\tau} + F(q) = 0 \tag{6}
\]

\[
F(q) = (1 + 2h^2 \zeta_1)q - 3\zeta_1 h q^3 + \zeta_1 q^5 - \frac{\beta_1 (1 + 2 R \cos(\omega_0 \tau))}{\sqrt{(1 + h - q)^2}}
\]

where \( R = V_{AC}/V_{DC} \). It is assumed here that the DC voltage is first applied and the system comes to rest in an equilibrium state. Then, a small AC voltage is applied to trigger oscillations around the static equilibrium position.

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Fig. 2. Typical potential energy functions for the parameter values given in Section 4.1, (a): for \( \beta = 90 \) out of the bistability region, (b): for \( \beta = 157 \) in the bistability region.
2.5. Model of vibrations around equilibrium points

According to the described sequence of loads, we introduce a change of variables in Eq. (6), as following:

\[ q(\tau) = q_0 + u(\tau) \]  

(7)

where \( q_0 \) represents either of the static fixed points due to the applied DC voltage. Substituting equation (7) into Eq. (6), expanding nonlinear terms in powers of \( u \) and collecting coefficients of like powers results in a general form of the governing equation in terms of \( u(\tau) \):

\[
\frac{d^2 u}{d\tau^2} + \mu \frac{du}{d\tau} + m_1 u + m_2 u^2 + m_3 u^3 + (s_0 + s_1 u + s_2 u^2 + s_3 u^3) R \cos(\omega_0 \tau) = 0
\]  

(8)

Note that in derivation of Eq. (8) the nonlinear electrostatic term in Eq. (6) is approximated by the 3rd order Taylor expansion. This leads to sufficient accuracy and will be justified later in the simulations. Also, coefficients given in Eq. (8) are obtained via basic calculations, as following:

\[
m_1 = 1 + 2h^2 x_1 - 6hx_1 q_0 + 3x_1 q_0^2 - \frac{3}{2} \beta_1 (1 + h - q_0)^{\frac{3}{2}}
\]  

(9a)

\[
m_2 = -3hx_1 - 3q_0 x_1 - \frac{15}{8} \beta_1 (1 + h - q_0)^{\frac{5}{2}}
\]  

(9b)

\[
m_3 = \alpha_1 - \frac{35}{16} \beta_1 (1 + h - q_0)^{\frac{7}{2}}
\]  

(9c)

\[
s_0 = -2\beta_1 (1 + h - q_0)^{\frac{3}{2}}
\]  

(9d)

\[
s_1 = -3\beta_1 (1 + h - q_0)^{\frac{5}{2}}
\]  

(9e)

\[
s_2 = -\frac{15}{4} \beta_1 (1 + h - q_0)^{\frac{7}{2}}
\]  

(9f)

\[
s_3 = \frac{35}{8} \beta_1 (1 + h - q_0)^{\frac{9}{2}}
\]  

(9g)

Note also that \( q_0 \) and \( \beta_1 \) satisfy the static equilibrium equation.

3. Analytical frequency response

In this section, we aim to obtain analytical solution for the frequency response of the general equation (8). To achieve this goal, we first suppose that \( \omega_0 \) is close to primary resonance frequency of the system:

\[
\omega_0 = \sqrt{m_1} + \sigma
\]  

(10)

where, \( \sigma \) is a detuning parameter. Prior to further progress in the solution procedure, we first introduce the following change of variables into Eq. (8):

\[
u(\tau) = \delta + y(\tau)
\]  

(11)

\[
\delta = \frac{1}{T} \int_0^T u(\tau) d\tau
\]  

(12)

where \( \delta \) denotes the temporal average of the nonlinear vibrations obtained by the averaging of \( u(\tau) \) in a period \( T \), and \( y(\tau) \) is a zero mean time series. This change of variables makes it feasible to account for the quadratic terms of Eq. (8) in the solution procedure. Thus, we are left by the following nonlinear equation which governs nonlinear vibrations of the micro arch near its equilibrium configuration:

\[
\frac{d^2 y}{d\tau^2} + \mu \frac{dy}{d\tau} + m_1 (y + \delta) + m_2 (y + \delta)^2 + m_3 (y + \delta)^3 + (s_0 + s_1 (y + \delta) + s_2 (y + \delta)^2 + s_3 (y + \delta)^3) R \cos(\omega_0 \tau) = 0
\]  

