Characterization of coupled-domain multi-layer microplates in pull-in phenomenon, vibrations and dynamics

M. Moghimi Zand*, M.T. Ahmadian

Center of Excellence in Design, Robotics and Automation, School of Mechanical Engineering, Sharif University of Technology, P.O. Box 11365-9567, Tehran, Iran

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Abstract
This paper presents a model to analyze pull-in phenomenon, vibrational behavior and dynamics of multi-layer microplates using coupled finite element and finite difference methods (FDM). First-order shear deformation theory (FSDT) is used to model dynamical system using finite element method, while FDM is applied to solve nonlinear Reynolds equation of squeeze film damping. Using this model, pull-in analysis of single- and multi-layer microplates are studied. Vibrational behavior of single- and multi-layer microplates are analyzed to compute resonance frequencies and mode shapes of the system. Also, an algorithm is presented to study dynamics of microplates under the actuation of nonlinear electrostatic force and squeeze film damping. Results for simplified single-layer microplates are validated and in good agreement with the published literature. This investigation can be implemented in the design of multi-layer microplates.

Keywords: Vibration; Squeeze film damping; Pull-in; Multi-layer microplates; Dynamics; Electrostatic

1. Introduction

Technology of microelectromechanical systems has experienced a lot of progress recently. Their light weight, low cost, small dimensions, low-energy consumption, and durability make them quite useful. Studying the design parameters of microelectromechanical systems by scientific techniques is of great importance; therefore simulation tools are being improved.

Among various methods, electrostatic actuation and sensing is widely used in many MEMS sensors and actuators, including acoustic resonators, RF switches and pressure sensors. In designing microplates, pull-in effect, as an important phenomenon, must be considered. Due to this importance, many researches have been done on this phenomenon. Osterberg [1] achieved several closed-form models for pull-in phenomena in these systems and has compared his findings with experiments.


Some researchers have paid attention to design-parameters study of micro systems. Zhao et al. [5] presented a model for plates considering electrostatic actuation and plate stretching. Using this model, they have discussed about design parameters. Abdullah et al. [6] considered the effect of microbeam dimensions on pull-in voltage.

MEMS devices could be of single-layer type or multi-layer one. Unfortunately, dynamical behavior and vibration analysis of multi-layer microplates are not addressed well in the literature. Rong et al. [7] presented an analytical
model to study pull-in phenomenon of multi-layer beams using energy method.

In this paper, pull-in analysis, vibrational behavior and transient dynamics of orthotropic multi-layer microplates are studied. Nine-node elements, with three or five degrees of freedom in each node are considered. The approach utilizes first-order shear deformation theory (FSDT) of multi-layer plates coupled with nonlinear electrostatic actuation and Reynolds equation.

Multi-layer model is used to verify pull-in results with the results published in the literature. Afterwards, vibrational analysis for a single-layer model is performed to compare results with literature.

Using these certifications an analysis for vibrational and dynamic behavior of multi-layer microplates are presented.

2. Equations of motion

In an $n$-layer prismatic microplate, there are one conductive and some dielectric layers (Fig. 1). It is assumed that the conductive layer always positioned at the top of the stack. A fixed planar electrode underlies the plate as a ground. The multi-layer microplate is deformable while the substrate plate is rigid. When a voltage applied between the plate and the substrate plate, an attractive electrostatic force causes the microplate to deform. The length of the multi-layer microplate is $l$, the width is $b$, the thickness of the $i$th layer is $h_i$, the plate total thickness is $h$, the relative permittivity of the $i$th layer is $\varepsilon_i$ and the air initial gap is $d_{gap}$. $x$ and $y$ are in-plane coordinates, $z$ is the coordinate along thickness, $t$ is the time, $I_i$ is the mass moments of inertia $(i = 1, 2, 3)$, $\varphi_x$ $(\varphi_y)$ are the rotations of a transverse normal about $y$-axis ($x$-axis), $N_{ij}$ is in-plane applied forces $(i, j = x, y)$ and $\bar{N}_{ij}^k$ is in-plane applied forces in each layer $(k = 1, \ldots, n)$, $\sigma_k$ is the stress in layer $k$ due to $\bar{N}_{ij}^k$, $\rho_k$ is the density of $k$th layer, $u_0$, $v_0$ and $w_0$ are midplane displacements along $x$, $y$ and $z$-axis, respectively, and $V$ is the applied voltage.

