Effect of geometric nonlinearity on dynamic pull-in behavior of coupled-domain microstructures based on classical and shear deformation plate theories

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ABSTRACT
This paper investigates the dynamic pull-in behavior of microplates actuated by a suddenly applied electrostatic force. Electrostatic, elastic and fluid domains are involved in modeling. First-order shear deformation plate theory and classical plate theory are used to model the geometrically nonlinear microplates. The equations of motion are discretized by the finite element method. The effects of nonlinearity, fluid pressure, initial stress and different geometric parameters on dynamic behavior are examined. In addition, the influences of initial stress and actuation voltage on oscillatory behavior of microplates are evaluated.

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1. Introduction

Electrostatically actuated microelectromechanical systems (MEMS) are extensively used as resonant sensors in different applications such as signal filtering and mass sensing (Batra et al., 2008a). One of the most important phenomena associated with microelectromechanical systems is the pull-in instability. The pull-in instability was observed experimentally by Nathanson et al. (1967) and Taylor (1968). The pull-in instability occurs when the electrostatic force exceeds the elastic restoring force of the structure, leading to contact between the actuated structure and substrate (Chao et al., 2008). The critical value of voltage corresponding to this instability is referred to as the pull-in voltage. Calculating the pull-in voltage is essential in design process to analyze sensitivity, frequency response and dynamic range of the microstructures (Chowdhury et al., 2005). When the rate of voltage variation is very low and consequently inertia has almost no influence on the microstructure behavior, the critical value of voltage is called the static pull-in voltage (V_{ps}). However, when the rate of voltage variation is not negligible, the effect of inertia has to be considered. The pull-in instability related to this situation is called the dynamic pull-in instability and the critical value of voltage, corresponding to the dynamic instability, is referred to as the dynamic pull-in voltage (V_{pd}) (Krylov, 2007). Several authors have investigated the dynamic behavior of microstructures so far.

Krylov and Maimon (2004) have studied the transient dynamics of an electrically actuated microbeam considering the electrostatic forces, squeeze film damping, and rotational inertia of a mass carried by the microbeam. Rochus et al. (2005) have proposed a finite element approach to study dynamic pull-in behavior in MEMS, considering geometrically nonlinear structural elements. Krylov (2007) has investigated the dynamic pull-in instability of microbeams subjected to nonlinear squeeze film damping using a reduced-order model. Batra et al. (2008b, c) have considered the von Kármán nonlinearity and the Casimir force to develop reduced-order models for prestressed clamped rectangular and circular microplates. Reduced-order models have been used to study pull-in parameters and vibrations about a predeformed configuration. Nevertheless, the effect of inertia on the pull-in parameters has not been analyzed. Chao et al. (2008) have investigated the dynamic pull-in instability for a generalized double-clamped microbeam based on a continuous model and bifurcation analysis.

Dynamic pull-in instability is affected by different system parameters such as damping, in-plane forces, electrostatic actuation, structure stiffness and inertial forces. There are several mechanisms of damping associated with MEMS devices including extrinsic and intrinsic losses (Younis, 2004). Among extrinsic damping effects, squeeze film damping is the most significant source of energy loss, which has been studied widely so far. McCarthy et al. (2002) have used a time-transient finite difference analysis to model the dynamic behavior of two different electrostatically actuated microswitch configurations. Younis (2004) has presented hybrid numerical–analytical approaches to simulate microelectromechanical systems in multi-physics field considering...
squeeze film damping. Younis and Nayfeh (2007) have simulated the squeeze film damping of microplates considering large electrostatic loads. Perturbation techniques have been used to derive pressure distribution around the deflected position. The scaling effect, stability, nonlinearity and modeling of electrostatically actuated MEMS have been reviewed by Zhang et al. (2007) and Batra et al. (2007).

The objective of the present paper is to study the effects of different parameters on the dynamic pull-in instability of microplates based on different plate theories. The approach utilizes first-order shear deformation plate theory (FSDT) and classical plate theory (CPT), coupled with nonlinear electrostatic force term and Reynolds equation. The finite element method is utilized to describe the microstructure equations. The effects of geometric nonlinearity, ambient pressure and physical parameters on dynamic pull-in instability and oscillatory behavior are investigated. Achievements are summarized in the conclusion.

2. Problem formulation

In a prismatic microstructure, a deformable microplate is suspended over a fixed rigid electrode (substrate), as shown in Fig. 1. The length of the microplate is l, the width is b, the thickness is h, the constant initial gap is d_gap, x and y are in-plane coordinates, z is the coordinate along thickness, u_0, v_0 and w_0 are midplane displacements along x-, y- and z-axes respectively, t is the time and l_i is the mass inertia (i = 1, 2, 3).

In designing an electrostatically actuated microstructure, the elastic-fluid-electrostatic coupling has to be taken into account. Therefore, it is important to model various effects in MEMS. Electrostatic force, squeeze film damping and midplane stretching are some of these effects, which are all nonlinear.