(13)

Note also that since we are interested in the steady state forced vibrations of the system, the transient behaviors imposed by initial conditions can be disregarded.
3.1. Homotopy Analysis Method (HAM)

Basic ideas of the Homotopy Analysis Method (HAM) are already described in the literature [34–36]. We make use of this method to find approximate analytical solution for the frequency response of the system governed by Eq. (13). Since the system is harmonically excited, we expect the solution of the given nonlinear equation to be expressed by the following series:

$$y(\tau) = \sum_{k=1}^{\infty} U_k e^{i\nu_k \tau} + \bar{U}_k e^{-i\nu_k \tau}$$  \hspace{1cm} (14)

where $U_k$ and $\bar{U}_k$ are complex conjugate constants. This gives the so-called “rule of solution expression” [35]. In order to construct basis of the HAM, we introduce an embedded parameter $p \in [0,1]$ and a variation $\phi(\tau, p)$: $\mathbb{R}^N \times [0,1] \to \mathbb{R}$ for which, as $p$ increases continuously from 0 to 1, $\phi(\tau, p)$ evolves from an initial guess $\phi(\tau, 0) = y_0(\tau)$ to the original solution $\phi(\tau, 1) = y(\tau)$. The initial guess $y_0(\tau)$, consistent with the rule of solution expression is given as follows:

$$y_0(\tau) = U e^{i\nu \tau} + \bar{U} e^{-i\nu \tau}$$  \hspace{1cm} (15)

We then choose the auxiliary linear operator:

$$L[\phi(\tau, p)] = \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \omega^2 \phi(\tau, p)$$  \hspace{1cm} (16)

which satisfies the following condition:

$$L[U e^{i\nu \tau} + \bar{U} e^{-i\nu \tau}] = 0$$  \hspace{1cm} (17)

Then, using Eq. (13), we can define the following nonlinear operator:

$$N(\phi(\tau, p)) = \frac{\partial^2 \phi(\tau, p)}{\partial \tau^2} + \mu \frac{\partial \phi(\tau, p)}{\partial \tau} + m_1(\phi(\tau, p) + \delta) + m_2(\phi(\tau, p) + \delta)^2 + m_3(\phi(\tau, p) + \delta)^3 + (s_0 + s_1(\phi(\tau, p) + \delta))$$

$$+ s_2(\phi(\tau, p) + \delta)^2 + s_3(\phi(\tau, p) + \delta)^3)R \cos(\omega_0 \tau)$$  \hspace{1cm} (18)

Using the given linear and nonlinear operators together with the embedding parameter $p$ and an auxiliary convergence-control parameter $h$, we can construct the following zeroth-order deformation equation:

$$1 - p)L[\phi(\tau, p) - y_0(\tau)] = h p N(\phi(\tau, p))$$  \hspace{1cm} (19)

Assuming deformation derivatives $y_k(\tau)$, we can construct a Maclaurin series in $p$ which converges to the variation $\phi(\tau, p)$:

$$y_k(\tau) = \frac{1}{k!} \left. \frac{\partial^k \phi(\tau, p)}{\partial p^k} \right|_{p=0}$$  \hspace{1cm} (20)

$$\phi(\tau, p) = \sum_{k=0}^{\infty} y_k(\tau) p^k$$  \hspace{1cm} (21)

On the other hand, it can be proved that deformation derivatives $y_k(\tau)$ are obtained by the solution of the following kth order deformation equations [35]:

$$L[y_k(\tau) - \chi_k y_{k-1}(\tau)] = h R_k(\tau)$$  \hspace{1cm} (22)

$$R_k(\tau) = \frac{1}{(k-1)!} \left. \frac{\partial^{k-1} N[\phi(\tau, p)]}{\partial p^{k-1}} \right|_{p=0}$$  \hspace{1cm} (23)

$$\chi_k = \begin{cases} 0 & k \leq 1 \\ 1 & k > 1 \end{cases}$$  \hspace{1cm} (24)