In the electrostatic actuated MEMS, the basic system is a parallel-plates capacitor. For a single-layer microplate the capacitance formed by the part of the plate with the surface of $\text{dx} \times \text{dy}$ and the fixed substrate plate is

$$dC_s = \frac{\varepsilon_0 \text{dx} \text{dy}}{d_{gap} - w_0(x,y)}. \quad (1)$$

Using Eq. (1), in a single-layer microplate, the electrostatic force per unit area takes the following form:

$$q_{es} = -\frac{1}{2} \frac{V^2}{(d_{gap} - w_0(x,y))^2}. \quad (2)$$

For a multi-layer microplate the capacitance formed by the part of a multi-layer microplate with the surface of $\text{dx} \times \text{dy}$ and the ground is [7]:

$$dC_m = \frac{\varepsilon_0 \text{dx} \text{dy}}{\bar{d}_{gap} - w_0(x,y)}, \quad (3)$$

where

$$\bar{d}_{gap} = d_{gap} + \sum_{i=1}^{n_d} h_i/\varepsilon_i.$$  

Here $\bar{d}_{gap}$ is called the effective gap [7] and $n_d$ is the number of layers which are underneath the conductive layer.

Using Eqs. (2) and (3) for a multi-layer microplate, electrostatic force per unit area of $\text{dx} \times \text{dy}$ can be written as

$$q_{es} = -\frac{1}{2} \frac{V^2}{(\bar{d}_{gap} - w_0(x,y))^2}. \quad (5)$$

This equation helps us to consider electrostatic force for multi-layer microplates.

Determining the electrostatic actuation force, it is possible to impose its effect on elasticity relations. Using constitutive law, in each layer, the stress–strain relations in the laminate coordinates can be written as [8]:

$$\begin{cases} 
\sigma_{xx} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} 
\end{bmatrix} \begin{bmatrix} e_{xx} \\
\varepsilon_{xy} \\
\varepsilon_{yz} \end{bmatrix}, \\
\sigma_{yy} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\
\bar{Q}_{45} & \bar{Q}_{55} 
\end{bmatrix} \begin{bmatrix} \varepsilon_{xz} \\
\varepsilon_{yz} \end{bmatrix},
\end{cases} \quad (6)$$

where $\varepsilon_{ij}$, $\sigma_{ij}$ and $\bar{Q}_{ij}$ are strains, stresses and plane stress-reduced stiffnesses, respectively.

Fig. 1. Schematic diagram of multi-layer microplates.
Using equivalent single-layer (ESL) theories, by integrating Eq. (6) over the plate thickness we have [8]:

\[
\begin{align*}
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{bmatrix}
&= \int_{-h/2}^{z=h/2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \, dz, \\
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix}
&= \int_{-h/2}^{z=h/2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \, z \, dz, \\
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix}
&= K \int_{-h/2}^{z=h/2} \begin{bmatrix}
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} \, dz,
\end{align*}
\tag{7}
\]

where \(N_{ij}\), \(M_{ij}\) and \(Q_{ij}\) are in-plane force resultants, moment resultants and transverse force resultants, respectively.

By replacing the values of stresses from elasticity theorem in Eq. (7) we obtain:

\[
\begin{align*}
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{bmatrix}
&= \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} \bigg|_0 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
\frac{\partial v}{\partial y} \bigg|_0 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\
\frac{\partial w}{\partial y} \bigg|_0 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\end{bmatrix}, \\
\begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix}
&= \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \varphi}{\partial x} \bigg|_0 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2 \\
\frac{\partial \varphi}{\partial y} \bigg|_0 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial y} \right)^2 \\
\frac{\partial \varphi}{\partial y} \bigg|_0 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2
\end{bmatrix}, \\
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix}
&= K \begin{bmatrix}
A_{44} & A_{45} \\
A_{45} & A_{55}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial w}{\partial y} \bigg|_0 + \varphi_y \\
\frac{\partial w}{\partial y} \bigg|_0 + \varphi_x
\end{bmatrix},
\end{align*}
\tag{8}
\]

where \(N_{ij}\) and \(Q_{ij}\) are shear correction coefficient, extensional stiffnesses, bending-extensional stiffnesses and bending stiffnesses, respectively [8].

Assuming FSDT, equations of motion are derived using dynamic version of the principle of virtual displacements to model deflections of the plate. To consider squeeze film damping, nonlinear Reynolds equation is used.