The effect of electrostatic force is significant in micro domain. When a voltage V is applied between the microplate and the substrate, an attractive electrostatic force causes the microplate to deflect. Electrostatic actuation is popular due to its advantages such as fast response, the ability to achieve rotary motion, and low power consumption (Zhang et al., 2007). The electrostatic actuation can be simplified by utilizing the parallel plate approximation and neglecting the fringing field effects as

$$q_e = \frac{1}{2} \frac{\varepsilon V^2}{(d_{\text{gap}} - w(x, y, t))^2}$$

where $\varepsilon$ is the vacuum permittivity.

The squeeze film damping occurs as a result of the massive movement of the fluid underneath the plate (Younis, 2004). The effect of squeeze film damping can be modeled by compressible (nonlinear) Reynolds equation as

$$\frac{\partial}{\partial x}(H^2 P \frac{\partial P}{\partial x}) + \frac{\partial}{\partial y}(H^2 P \frac{\partial P}{\partial y}) = 12 \mu_{\text{eff}} \left( \frac{\partial^2 P}{\partial t^2} + P \frac{\partial H}{\partial t} \right)$$

where P, H and $\mu_{\text{eff}}$ are the total pressure, the variable distance between the two plates (i.e. $H = d_{\text{gap}} - w_0$) and the effective viscosity of the fluid calculated using the model of Veijola et al. (1995), respectively. It should be noted that the effective viscosity of the fluid accounts for the rarefied gas effect.

Depending on the thickness-to-length ratio of the plate, FSDT or CPT can be utilized to model the microplate deflections (Reddy, 2004). The displacement field of the plate based on FSDT is

$$u(x, y, z, t) = u_0(x, y, t) + z \phi_x(x, y, t)$$
$$v(x, y, z, t) = v_0(x, y, t) + z \phi_y(x, y, t)$$
$$w(x, y, z, t) = w_0(x, y, t)$$

where $u$, $v$ and $w$ are displacements in the x-, y- and z-axes, and $\phi_x$ and $\phi_y$ are the rotations of a transverse normal about y- and x-axes, respectively. Using Eq. (3), the strain components are defined as

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 + z \frac{\partial \phi_x}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 + z \frac{\partial \phi_y}{\partial y}$$
$$\varepsilon_{zz} = 0$$
$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial \phi_y}{\partial x} + z \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)$$

Here $(1/2)(\partial w_0/\partial y)^2$, $(1/2)(\partial w_0/\partial x)^2$ and $(\partial w_0/\partial x)(\partial w_0/\partial y)$ are the von Kármán nonlinear strains, which are considered to account for midplane stretching (Younis and Nayfeh, 2007).

Using the dynamic version of the von Kármán equations for isotropic plates and assuming FSDT model, squeeze film damping and electrostatic actuation, the equations of motion are given by

$$\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2}$$
$$\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2}$$
$$\delta w_0 : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N + 1 \frac{\partial^2}{} \frac{\partial (\partial w_0 - P)}{\partial t^2} = \frac{1}{2} \frac{(d_{\text{gap}} - w_0)^2}{(d_{\text{gap}} - w_0)^2}$$

subject to zero initial conditions. Here $P_0$, $N_0$, $M_i$ and $Q_i$ are the ambient pressure, in-plane force resultants, moment resultants and transverse force resultants, respectively, which have been presented by Reddy (2004). It is to be noted that $N = N_{xx}(\partial^2 w_0/\partial x^2)$ is considered to take into account the effect of initial stresses in the x-direction. The structural boundary conditions are expressed as

$$x = 0 \text{ : } u_0 = v_0 = w_0 = \phi_x = \phi_y = 0$$
$$y = 0 \text{ : } N_{yy} = N_{xy} = M_{yy} = M_{xy} = Q_y = 0$$

Subject to the boundary condition $w(x, y, t) = 0$ at $y = 0$ and $x = 0$, and initial condition $u_0(x, y, t) = v_0(x, y, t) = w_0(x, y, t) = 0$ at $t = 0$. The displacement field of the plate based on CPT is

$$u(x, y, z, t) = u_0(x, y, t) + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 + z \frac{\partial \phi_x}{\partial x}$$
$$v(x, y, z, t) = v_0(x, y, t) + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 + z \frac{\partial \phi_y}{\partial y}$$
$$w(x, y, z, t) = w_0(x, y, t)$$

(6)

Fig. 1. Schematic view of a double-clamped microplate, similar to Younis and Nayfeh (2007).
The pressure boundary conditions are expressed as

\[
\begin{align*}
x = 0, l: & \quad \frac{\partial P}{\partial x} = 0 \\
y = 0, b: & \quad P = P_0
\end{align*}
\]

(7)