However, using initial guess of Eq. (15) together with Eqs. (22)–(24) we are left with the following equation for $y_1(\tau)$:

$$\frac{\partial^2 y_1(\tau)}{\partial \tau^2} + \omega^2 y_1(\tau) = h N[y_0(\tau)]$$  \hspace{1cm} (25)

In order to have bounded solution for Eq. (25), secular terms should be eliminated. These secular terms are clearly discovered if we rewrite the right hand of Eq. (25) in the form of Fourier transformation:

$$N[y_0(\tau)] = f_0(\cdot) + f_1(\cdot) e^{i\nu \tau} + f_2(\cdot) e^{2i\nu \tau} + \cdots$$  \hspace{1cm} (26)

According to this representation, $f_0(\cdot)$ should be equal to zero, for the temporal average of $y(\tau)$ is zero. Moreover, secular terms should be eliminated. Thus, the two following equations are obtained:
\[ f_0(\cdot) = 0 \Rightarrow m_1 \delta + m_2 \delta^2 + m_3 \delta^3 + 2m_2 \mathcal{U} + 6 \delta \mathcal{M} \mathcal{U} = 0 \] (27)

\[ f_1(\cdot) = 0 \Rightarrow \left( \frac{g}{2} \left( 2s_0 + \delta s_1 + \delta^2 s_2 + \delta^3 s_3 + 2s_2 \mathcal{U} + 6 \delta s_1 \mathcal{U} \right) e^{i \omega t} + \frac{g}{3} \left( s_2 \mathcal{U}^2 + 3 \delta s_3 \mathcal{U}^2 \right) e^{-i \omega t} + 2m_2 \mathcal{U} + 3 \delta^2 m_1 \mathcal{U} + \mu \omega \mathcal{U} i + i \omega \omega_0 \mathcal{U} \right) \] (28)

Note that in derivation of Eqs. (27) and (28), \( \omega_0 \) is replaced by \( \omega + \sigma \). Supposing \( a \) and \( b \) to be real parameters representing amplitude and phase shift of the response, \( \mathcal{U} = \frac{1}{2} e^{i \omega t} + \frac{1}{2} e^{-i \omega t} \) are introduced in Eqs. (27) and (28). In aftermath of common mathematical operations and introducing \( \mathcal{U} = \sigma \tau - b \), Eqs. (27) and (28) are converted to Eqs. (29) and (30), respectively.

\[ m_1 \delta + m_2 \left( \delta^2 + \frac{1}{2} a^2 \right) + m_3 \left( \delta^3 + \frac{3}{2} a^2 \delta \right) = 0 \] (29)

\[ \left( \frac{g}{2} \left( s_0 + \delta s_1 + \frac{1}{2} a^2 s_2 + \delta^3 s_3 + 3 a^2 \delta s_3 \right) e^{i \omega t} + \left( s_2 \mathcal{U}^2 + 3 \delta s_3 \mathcal{U}^2 \right) e^{-i \omega t} + a \delta m_2 + \frac{3}{2} a^3 m_3 + \frac{3}{2} a \delta^2 m_3 + \frac{1}{2} a \mu \omega i + i \omega (m_1 - \omega^2) \right) = 0 \] (30)

Equating real and imaginary parts of Eq. (30) to zero results in two simultaneous equations for \( \cos(\theta) \) and \( \sin(\theta) \). Using basic trigonometric operations eventuates in the following system of coupled nonlinear equations:

\[ \frac{1}{2} a^2 (m_2 + 3 m_3) \delta + m_1 \delta + m_2 \delta^2 + m_3 \delta^3 = 0 \] (31)

\[ \left( \frac{8 \delta m_2 + 3 a^2 m_3 + 12 \delta^2 m_1 + 4 (m_1 - \omega^2)}{4 s_0 + 4 \delta s_1 + a^2 s_2 + 4 \delta^2 s_2 + 3 a^2 \delta s_3} \right)^2 + \left( \frac{4 \mu \omega}{3 a \delta^2 m_3 + 2 a \delta^2 m_3 + \frac{1}{2} a \mu \omega + \frac{1}{2} a (m_1 - \omega^2)} \right)^2 = \left( \frac{R}{a} \right)^2 \] (32)