Governing equations are:

\[
\begin{align*}
\delta u_0 : \quad & \frac{\partial^2 N_{xx}}{\partial x^2} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^4 \phi_x}{\partial t^4}, \\
\delta v_0 : \quad & \frac{\partial^2 N_{xy}}{\partial x \partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^4 \phi_y}{\partial t^4}, \\
\delta w_0 : \quad & \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 + (P - P_0) = I_0 \frac{\partial^2 w_0}{\partial t^2}, \\
\delta \phi_x : \quad & \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = I_1 \frac{\partial^2 \phi_x}{\partial t^2} + I_2 \frac{\partial^4 \phi_x}{\partial t^4}, \\
\delta \phi_y : \quad & \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = I_1 \frac{\partial^2 \phi_y}{\partial t^2} + I_2 \frac{\partial^4 \phi_y}{\partial t^4},
\end{align*}
\tag{12}
\]

Here \(P = P(x,y)\) is the total pressure in the gap, \(P_0\) is the ambient pressure and \(\eta\) is the viscosity of the fluid in the gap. Eq. (12f) is called the compressible Reynolds equation that is commonly used to model squeeze film damping in microstructures. Replacing \(N_{xx}, N_{yy}, Q_x, Q_y, M_{xx}, M_{xy},\) and \(M_{yy}\) from Eqs. (8)–(10) into Eq. (12) governing equations can be expressed in terms of displacements. Keeping the nonlinearity of the electrostatic force and Reynolds equation, and linearizing the rest of the terms, the equations become simplified.

3. Finite element model of the plate

Multiplying Eqs. (12a)–(12e) with \(\delta u_0, \delta v_0, \delta w_0, \delta \phi_x\) and \(\delta \phi_y\), respectively, and integrating over the domain, an integrated form of governing equations are obtained. Integration by parts, results in the weakened expressions.

Assuming displacements and rotations as [8]:

\[
\begin{align*}
u_0(x,y,t) &= \sum_{j=1}^{m} u_j(t) \psi_j(x,y), \\
v_0(x,y,t) &= \sum_{j=1}^{m} v_j(t) \psi_j(x,y), \\
w_0(x,y,t) &= \sum_{j=1}^{m} w_j(t) \psi_j(x,y), \\
\phi_x(x,y,t) &= \sum_{j=1}^{m} \phi_{x}^j(t) \psi_j(x,y), \\
\phi_y(x,y,t) &= \sum_{j=1}^{m} \phi_{y}^j(t) \psi_j(x,y),
\end{align*}
\tag{13}
\]
\[ \varphi_i(x, y, t) = \sum_{j=1}^{m} S_j^i(t) \psi_j(x, y), \]  

(13e)

where \( \psi_j \) and \( m \) are Lagrange interpolation functions and number of nodes respectively. Considering Lagrange interpolation functions for a nine-node element in the element coordinates \((\xi, \eta)\) as [8]:

\[
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\psi_5 \\
\psi_6 \\
\psi_7 \\
\psi_8 \\
\psi_9
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
(1 - \xi)(1 - \eta)(-\eta - \xi - 1) + (1 - \xi^2)(1 - \eta^2) \\
2(1 - \xi^2)(1 - \eta) - (1 - \xi^3)(1 - \eta^2) \\
(1 + \xi)(1 - \eta)(-\eta + \xi - 1) + (1 - \xi^2)(1 - \eta^2) \\
2(1 - \xi)(1 - \eta^2) - (1 - \xi^2)(1 - \eta^2) \\
4(1 - \xi^3)(1 - \eta^2) \\
2(1 + \xi)(1 - \eta^2) - (1 - \xi^3)(1 - \eta^2) \\
(1 - \xi)(1 + \eta)(\eta - \xi - 1) + (1 - \xi^2)(1 - \eta^2) \\
2(1 - \xi^3)(1 + \eta) - (1 - \xi^3)(1 - \eta^2) \\
(1 + \xi)(1 + \eta)(\eta + \xi - 1) + (1 - \xi^2)(1 - \eta^2)
\end{bmatrix}
\]

(14)

The semi-discrete linear finite element model is obtained as

\[
\begin{bmatrix}
(K_e) + \{G_e\}\{W_e\} + \{M_e\}\{\ddot{W}_e\} = \{F_e\},
\end{bmatrix}
\]

(15)

where \( \{K_e\}, \{G_e\}, \{W_e\}, \{F_e\} \) and \( \{M_e\} \) are stiffness, force, in-plane force, deflection, force and mass matrices. Node numbering in the finite element problem is presented in Fig. 2a.