Similar equations can be derived based on CPT. Therefore, one can obtain

\[
\begin{align*}
\delta u_0: & \quad \frac{\partial^2 N_{xx}}{\partial x^2} + \frac{\partial N_{xy}}{\partial y} + \frac{\partial^2 N_{yy}}{\partial y^2} + \frac{1}{2} \left( \frac{d_{gap} - W_0}{\nu} \right)^2 \\
\delta v_0: & \quad \frac{\partial^2 N_{xy}}{\partial x^2} + \frac{\partial^2 N_{yy}}{\partial y^2} + \frac{1}{2} \left( \frac{d_{gap} - W_0}{\nu} \right)^2 \\
\delta W_0: & \quad \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + \tilde{N} + \frac{1}{2} \left( \frac{d_{gap} - W_0}{\nu} \right)^2 \\
& \quad -(P - P_0) = \frac{\partial^2 \sigma}{\partial t^2} - \int_0^l \frac{\partial^2 \sigma}{\partial y^2} \left( \frac{d_{gap} - W_0}{\nu} \right) dy \\
& \quad = 12 \mu_{eff} \left( \frac{d_{gap} - W_0}{\nu} \right) \left( \frac{\partial \sigma}{\partial t} - P \frac{\partial W_0}{\partial t} \right)
\end{align*}
\]

subject to zero initial conditions. Here \(N_{ij}\) and \(M_{ij}\) are in-plane force resultants and moment resultants, respectively (Reddy, 2004).

The structural boundary conditions are expressed as

\[
\begin{align*}
x = 0, l: & \quad u_0 = v_0 = W_0 = \frac{\partial W_0}{\partial x} = \frac{\partial W_0}{\partial y} = 0 \\
y = 0, b: & \quad N_{xy} = N_{yy} = M_{xy} = M_{yy} = Q_y = 0
\end{align*}
\]

(9)

The pressure boundary conditions are expressed by Eq. (7).

2.1. Finite element modeling and computational algorithm

Based on FSDT, the variables \(u_0, v_0, W_0, \psi_x, \psi_y\) and \(P\) can be approximated using the quadratic Lagrange interpolation functions \(\psi_i\) (Reddy, 2004),

\[
\begin{align*}
u_0(x, y, t) = & \quad \sum_{i=1}^{n} U_i(t) \psi_i(x, y), \quad n = 9 \\
v_0(x, y, t) = & \quad \sum_{i=1}^{n} V_i(t) \psi_i(x, y), \quad n = 9 \\
w_0(x, y, t) = & \quad \sum_{i=1}^{n} W_i(t) \psi_i(x, y), \quad n = 9 \\
\phi_x(x, y, t) = & \quad \sum_{i=1}^{n} S_{1i}^x(t) \psi_i(x, y), \quad n = 9 \\
\phi_y(x, y, t) = & \quad \sum_{i=1}^{n} S_{2i}^y(t) \psi_i(x, y), \quad n = 9 \\
P(x, y, t) = & \quad \sum_{i=1}^{n} P_i(t) \psi_i(x, y), \quad n = 9
\end{align*}
\]

(10)

where \(U_i, V_i, W_i, S_i^1, S_i^2, P\) are the nodal values of \(u_0, v_0, W_0, \phi_x, \phi_y, P\), respectively. The quadratic Lagrange interpolation functions of a nine-node element in terms of the element coordinates are (Reddy, 2004)

\[
\begin{align*}
\psi_1 & = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1) + \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_2 & = 2(1 - \xi^2)(1 - \eta) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_3 & = (1 + \xi)(1 - \eta)(-\xi + \eta - 1) + \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_4 & = 2(1 + \xi)(1 - \eta^2) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_5 & = \frac{1}{4}(1 + \xi^2)(1 - \eta^2) \\
\psi_6 & = 2(1 + \xi)(1 - \eta^2) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_7 & = (1 - \xi)(1 + \eta)(-\xi + \eta - 1) + \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_8 & = 2(1 - \xi)(1 + \eta^2) - \frac{1}{4}(1 - \xi^2)(1 - \eta^2) \\
\psi_9 & = (1 + \xi)(1 + \eta)(\xi + \eta - 1) + \frac{1}{4}(1 - \xi^2)(1 - \eta^2)
\end{align*}
\]

(11)

The node numbering and local coordinates are shown in Fig. 2. In order to generate the finite element model of the squeeze film damping, the weak form associated with Eq. (2) has to be derived. Due to the nonlinear nature of the problem, iterative methods can be used to solve the equations. For this purpose, Eq. (2) is rewritten as

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \int_{\Gamma} \left( H^{(k)} \right)^3 p(k) \frac{\partial p^{(k+1)}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \int_{\Gamma} \left( H^{(k)} \right)^3 p(k) \frac{\partial p^{(k+1)}}{\partial y} \right) \\
& = 12 \mu_{eff} \left( \int_{\Gamma} \left( H^{(k)} \right)^3 p^{(k+1)} \frac{\partial w^{(k+1)}}{\partial t} \right)
\end{align*}
\]