Note that terms containing \( \delta^4 \) are neglected due to its small value comparing to other terms. Equations (31) and (32) constitute a system of two nonlinear coupled equations with unknowns \( \delta \) and \( \omega \) as the response temporal average and amplitude respectively. Other parameters in these coupled equations are known for definite values of initial DC load, viscous damping, geometrical parameters, amplitude of the AC load, and the frequency of the applied load represented by \( \omega = \omega_0 \). Supposing all these loading and physical parameters to be known, steady state amplitude of the response is obtained by simultaneous solution of Eqs. (31) and (32). This will be discussed in the next section.

We are now finished with the first step of the HAM which provides the needed analytical solution for the frequency response of the system. Note that the convergence-control parameter \( \hbar \) appears as a non-zero coefficient in the left hand of Eqs. (27) and (28); thus it does not enter the corresponding frequency response equations. One can pursue the procedure by solving equation (25) for \( y_l(\tau) \), stepping forward and constructing appropriate number of additional terms of the series solution which include parameter \( \hbar \), and finally, ensuring the convergence of the series by tuning parameter \( \hbar \). Since we are satisfied with the obtained analytical frequency response from the first step, we terminate the procedure in this step.

4. Simulations and results

In this section we aim to solve the obtained equations of frequency response (Eqs. (31) and (32)). The results are validated through comparison with the numerical simulations. Also, two types of nonlinear behaviors of the arch shaped MEMS resonator are introduced and discussed in this section.

4.1. Frequency response for small actuations

We first suppose all geometric, DC load and damping parameters to be known. Then for a given value of the AC voltage amplitude (known \( R \)) and a given value for the AC actuation frequency (known \( \omega \)) equations (31) and (32) are solved numerically. For the sake of comparison, Eq. (6) is also solved numerically under the same loading conditions. In order to obtain the frequency response via numerical solution of Eq. (6), frequency of the applied AC load is increased step by step, and then, it is decreased back. At each step, simulations are performed for long time (\( \tau = 2500 \) for instance) and the resulting steady state amplitude is recorded.

An arch shaped micro beam of length \( L = 1000 \mu m \), thickness \( d = 3 \mu m \), width \( b = 30 \mu m \), gap \( g_0 = 10 \mu m \) and initial rise \( h = 0.38 \) is studied in the following simulations. Frequency response of the arch MEMS resonator, for loading and damping parameters of \( \mu = 90, R = 0.02 \) and \( \mu = 0.02 \) is shown in Fig. 3, using both HAM and numerical simulation of Eq. (6).

Due to the shape of the potential energy function of this case (see Fig. 2(a)), the arch is not in the bistability region for parameter values associated with simulations depicted Fig. 3. In this case, the arch MEMS resonator acts similar to a typical resonator, showing a softening behavior. Once the arch is in the bistability region, provided that the actuation parameters are sufficiently high, the arch undergoes dynamic snap-through instability. In this case, interesting sequential jumps between stable fixed points will be feasible. Frequency responses of these conditions are studied in the following.
4.2. Frequency response in bistability region

As discussed earlier, the system contains two stable fixed points in the bistability region. All parameters including $\beta$ are the same for the two fixed points and only the values of $q_0$ differ between them. Thus, for each value of $q_0$ we can calculate parameters given by Eq. (9); consequently, we can rewrite equation (8) for each of these two points. This yields into two equations in the form of Eq. (8) governing vibrations in the vicinity of each of the fixed points. It is clear that the frequency responses for each of these fixed points can be obtained by Eqs. (31) and (32).

Fig. 4 displays the two frequency responses for parameter values given in Section 4.1 with $\beta = 157$, $\mu = 0.03$ and $R = 0.02$. These set of parameters result in two stable static fixed points, $q_{01} = 0.12$ and $q_{02} = 0.79$. It is clear in this figure that the two frequency responses corresponding to the so called stable fixed points merge together in the whole frequency domain. Moreover, as illustrated in Fig. 4, for some range of frequencies, two or even three stable solutions are possible. The main question arising here is that which of the solutions is possible and takes place in practice.