Matrices \( \{K_e\}, \{G_e\}, \{W_e\}, \{F_e\} \) and \( \{M_e\} \) in detailed form can be written as [8]:

\[
\{K_e\} = \begin{bmatrix}
\{K^{11}\} & \{K^{12}\} & \{K^{13}\} & \{K^{14}\} & \{K^{15}\} \\
\{K^{12}\}^T & \{K^{22}\} & \{K^{23}\} & \{K^{24}\} & \{K^{25}\} \\
\{K^{13}\}^T & \{K^{23}\}^T & \{K^{33}\} & \{K^{34}\} & \{K^{35}\} \\
\{K^{14}\}^T & \{K^{24}\}^T & \{K^{34}\}^T & \{K^{44}\} & \{K^{45}\} \\
\{K^{15}\}^T & \{K^{25}\}^T & \{K^{35}\}^T & \{K^{45}\}^T & \{K^{55}\}
\end{bmatrix}
\]

(17)

\[
\{G_e\} = \begin{bmatrix}
\{0\} & \{0\} & \{0\} & \{0\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} & \{0\} \\
\{0\} & \{0\} & \{G\} & \{0\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} & \{0\} \\
\{0\} & \{0\} & \{0\} & \{0\} & \{0\}
\end{bmatrix}
\]

(18)

\[
\{F_e\} = \begin{bmatrix}
\{F^1\} \\
\{F^2\} \\
\{F^3\} \\
\{F^4\} \\
\{F^5\}
\end{bmatrix}, \quad \{W_e\} = \begin{bmatrix}
\{u\} \\
\{v\} \\
\{w\} \\
\{S^1\} \\
\{S^2\}
\end{bmatrix}
\]

(19)

The coefficients of sub matrix \( \{K^{ij}\} \) are defined for \((x, \beta = 1, 2, \ldots, 5)\) by the expressions [8]:

\[
K_{ij}^{1x} = \int_{\Omega} \left( \frac{\partial \psi^i}{\partial x} N_{ij}^x + \frac{\partial \psi^j}{\partial y} N_{ij}^y \right) dx dy,
\]

(20a)

\[
K_{ij}^{2y} = \int_{\Omega} \left( \frac{\partial \psi^i}{\partial x} N_{ij}^y + \frac{\partial \psi^j}{\partial y} N_{ij}^x \right) dx dy,
\]

(20b)

\[
K_{ij}^{3z} = \int_{\Omega} \left( \frac{\partial \psi^i}{\partial x} Q_{ij}^z + \frac{\partial \psi^j}{\partial y} Q_{ij}^z \right) dx dy,
\]

(20c)

\[
K_{ij}^{4s} = \int_{\Omega} \left( \frac{\partial \psi^i}{\partial x} M_{ij}^x + \frac{\partial \psi^j}{\partial y} M_{ij}^y + \psi^i Q_{ij}^z \right) dx dy,
\]

(20d)

\[
K_{ij}^{5z} = \int_{\Omega} \left( \frac{\partial \psi^i}{\partial x} M_{ij}^y + \frac{\partial \psi^j}{\partial y} M_{ij}^y + \psi^i Q_{ij}^z \right) dx dy,
\]

(20e)

where

\[
N_{ij}^x = A_{11} \frac{\partial \psi^i}{\partial x} + A_{16} \frac{\partial \psi^j}{\partial y}, \quad N_{ij}^y = A_{12} \frac{\partial \psi^i}{\partial y} + A_{16} \frac{\partial \psi^j}{\partial x},
\]

(20f)

\[
N_{ij}^1 = B_{11} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad N_{ij}^2 = B_{12} \frac{\partial \psi^i}{\partial y} + B_{16} \frac{\partial \psi^j}{\partial x},
\]

(20g)

\[
N_{ij}^4 = B_{12} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad N_{ij}^5 = B_{22} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20h)

\[
N_{ij}^7 = B_{16} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad N_{ij}^8 = B_{26} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20i)

\[
N_{ij}^6 = B_{16} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad N_{ij}^9 = B_{26} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20j)

\[
M_{ij}^1 = B_{11} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad M_{ij}^2 = B_{12} \frac{\partial \psi^i}{\partial y} + B_{16} \frac{\partial \psi^j}{\partial x},
\]

(20k)

\[
M_{ij}^4 = B_{12} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad M_{ij}^5 = B_{22} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20l)

\[
M_{ij}^7 = B_{16} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad M_{ij}^8 = B_{26} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20m)

\[
M_{ij}^6 = B_{16} \frac{\partial \psi^i}{\partial x} + B_{16} \frac{\partial \psi^j}{\partial y}, \quad M_{ij}^9 = B_{26} \frac{\partial \psi^i}{\partial y} + B_{26} \frac{\partial \psi^j}{\partial x},
\]