(12)

where \(k\) is the iteration number. By multiplying Eq. (12) by the weight function \(\psi_i\), integrating by parts over the area of an arbitrary element, and substituting Eq. (10) into the resulted equation one obtains the following weak form

\[
\begin{align*}
& - \int_{\Gamma} \left( \int \left( \psi_i \psi_j (\psi_j \psi_j + \psi_j \psi_j) \left( H^{(k)} \right)^3 p^{(k)} \right) dA \right) P_j^{(k+1)} \\
& + \int_{\Gamma} \left( \int \left( \psi_i \left( H^{(k)} \right)^3 p(k) \frac{\partial p^{(k+1)}}{\partial y} \right) \right) dA \\
& = 12 \mu_{eff} \left( \int_{\Gamma} \left( \psi_i \psi_j-H^{(k)} \right) dA \right) P_j^{(k+1)} \\
& - \int_{\Gamma} \left( \int \left( \psi_i \psi_j \frac{\partial p^{(k)}}{\partial y} \right) \right) dA
\end{align*}
\]

(13)

By combining the finite element model of plates based on FSDT, presented by Reddy (2004), and Eq. (13), one can develop the following semi-discrete finite element model for an element of the coupled system

\[
\begin{align*}
[M^{(k)}]^{(k+1)} \{X^{(k+1)}\} + [C^{(k)}]^{(k+1)} \{X^{(k+1)}\} + [K^{(k)}]^{(k+1)} \{X^{(k+1)}\} = \{F^{(k)}\}
\end{align*}
\]

(14)

where

\[
\begin{align*}
M^{e} = & \begin{bmatrix} M_{45}\times45 & 0_{45}\times9 \vline 0_{45}\times9 \\ 0_{45}\times9 & 0_{45}\times9 \end{bmatrix}, \quad K^{e} = \begin{bmatrix} K + G_{45}\times45 & 0_{45}\times9 \\ 0_{45}\times9 & K_{9}\times9 \end{bmatrix} \\
C^{e} = & \begin{bmatrix} 0_{45}\times45 & 0_{45}\times9 \vline 0_{45}\times9 \\ 0_{45}\times9 & 0_{45}\times9 \end{bmatrix}, \quad F^{e} = \begin{bmatrix} \{F\} \\ \{1\} \end{bmatrix}
\end{align*}
\]

(15)
Here [K], [G] and [M] are stiffness, in-plane force and mass matrices based on FSDT, which have been presented by Reddy (2004) and Ng et al. (2004). It should be noted that the vector \( \{\chi\} = \{u_0^T, u_0^T, w_0^T, \phi_x^T, \phi_y^T\}^T \) is the deflection vector of the microstructure, based on FSDT. The detailed components of matrices \([C^w], [C^p]\) and \([K^p]\), and vectors \(F\) and \(\{Q^s\}\) are expressed as

\[
C^w_{ij} = -12\mu_{eff} \int \psi_i \psi_j P^{(k)} \, dA, \quad C^p_{ij} = 12\mu_{eff} \int \psi_i \psi_j H^{(k)} \, dA
\]

\[
K^p_{ij} = \int \int (H_i^{(k)})^3 p^{(k)} \, dA, \quad Q^s_{i} = \int \psi_i (H_i^{(k)})^3 p^{(k)} (\nabla p^{(k)} + 1) \cdot dS
\]

\[
F_1^j = \int P_x \psi_j \, dA, \quad F_2^j = \int P_y \psi_j \, dA, \quad F_3^j = \int \psi_j (q_e - (p_i - P_0)) \, dA + \int Q_0 \psi_j \, ds
\]

\[
F_4^j = \int T_x \psi_j \, dA, \quad F_5^j = \int T_y \psi_j \, dA
\]

where \(P_x, P_y, T_x, T_y\) and \(Q_0\) are the secondary variables. It is noteworthy that the linear FSDT model can be obtained by omitting the von Kármán nonlinear strains.

A similar procedure can be performed to derive the finite element model of the coupled system based on CPT.

To this end, the variable \(w_0\) is approximated using the Hermite interpolation functions \(\Phi_i\) of a nonconforming rectangular element as (Younis and Nayfeh, 2007)

\[
w_0(x, y, t) = \sum_{i=1}^{n} \Delta_i(t) \Phi_i(x, y) \quad n = 12
\]

where the \(\Delta_i\) are the nodal values of \(w_0\) and its derivatives. The variables \(u_0, v_0\), and \(P\) can be approximated using the linear Lagrange interpolation functions \(\Psi_i\),

\[
u_0(x, y, t) = \sum_{i=1}^{n} U_i(t) \Psi_i(x, y) \quad n = 4
\]

\[
v_0(x, y, t) = \sum_{i=1}^{n} V_i(t) \Psi_i(x, y) \quad n = 4
\]

\[
P(x, y, t) = \sum_{i=1}^{n} P_i(t) \Psi_i(x, y) \quad n = 4
\]

Further information on linear Lagrange and Hermite interpolation functions can be found in Reddy (2004).