In order to discuss the merging behavior in the bistable region we define four types of simulations, for parameter values associated with Fig. 4. These simulations differ only in the initial conditions, as following:

1. Around the lower center starting with large frequencies.
2. Around the higher center starting with large frequencies.
3. Around the lower center starting with small frequencies.
4. Around the higher center starting with small frequencies.

In all four types of simulations, both backward and forward frequency sweeps are performed. Thus, each simulation starts in a fixed frequency, and ends in the same frequency. Fig. 5 displays the results of the above four types of simulations together with the HAM-derived frequency responses for lower and higher fixed points.

Before further discussions in the details of the system behavior in the above four types of simulations, it is worthy to be noted that in some specific frequencies, the vibrations are switched between the two frequency responses. Recalling that these switching behaviors represent transition between the two configurations due to dynamic snap-through, we introduce four switching frequencies in Table 2.

Also note that the following mechanisms of sudden jumps are recognized in Fig. 5:

1. Jumps taking place on one of the frequency responses due to softening behavior without switching between the two graphs; we name this jump mechanism as “simple jump”.
2. Jumps between the two frequency responses which result in the transitions between the two configurations. This type of jumps takes place in the switching frequency responses given in Table 2, and we name this jump mechanism as “snap-through jump”.

With these definitions we begin to discuss in details the above four types of simulations.

Type 1: the system which is initially at rest in the lower fixed point is excited with a frequency which is larger than the fundamental natural frequency of the arch (Fig. 5a). Backward frequency sweep results in the growth of the response amplitude due to resonance up to point B with frequency $\omega_{LR}$. At this point, dynamic response amplitude reaches the critical value, and then, the dynamic snap-through occurs reaching point C and forcing the vibrations to continue in the domain of attraction of the higher fixed point. Further decrease in the excitation frequency causes decrease in the response amplitude up to point D.
Reversing the frequency path from backward to forward at point D, increases the response amplitude up to point E with frequency $x_{HLL}$ where the system tends to undergo "simple jump" due to softening behavior in the vicinity of the higher fixed point. But in this case of undergoing "simple jump", the resulting high amplitude is not possible since it would be larger than the critical snap-through amplitude in the neighborhood of the higher fixed point. Thus, the system undergoes "snap-through jump" and point F in the neighborhood of the lower fixed point is reached. Further increasing the applied frequency moves the system to point A. Note that after a backward and forward sweeps of the forcing frequency, the system comes back to its original starting point.

Type 2: simulation starts at point A in the neighborhood of higher fixed point with large frequency (Fig. 5b). Backward frequency sweep results in the growth of the response amplitude up to point B with frequency $x_{HLR}$. Since the response amplitude reaches the critical snap-through amplitude in the vicinity of the higher fixed point, the system undergoes a "snap-through jump" and reaches point C near the lower fixed point. Then the system passes through points D–E–F in the backward frequency sweep just like the path of points B–C–D in Fig. 5a. From point F up to end point I in forward frequency path, points F–G–H–I are passed, just like the corresponding path of D–E–F–A in Fig. 5a. Note that although a closed frequency cycle is applied, in this simulation starting point A is not the same as the end point I.

Type 3: simulations start at point A near the lower fixed point with small frequency (Fig. 5c). Sweeping forward the forcing frequency, the response amplitude increases due to resonance up to point B with frequency $x_{LHL}$ where it tends to undergo "simple jump" due to softening behavior. But, if the system undergoes "simple jump" at this point, the resulting high amplitude would be larger than the critical snap-through amplitude in the neighborhood of lower fixed point. Thus, the system undergoes "snap-through jump" instead of "simple jump" at this frequency and point C in the domain of attraction of the higher fixed point is reached out. Further increasing the excitation frequency leads the system to point F near the lower fixed point through the path D–E–F just like the path E–F–A in Fig. 5a. Also, backward frequency path from point F to point I through points G and H is the same as corresponding path A–B–C–D in Fig. 5a. Note that in this type of simulations, start and end points are not the same (as in Type 2).