(20n)

\[
M_{ij}^3 = M_{ij}^1 - M_{ij}^6, \quad N_{ij}^3 = N_{ij}^1 - N_{ij}^6 = N_{ij}^9 = 0.
\]

(21)
The coefficients of sub matrix \( \{ M^{\alpha \beta} \} \) are defined for \((\alpha, \beta = 1,2,\ldots, 5)\) by \[ M_{ij} = \int_{\Omega_i} \psi^\alpha_i \psi^\beta_j \, dx \, dy. \] (22)

The coefficients of sub matrix \( \{ G \} \) are defined by \[ G_{ij} = \int_{\Omega_i} \left( \frac{\partial \psi^\alpha_i \psi^\beta_j}{\partial x} \tilde{N}_{xx} + \left( \frac{\partial \psi^\alpha_i \psi^\beta_j}{\partial x} \frac{\partial}{\partial y} + \frac{\partial \psi^\alpha_i \psi^\beta_j}{\partial y} \frac{\partial}{\partial x} \right) \tilde{N}_{xy} \right) \, dx \, dy, \] (23)

where in-plane applied forces can be obtained as
\[
\tilde{N}_{xx} = \frac{\sum_{i=1}^{n} \tilde{N}_{xx} \, h_i}{\sum_{i=1}^{n} h_i}, \quad \tilde{N}_{xy} = \frac{\sum_{i=1}^{n} \tilde{N}_{xy} \, h_i}{\sum_{i=1}^{n} h_i},
\]
\[
\tilde{N}_{yy} = \frac{\sum_{i=1}^{n} \tilde{N}_{yy} \, h_i}{\sum_{i=1}^{n} h_i}. \]
(24)

Finally, the coefficients of sub matrix \( \{ F^2 \} \) are defined by \[ F^1_i = \int_{\Gamma_s} P \psi^\alpha_i \, dx \, dy, \quad F^2_i = \int_{\Gamma_s} P \psi^\beta_i \, dx \, dy, \]
\[ F^3_i = \int_{\Omega_e} q_{em} \psi^\alpha_i \, dx \, dy + \int_{\Omega_e} q_y \psi^\beta_i \, dx \, dy + \int_{\Gamma_s} Q_n \psi^\beta_i \, dx, \]
\[ F^4_i = \int_{\Gamma_s} T_x \psi^\alpha_i \, dx \, dy, \quad F^5_i = \int_{\Gamma_s} T_y \psi^\beta_i \, dx \, dy, \] (25)

where
\[ q_p = P - P_0 = \bar{P}. \] (26)

Here \( Q_n, P_x, P_y, T_x \) and \( T_y \) are secondary variables, \( q_{em} \) is the electrostatic force per unit area of multi-layer microplates, \( q_p \) is the fluid pressure, \( \bar{P} \) is the dynamic pressure and \( \Gamma_s \) and \( \Omega_e \) are surface and boundary of the elements, respectively. A schematic diagram of nodes configuration after element assemblage is shown in Fig. 2b. It is to be noted that the same nodes configuration of the finite element problem will be used for the finite difference problem.

4. Finite difference model for squeeze film damping

In studying transient behavior of microplates, finite difference method can be used to model squeeze film damping. Depending on the magnitudes of deflection, nonlinear or linearized Reynolds equation can be used. For problems with small deflection, either nonlinear or linearized Reynolds equation and for problems with large deflection, nonlinear Reynolds equation is used.

Reynolds equation has the form:
\[
\frac{\partial}{\partial x} \left( (d_{gap} - w_0)^2 \frac{\partial P}{\partial x} \right) + \frac{\partial}{\partial y} \left( (d_{gap} - w_0)^2 \frac{\partial P}{\partial y} \right) = 12\eta \left( (d_{gap} - w_0) \frac{\partial P}{\partial t} - P \frac{\partial w_0}{\partial t} \right). \] (27)

Assuming
\[ P(x, y, t) = P_0 + \bar{P}(x, y, t). \] (28)

For linearized Reynolds equation we have:
\[
\frac{\partial^2 \bar{P}}{\partial x^2} + \frac{\partial^2 \bar{P}}{\partial y^2} = \frac{12\eta}{d_{gap}^3 P_0} \left( d_{gap} \frac{\partial \bar{P}}{\partial t} - P_0 \frac{\partial w_0}{\partial t} \right), \] (29)

where \( P, P_0 \) and \( \bar{P} \) are the absolute, ambient and dynamic pressure, respectively.