By multiplying Eq. (12) by the weight function \(\Psi_i\), integrating by parts over the area of an arbitrary element, and introducing Eqs. (17) and (18) into the resulted equation, one can obtain

\[
- \int \int (\Psi_{i,x}\Psi_{j,x} + \Psi_{i,y}\Psi_{j,y}) (H^{(k)})^3 p^{(k)} \, dA \left[ F_{j}^{(k+1)} \right]
\]

\[
+ \int \psi_i (H^{(k)})^3 p^{(k)} (\nabla p^{(k)} + 1) \cdot dS
\]

\[
\text{Boundary Condition}
\]

\[
= 12\mu_{eff} \int \psi_i \psi_j H^{(k)} \, dA \left[ F_{j}^{(k+1)} \right] - 12\mu_{eff} \int \psi_i \phi_i p^{(k)} \, dA \left[ W_{j}^{(k+1)} \right]
\]

By combining Eq. (19) and the finite element model of plates based on CPT, one can find the following stiffness, in-plane force and mass matrices for an element of the coupled system

\[
M^e = \begin{bmatrix}
[M]_{20 \times 20} & [0]_{20 \times 4} \\
[0]_{4 \times 20} & [0]_{4 \times 4}
\end{bmatrix}
\]

\[
C^e = \begin{bmatrix}
[0]_{20 \times 20} & [0]_{20 \times 4} \\
[0]_{4 \times 20} & [C^w]_{4 \times 12}
\end{bmatrix}
\]

\[
K^e = \begin{bmatrix}
[K + G]_{20 \times 20} & [0]_{20 \times 4} \\
[0]_{4 \times 20} & [K^p]_{4 \times 4}
\end{bmatrix}
\]

where \([K], [G]\) and \([M]\) are stiffness, in-plane force and mass matrices based on CPT (Reddy, 2004). In addition, the deflection and force vectors for an element of the coupled system can be expressed as

\[
X^e = \begin{bmatrix}
\{x\} \\
\{p\}
\end{bmatrix}, \quad F^e = \begin{bmatrix}
\{F\} \\
\{Q^s\}
\end{bmatrix}
\]

where the vector \(\{x\} = \{u_0^T, v_0^T, w_0^T\}^T\) is the deflection vector of the microstructure, based on CPT. The detailed components of matrices \([C^w], [C^p]\) and \([K^p]\) and vectors \(F\) and \(\{Q^s\}\) are expressed as

\[
C^w_{ij} = -12\mu_{eff} \int \psi_i \psi_j \phi_i p^{(k)} \, dA, \quad C^p_{ij} = 12\mu_{eff} \int \psi_i \psi_j H^{(k)} \, dA
\]

\[
K^p_{ij} = \int \int \psi_i (H^{(k)})^3 p^{(k)} (\nabla p^{(k)} + 1) \cdot dS
\]

\[
F_1^j = \int P_x \psi_j \, dA, \quad F_2^j = \int P_y \psi_j \, dA, \quad F_3^j = \int \psi_j (q_e - (p_i - P_0)) \, dA + \int Q_0 \psi_j \, ds
\]

\[
F_4^j = \int T_x \psi_j \, dA, \quad F_5^j = \int T_y \psi_j \, dA
\]

where \(P_x, P_y\) and \(Q_0\) are the secondary variables. The numerical integration over the area of an arbitrary element is performed using

\begin{table}[h]
\centering
\caption{Comparison between static pull-in voltages of different microplates (b = 50 μm, h = 3 μm, \(d_{gap} = 1 μm\)).}
\begin{tabular}{|c|c|c|c|c|}
\hline
Length, \(l\) (μm) & Initial stress, \(\sigma\) (MPa) & \(V_{pull-in}\) Linear FSDT (Volt) & \(V_{pull-in}\) Nonlinear FSDT (Volt) & \(V_{pull-in}\) Osterberg (1995) (Volt) \\
\hline
250 & 0 & 39.41 & 39.64 & 40.1 \\
250 & -25 & 33.30 & 33.58 & 33.6 \\
250 & 100 & 57.32 & 57.46 & 57.6 \\
350 & 0 & 20.23 & 20.25 & 20.3 \\
350 & -25 & 13.30 & 13.49 & 13.7 \\
350 & 100 & 35.75 & 35.80 & 35.8 \\
\hline
\end{tabular}
\end{table}
Gaussian quadrature method. Afterward, by assembling the equations of various elements, the finite element model of the total coupled problem is found. The fully discretized form of the total problem is obtained using Newmark time discretization. There are some well-known schemes for Newmark method. In this paper, the constant-average acceleration method is utilized. It is noteworthy that, due to the nonlinearity of the problem, the direct iteration method is utilized to find deflections in each time step. The iteration process in each time step is continued until the convergence criteria of deflections are satisfied. For more study on direct iteration method, see Reddy (2004).