Type 4: simulations start near the higher fixed point with small frequency (Fig. 5d). It is clear that the path of A–B–C–D in the forward frequency sweep is the same as path D–E–F–A in Fig. 5a. Also, the path D–E–F–A in the backward frequency path is the same as the path A–B–C–D in Fig. 5a. Thus, simulations of Type 1 and Type 4 result in the same closed response loop.

- Following results are also concluded by detailed analysis of the above four types of simulations.
- The total frequency responses given in Fig. 5a and d are the same, and form a closed hysteresis loop which can be repeated over and over only by varying the forcing frequency. However, frequency responses given in Fig. 5b and c do not form a closed loop. The more interesting result is that if we repeat the frequency cycle for these two types of simulations, they are converted to the one shown in Fig. 5a or 5d. This means that the open loops given in Fig. 5b and c are transient and the dominant frequency response is as shown in Fig. 5a.
System parameters determine how close the two frequency responses are in the bistability region. In a case where the two frequency responses are far from each other, the switching behaviors diminish. While, in the case they are very close together the system can unpredictability undergo consequent “snap-through jumps”. This may result in chaotic vibrations.

As stated earlier, jumps between the stable fixed points occur when the dynamic response amplitude reaches the critical amplitude of the homoclinic orbit. In the numerical simulations this happens slightly earlier. This is mainly due to the transient behaviors which cause an overshoot in the overall dynamic response which is larger than the steady state behavior.

The four critical frequencies summarized in Table 2 determine the total behavior of the system. Also, the dominant critical frequencies $\omega_{HLL}$ and $\omega_{LHR}$ play an important role in the initiation of the steady-state arch bistable dynamic behavior. These frequencies mainly depend on the values of the system operating parameters and the created hysteresis loop depends on these two critical frequencies.

5. Conclusions

During recent years, arch shape bistable microbeams have been evolved into a useful tool in different applications including microrelays, microvalves, switches and MEMS based memories. In spite of considerable progress made in the study of the dynamic behavior of bistable microbeams, there are still many issues under question. In this work, the nonlinear dynamics of an initially curved double clamped microbeam under combined static (DC) and harmonic (AC) distributed electrostatic actuation is studied. The nonlinear equation of motion of the shallow arch described in the framework of Euler–Bernoulli beam theory is reduced to a nonlinear ordinary differential equation by means of first mode Galerkin’s decomposition method. This single degree of freedom model is implemented for the analytical and numerical investigation of the nonlinear dynamics of the bistable MEMS resonator near its primary resonance frequency.

In order to account for the dynamic snap-through that is equivalent to abrupt variation in the average of the dynamic response, we described the overall displacement of the arch using two independent unknowns, namely the amplitude and the temporal average of dynamic vibrations. Then, we used HAM method in order to obtain analytical descriptions for these two unknowns. Obtained analytical expressions are then validated through numerical simulations.
The results show that the dynamic response of the presented arch is of softening type. Moreover, analytical results show that in the bistability region the overall dynamic response of the system is described by means of a couple of softening type frequency responses merging together in a specific frequency band. The dynamic snap-through is then described by transitions between these two frequency responses each of which corresponding to one of the stable configurations of the arch. This overcomes the previously reported limitations [17] in the analytical description of the dynamic snap-through using perturbation techniques.

The presented analytical expressions validated by numerical simulations provide a fresh insight into the problem of nonlinear dynamic response of the arch resonator within its bistability region. Using appropriate simulations, we showed that the overall dynamic response of the system may depend on the loading conditions. This suggests that the dynamic snap-through band may differ in the forward and backward frequency sweeps. This is an important result which should be noted in the application of arch MEMS as bandpass filters, an application which is recently discussed in detail by Ouakad and Younis [30, 31].

The analytical results presented in this work provide appropriate tool for engineers for the design and improvement of the bistable shallow micro/nano arches. The deep perception of the nonlinear vibrations of presented bistable MEMS resonator, specially understanding the dynamics of sequential snap-through instabilities, opens possibilities for various applications such as filters, resonant sensors, actuators, and memories. Future works are needed to investigate the effects of various parameters and nonlinearities on the bistable vibrations of the MEMS/NEMS resonators.

References