Discrete linearized Reynolds equation is in the form:
\[
\frac{P_{i+1,j} - P_{i-1,j} - 2P_{ij}}{\Delta x^2} + \frac{P_{i,j+1} - P_{i,j-1} - 2P_{ij}}{\Delta y^2} + \frac{12\eta}{d_{gap}^3 P_0} \left( d_{gap} \frac{P_{i,j} - P_{i,j}}{\Delta t} - P_0 \frac{w_{ij}^{t+1} - w_{ij}}{\Delta t} \right), \] (30)

where indices \( i \) and \( j \) represent nodes of \( x \)- and \( y \)-axis, respectively (Fig. 3a). Superscript \( t \) represents properties of time \( t \).

Using discrete linearized Reynolds equation, values of dynamic pressure in each time step for all nodes can be
found by

\[
P_{i,j}^{t+1} = P_{i,j}^t + P_0 d_{gap} (w_{i,j}^{t+1} - w_{i,j}^t) + \frac{d_{gap}^2 P_0 \Delta t}{12 \eta} \times \left[ \left( P_{i+1,j}^t + P_{i-1,j}^t - 2P_{i,j}^t \right) \frac{\partial^2 (P^t)}{\partial x^2} + \left( P_{i,j+1}^t + P_{i,j-1}^t - 2P_{i,j}^t \right) \frac{\partial^2 (P^t)}{\partial y^2} \right].
\] (31)

For a successful solution of this problem, stability analysis is essential. Using Von Neumann stability analysis we have [9]:

\[
P_0 d_{gap}^2 \Delta t ((1/\Delta x^2) + (1/\Delta y^2))^2 < 1.
\] (32)

This equation imposes limitations on the time step sizes. For nonlinear Reynolds equation, we can rewrite Eq. (27) in the form:

\[
(d_{gap} - w_0)^3 \left( \frac{\partial^2 (P^t)}{\partial x^2} \right) + 3(d_{gap} - w_0)^2 \frac{\partial}{\partial x} (d_{gap} - w_0) \times \frac{\partial (P^t)}{\partial x} + (d_{gap} - w_0)^3 \left( \frac{\partial^2 (P^t)}{\partial y^2} \right) + 3(d_{gap} - w_0)^2 \times \frac{\partial}{\partial y} (d_{gap} - w_0) \frac{\partial (P^t)}{\partial y} = 24 \eta \left( (d_{gap} - w_0) \frac{\partial P}{\partial t} - P \frac{\partial w_0}{\partial t} \right).
\] (33)

Discretized nonlinear Reynolds equation is in the form:

\[
(d_{gap} - w_{i,j}^t)^3 \left[ \left( \frac{(P_{i+1,j}^t)^2 + (P_{i-1,j}^t)^2 - 2(P_{i,j}^t)^2}{\Delta x^2} \right) \right] + \left( \frac{(P_{i,j+1}^t)^2 + (P_{i,j-1}^t)^2 - 2(P_{i,j}^t)^2}{\Delta y^2} \right) \]

\[- 3(d_{gap} - w_{i,j}^t)^2 \left( w_{i+1,j}^t - w_{i,j+1}^t \right) \left( \frac{(P_{i+1,j}^t)^2 - (P_{i,j+1}^t)^2}{\Delta x} \right) \]

\[- 3(d_{gap} - w_{i,j}^t)^2 \left( w_{i,j-1}^t - w_{i,j}^t \right) \left( \frac{(P_{i,j-1}^t)^2 - (P_{i,j}^t)^2}{\Delta y} \right) \]

\[= 24 \eta \left( (d_{gap} - w_{i,j}^t) \frac{P_{i+1,j}^t - P_{i,j+1}^t}{\Delta t} - P_{i,j} \frac{w_{i,j+1}^t - w_{i,j}^t}{\Delta t} \right).
\] (34)

Using discretized nonlinear Reynolds equation, values of pressure in each time step for all nodes can be found by

\[
P_{i,j}^{t+1} = P_{i,j}^t + \frac{P_{i,j}^t (w_{i,j}^{t+1} - w_{i,j}^t)}{(d_{gap} - w_{i,j}^t)} + \Delta t (d_{gap} - w_{i,j}^t)^2
\]

\[x \left[ \left( \frac{(P_{i+1,j}^t)^2 + (P_{i-1,j}^t)^2 - 2(P_{i,j}^t)^2}{\Delta x^2} \right) \right]

\[+ \left( \frac{(P_{i,j+1}^t)^2 + (P_{i,j-1}^t)^2 - 2(P_{i,j}^t)^2}{\Delta y^2} \right) \]