3. Simulation and discussion

In order to validate the model, the results can be compared with the data presented for the static pull-in instability in the literature. It should be noted that the equations of the static behavior are obtained by ignoring the effects of inertia and fluid pressure. The values of static pull-in voltage \( V_{pi} \) are presented in Table 1. In accordance with this table, the pull-in voltages are elevated with an increase in the initial stress. It can be seen that the values of \( V_{pi} \) are consistent with those reported by Osterberg (1995).

In order to study the dynamic behavior of microstructures, first the convergence of the model should be examined. To this end, one can consider a microstructure, subjected to zero initial conditions and a suddenly applied voltage of \( V = 60 \) V (\( l = 250 \) μm, \( b = 50 \) μm, \( h = 2 \) μm, \( E = 169 \) GPa, \( \rho = 2330 \) kg/m\(^3\), \( d_{gap} = 2 \) μm, \( \sigma = 0 \) MPa, \( P_0 = 0.2 \) bar, \( r = 0.22 \)). The midpoint deflection time history for various time increments (\( \Delta t \)) and different numbers of elements along the length (\( n \)) is shown in Fig. 3(a) and (b), respectively. The diagrams have been plotted for microplates with six elements along the width. It can be concluded that using \( \Delta t < 2 \times 10^{-8} \) and \( n > 40 \) results in good convergence. It should be noted that deflections are considered negative in the positive direction of the \( z \)-axis.
Another comparison is performed using an undamped microstructure, which has been studied by Rochus et al. (2005). The microstructure is composed of a substrate and a double-clamped silicon microplate with density $\rho = 2648.38 \text{ kg/m}^3$ and Young’s modulus $E = 77 \text{ GPa}$. And the thickness and length of the microplate are $h = 0.5 \mu\text{m}$ and $l = 300 \mu\text{m}$, respectively. The initial air gap is $d_{\text{gap}} = 6 \mu\text{m}$, which is not small compared to the plate length.

Therefore, the nonlinear FSDT model is utilized. A voltage $V$ is suddenly applied between the microplate and the substrate. The phase portrait is plotted for various voltages in Fig. 4(a). As shown in this figure, the dynamic instability appears at $V = 85.0 \text{ V}$. In addition, the phase portrait for an input of $V = 70 \text{ V}$ is plotted in Fig. 5(a) for several cycles. These figures show consistency with the results presented by Rochus et al. (Figs. 4(b), 5(b)).

The midpoint deflections of another microplate are shown in Fig. 6(a) and (b) ($l = 300 \mu\text{m}$, $b = 20 \mu\text{m}$, $h = 2 \mu\text{m}$, $E = 169 \text{ GPa}$, $\rho = 2330 \text{ kg/m}^3$, $d_{\text{gap}} = 2 \mu\text{m}$, $P_0 = 1 \text{ bar}$, $\sigma = 0 \text{ MPa}$, $r = 0.28$). These figures can be compared with Fig. 6(c), which has been presented by Krylov (2007). It can be seen that, due to the midplane stretching, the dynamic pull-in voltage obtained based on nonlinear FSDT is higher than that obtained based on linear FSDT.

In Fig. 7, dynamic responses of microplates with various thicknesses are depicted, using FSDT and CPT ($l = 150 \mu\text{m}$, $b = 50 \mu\text{m}$, nonlinear CPT; nonlinear FSDT).
$E = 169 \text{ GPa}, \rho = 2330 \text{ kg/m}^3, d_{\text{gap}} = 2 \text{ mm}, P_0 = 0.1 \text{ bar}, \sigma = 0 \text{ MPa}, r = 0.28$. As seen in this figure, the dynamic responses of microplates using both plate theories are similar for small thicknesses, but by increasing the plate thickness, the dynamic responses differ from each other even more. These diagrams show that for large thickness-to-length ratios, the effect of rotary inertia and shear deformation is considerable. Therefore, FSDT should be utilized to model the microplate behavior.

Another important parameter which influences the dynamic behavior is the ambient pressure. Fig. 8(a) shows the midpoint deflection time history of a microplate for various values of the ambient pressure, based on nonlinear FSDT. In accordance with this

Fig. 9. Midpoint deflection time history of a microplate for various voltages; ---, nonlinear FSDT; ---, linear FSDT.

Fig. 10. Midpoint deflection time history of a microplate with different initial stresses based on (a) nonlinear FSDT; (b) linear FSDT.

Fig. 11. Phase portrait, corresponding to Fig. 10, based on (a) nonlinear FSDT; (b) linear FSDT.
figure, the amplitude of vibrations is decreased with an increase in the ambient pressure. In Fig. 8(b), linear and nonlinear FSDT models are compared \( (l = 250 \, \text{mm}, b = 50 \, \text{mm}, h = 2 \, \text{mm}, E = 169 \, \text{GPa}, \rho = 2330 \, \text{kg/m}^3, d_{\text{gap}} = 2 \, \text{um}, P_0 = 0.2 \text{ bar}, \sigma = 0 \, \text{MPa}, r = 0.22). \) Comparing the results of linear and nonlinear FSDT, it is observed that the response amplitudes obtained using nonlinear FSDT are smaller. Therefore, neglecting the midplane stretching underestimates the microplate stiffness and frequency.

In Fig. 10(a) and (b) the influence of initial stresses on the dynamic responses are plotted using linear and nonlinear FSDT models \( (l = 250 \, \text{mm}, b = 50 \, \text{mm}, h = 2 \, \text{mm}, E = 169 \, \text{Gpa}, \rho = 2330 \, \text{kg/m}^3, d_{\text{gap}} = 2 \, \text{um}, V = 30 \, \text{V}). \) The initial stress may be induced during the fabrication process. Comparing Fig. 10(a) and (b) indicates that as the response amplitude of the microplate is in the applicability range of linear FSDT, the results of both linear and nonlinear FSDT models are approximately coincident. However, by increasing the initial compressive stress to \( \sigma = -25 \, \text{MPa}, \) linear FSDT modeling is no longer reliable. These diagrams signify the fact that the larger values of the tensile initial stress lead to increasing the stiffness of the plate.

Fig. 12. Midpoint deflection time history of microplates with different air gaps: (a) \( d_{\text{gap}} = 0.5 \, \text{um}; \) (b) \( d_{\text{gap}} = 1 \, \text{um}; \) (c) \( d_{\text{gap}} = 2 \, \text{um}; \) \( -\ldots- \) nonlinear FSDT; \( - - - - \) linear FSDT.

In Fig. 12 midpoint deflections of a microplate versus time for various air gaps are depicted \( (l = 250 \, \text{um}, b = 50 \, \text{um}, h = 2 \, \text{um}, E = 169 \, \text{Gpa}, \rho = 2330 \, \text{kg/m}^3, P_0 = 0.1 \text{ bar}, \sigma = 0 \, \text{MPa}, r = 0.28). \) These diagrams show that as the actuation voltage is increased, the difference between the results of linear and nonlinear FSDT models increases. Therefore, for high voltages, it is necessary to utilize nonlinear FSDT model.

Table 2 shows the dynamic pull-in voltages of the microplates listed in Table 1. The \( V_{\text{pid}}/V_{\text{pu}} \) ratios of the microstructures in this table are in agreement with those reported by Krylov and Maimon (2004). One also observes that increasing the initial stress leads to an increase in the \( V_{\text{pid}}/V_{\text{pu}} \) ratio. It can be concluded that inertia has more influence on microstructures with lower initial stresses.

<table>
<thead>
<tr>
<th>Length, ( l ) (( \text{um} ))</th>
<th>Initial stress, ( \sigma ) (( \text{MPa} ))</th>
<th>( V_{\text{pu}} ), Linear FSDT (Volt)</th>
<th>( V_{\text{pu}} ), Nonlinear FSDT (Volt)</th>
<th>( V_{\text{pid}}/V_{\text{pu}} ), Linear FSDT</th>
<th>( V_{\text{pid}}/V_{\text{pu}} ), Nonlinear FSDT</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>0</td>
<td>35.90</td>
<td>36.16</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td>250</td>
<td>-25</td>
<td>29.93</td>
<td>30.24</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>250</td>
<td>100</td>
<td>53.09</td>
<td>53.24</td>
<td>0.93</td>
<td>0.93</td>
</tr>
<tr>
<td>350</td>
<td>0</td>
<td>18.33</td>
<td>18.45</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td>350</td>
<td>-25</td>
<td>11.56</td>
<td>11.77</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>350</td>
<td>100</td>
<td>33.25</td>
<td>33.31</td>
<td>0.93</td>
<td>0.93</td>
</tr>
</tbody>
</table>

**Fig. 13.** Variation of frequency with voltage for different initial stresses; \( - - - - \), nonlinear FSDT; \( - - - - \), linear FSDT.
The vibrational behavior of microstructures near dynamic pull-in point is also of interest. In Fig. 13 the variation of frequency with the actuation voltage for various initial stresses is depicted (\( l = 250 \mu m, b = 50 \mu m, h = 2 \mu m, E = 169 Gpa, \rho = 2330 kg/m^3, d_{gap} = 2 \mu m, P_0 = 0.1 bar, \sigma = 0.22 \)). As shown in Fig. 13, there are significant differences between the frequencies computed with linear and nonlinear FSDT. The difference is elevated by increasing the compressive initial stress. It is noteworthy that for linear FSDT model, the frequency versus voltage curve monotonically decreases for all initial values. For nonlinear FSDT model, the frequency versus voltage curves monotonically decrease for the cases with \( \sigma = -10 \) and \( +10 \) MPa, while this does not hold for the case with \( \sigma = -30 \) MPa. This behavior is consistent with the behavior reported by Batra et al. (2008a–c).