\[- 3(d_{gap} - w_{i,j}^t)^2 \left( w_{i+1,j}^t - w_{i,j+1}^t \right) \left( \frac{(P_{i+1,j}^t)^2 - (P_{i,j+1}^t)^2}{\Delta x} \right) \]

\[- 3(d_{gap} - w_{i,j}^t)^2 \left( w_{i,j-1}^t - w_{i,j}^t \right) \left( \frac{(P_{i,j-1}^t)^2 - (P_{i,j}^t)^2}{\Delta y} \right) \]

\[+ 24 \eta \left( (d_{gap} - w_{i,j}^t) \frac{P_{i+1,j}^t - P_{i,j+1}^t}{\Delta t} - P_{i,j} \frac{w_{i,j+1}^t - w_{i,j}^t}{\Delta t} \right).\]

(35)

It is to be noted that for C–F–C–F plates (two opposite clamped sides and two free sides), boundary conditions are zero flux for clamped sides and zero-dynamic pressure for free sides. Therefore, initially, values of pressure for free sides (zone 1 in Fig. 3b) are assumed to be equal to ambient pressure. Afterwards, using Eqs. (31) or (35) values of pressure for internal nodes (zone 2) are obtained. Finally, values of pressure for clamped boundaries (zone 3) using zero-flux condition can be calculated. For the sake of simplicity, effective forces due to pressure and electrostatic actuation in all nodes of each element may be assumed to be equal. For instance, magnitudes of fluid pressure and electrostatic force of the central node (node 5) may be considered for all nodes of the element. However, in this case, it is essential to consider finer meshes, which results in applying smaller time steps to assure convergence due to Eq. (32).
5. Simulation and discussion

5.1. The pull-in analysis of single- and multi-layer microplates

With increasing applied voltage, the plate will bend further, but the system is stable and there will be an equilibrium state due to the equivalence of the electrostatic force and plate restoring force. However, there is a critical voltage in which, system becomes unstable. In other words, if the applied voltage reaches this critical value or beyond, there will be no static equilibrium in the system. This value is called the pull-in voltage ($V_{pi}$) and this phenomenon is called pull-in phenomenon. Determination of the pull-in voltages of the system is essential in design process to analyze sensitivity, frequency response, instability and dynamic range of the system [10]. For an applied voltage lower than $V_{pi}$ we get:

$$\{K_e + G_e\}\{W_e\} = \{F_e\}.$$  \hspace{1cm} (36)

We note that $\{F_e(W_e)\}$ is the electrostatic force that is a function of $\{W_e\}$. It is to be noted that fluid pressure force is not involved in the pull-in analysis.

Using Eq. (36) we obtain:

$$\{W_{e}^{k+1}\} = (\{K_e + G_e\})^{-1}\{F_e(W_{e}^{k})\},$$  \hspace{1cm} (37)

where superscript $k$ is the iteration number. In the first step, $\{W_{e}\}$ is an array of zeros. For voltages lower than $V_{pi}$, using Eq. (37) and imposing initial conditions, after a number of iterations, the equilibrium state can be found. But for the values of voltage higher than $V_{pi}$, there is no equilibrium. Using this point, increasing voltage gradually, the value of $V_{pi}$ can be determined. A computer code is used to calculate pull-in voltage by applying this approach. In order to find the appropriate mesh size, computation performed with finer meshes to approach approximately constant results. Imposing this technique, values of pull-in voltages for two single-layer microplates are calculated and compared in Table 1. As another example, variations of pull-in voltage vs. length of single-layer microplates are presented in Fig. 4. As it can be seen, results of this study ($V_{pi}$) are in good agreement with results reported by Osterberg ($V_{pi}'$) [1]. Values of $V_{pi}$ for two-layer microplates are presented in Figs. 5a–5b (dielectric layer is Si$_3$N$_4$ and conductive layer is gold). $E_i$ and $\rho_i$ are the modulus and density of each layer respectively. It can be seen from Figs. 5a and 5b, values of $V_{pi}$ are in good agreement with values of $V_{pi}'$ and $V_{pi}''$ as reported by Rong et al. [7].