4. Conclusion

In this study, the effect of nonlinearity on the dynamic pull-in behavior of microstructures has been investigated based on classical and first-order shear deformation plate theories. Finite element models were constructed to study the dynamic behavior of microstructures, actuated by step input voltages. The models have considered midplane stretching, residual stress, nonlinear electrostatic actuation and fluid pressure. It was found that by increasing the input voltage, the effect of midplane stretching increases. Results indicated that a qualitative change in the character of microplate response is observed when the input voltage exceeds the dynamic pull-in value. It was noticed that for large thickness-to-length ratios, first-order shear deformation theory should be utilized, especially near the dynamic pull-in point. Furthermore, it was shown that while for the linear FSDT model, the frequency versus voltage curves monotonically decrease for all initial stresses, this does not hold for the nonlinear FSDT model. This study is useful in modeling the dynamic behavior of microstructures.

Acknowledgement

The authors would like to thank Dr. M. Mahzoon for his valuable suggestions.

Appendix A. Plate constitutive equations

The displacement field of the plate based on classical plate theory is (Reddy, 2004)

\[
\begin{align*}
u(x, y, z, t) & = u_0(x, y, t) - z \frac{\partial w_0}{\partial y} \\
\gamma(x, y, z, t) & = v_0(x, y, t) - z \frac{\partial w_0}{\partial x} \\
u(x, y, z, t) & = w_0(x, y, t)
\end{align*}
\]

(A1)

The strain–displacement relations are expressed as

\[
\begin{pmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\gamma_{xy}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\left( \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial y} \right) + z \left( \frac{\partial^2 w_0}{\partial x \partial y} \right)
\end{pmatrix} + z \begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

(A2)

Therefore, the plate constitutive equations for isotropic plates take the form

\[
\begin{pmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{pmatrix} = \int_{-h/2}^{z-h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} dz
\]

and

\[
\begin{pmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{pmatrix} = \int_{-h/2}^{z-h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} z dz
\]

\[
\begin{pmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{pmatrix}\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\left( \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial y} \right) + z \left( \frac{\partial^2 w_0}{\partial x \partial y} \right)
\end{pmatrix} + z \begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

(A3)

\[
\begin{pmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{pmatrix}\begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix}
\]

(A4)

where

\[
(A_{ij}, D_{ij}) = \int_{z-h/2}^{z+h/2} Q_{ij}(z, 1, z^2) dz
\]

(A5)

Here \( N_i \) and \( M_{ij} \) are in-plane force resultants and moment resultants, respectively.

Considering Eqs. (4) and (5), based on first-order shear deformation plate theory, the plate constitutive equations for isotropic plates take the form

\[
\begin{pmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{pmatrix} = \int_{-h/2}^{z-h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} dz
\]

\[
\begin{pmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{pmatrix} = \int_{-h/2}^{z-h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} z dz
\]

\[
\begin{pmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{pmatrix}\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\left( \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial y} \right) + z \left( \frac{\partial^2 w_0}{\partial x \partial y} \right)
\end{pmatrix} + z \begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

(A6)

\[
\begin{pmatrix}
Q_{xx} \\
Q_{yy} \\
Q_{xy}
\end{pmatrix} = K \int_{z-h/2}^{z+h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} dz = K \int_{z-h/2}^{z+h/2} \begin{pmatrix}
Q_{44} & 0 \\
0 & Q_{55}
\end{pmatrix} \begin{pmatrix}
\gamma_{yz} \\
\gamma_{xz}
\end{pmatrix} dz
\]

(A7)

\[
\begin{pmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{pmatrix} = \int_{-h/2}^{z-h/2} \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{pmatrix} z dz
\]

\[
\begin{pmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{pmatrix}\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\left( \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial y} \right) + z \left( \frac{\partial^2 w_0}{\partial x \partial y} \right)
\end{pmatrix} + z \begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

(A8)
where $K$ is the shear correction coefficient. It should be noted that in an isotropic plate, one obtains

$$Q_{11}, Q_{22} = \frac{E}{1 - \nu^2} Q_{12} = \frac{\nu E}{1 - \nu^2} Q_{44}, Q_{55}, Q_{66} = G$$ \hfill (A9)

References