![Fig. 4. Pull-in voltage of single-layer microplates vs. length of the plate ($b = 50\,\mu m$, $h = 14.4\,\mu m$, $d_{gap} = 1\,\mu m$, $E = 169\,GPa$).](image)

![Fig. 5. (a) Pull-in voltage vs. thickness of Si$_3$N$_4$ layer ($l = 500\,\mu m$, $h_2 = 0.5\,\mu m$, $d_{gap} = 1\,\mu m$, $E_1 = 250\,GPa$, $E_2 = 53\,GPa$, $\sigma_1 = 500\,MPa$, $\sigma_2 = 12\,MPa$, $\varepsilon_1' = 8$). (b) Pull-in voltage vs. length of the plate ($b = 50\,\mu m$, $h_1 = 1\,\mu m$, $h_2 = 0.5\,\mu m$, $d_{gap} = 1\,\mu m$, $E_1 = 250\,GPa$, $E_2 = 53\,GPa$, $\sigma_1 = 500\,MPa$, $\sigma_2 = 12\,MPa$, $\varepsilon_1' = 8$).](image)

Table 1

<table>
<thead>
<tr>
<th>Properties</th>
<th>Pull-in voltages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ($\mu m$)</td>
<td>Stress (MPa)</td>
</tr>
<tr>
<td>-----------</td>
<td>--------------</td>
</tr>
<tr>
<td>250</td>
<td>0</td>
</tr>
<tr>
<td>350</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1

Pull-in voltages of single layer microplates with C–F–C–F boundary conditions ($h_1 = 3\,\mu m$, $h = 50\,\mu m$, $E_i = 169\,GPa$, $\nu = 0.06$, $d_{gap} = 1\,\mu m$)
Eq. (24) is used to consider in-plane forces in multi-layer microplates. From these results, it can be seen that the method is able to calculate pull-in voltages correctly. Consequently, stiffness, in-plane force, deflection and force matrices are all correct. In fact, pull-in analysis is used to validate our model.

5.2. The vibrational analysis of single- and multi-layer microplates

For the vibrational analysis we obtain:

\[
\det \left\{ (K_e) + (G_e) - (M_e)\omega^2 \right\} \{W_e\} = \{0\}
\]

\[
\det \left\{ (K_e) + (G_e) - (M_e)\omega^2 \right\} = 0
\]

\[
\det \left\{ (M_e)^{-1}((K_e) + (G_e)) - \omega^2(I) \right\} = 0,
\]

where \(I\) is an identity matrix and \(\omega\) is the natural frequency. Eq. (38) is a standard eigenvalue problem that can be solved easily.

As before, we can assume five degrees of freedom \((u_0, v_0, \varphi_x, \varphi_y)\) for each node. But we use nodes with three degrees of freedom \((u_0, \varphi_x, \varphi_y)\) in each node, because out of plane vibrations of microplates are much more important than in-plane vibrations of these systems. Reduction in degrees of freedom, results in reduction of processing time.

To validate the model, we calculate the resonance frequencies of a plate with boundary conditions of C–F–F–F (one side clamped and other sides free). Results are presented in Table 2. From this table, it can be deduced that, results are in good agreement with results in literature. It is to be noted that there are some uncertainties in the computations arising from geometry (width and thickness) and material properties (modulus of elasticity).

Validating our model by pull-in analysis and vibrational analysis of single-layer plates, we can present vibrational analysis of multi-layer plates. Therefore, using ESL theories we find natural frequencies and mode shapes of multi-layer microplates for boundary conditions of C–F–F–F (two opposite sides clamped and other sides free) for wide and narrow plates (i.e., clamped side is wide or narrow). Consider a two-layer microplate, in which the upper layer is gold and the bottom layer is Si3N4. In C–F–F–F plate, existence of in-plane forces is not probable; but in C–F–C–F plate in-plane forces may exist. For the sake of simplicity, it is assumed that there are no in-plane forces in the plates. Using Eq. (38) and performing the same procedure as vibrational analysis of single-layer plates, values of resonance frequencies and mode shapes can be calculated. In Table 3 values of resonance frequencies for plates with three different widths are presented.

In Fig. 6, first eight mode shapes of these plates are presented. As it can be seen, two types of mode shapes exist. First type is related to the mode shapes that can be predicted by beam theory. Values of frequencies related to these mode shapes increase by increasing the width, but the magnitude of the increase is small. Second type is related to the twisting mode shapes. By increasing the width, magnitudes of the frequencies related to these mode shapes strongly decrease. As it can be seen in Fig. 6, among presented mode shapes, one, two and three twisting mode shapes exist in narrow, average and wide plates, respectively. It can be concluded that twisting mode shapes of out of plane vibrations, for wider plates, are more important than narrower plates. Consequently, for narrow plates \((l/20 > b)\), first, second and third out of plane mode shapes can easily be calculated by beam theory and there is no need to consider twisting mode shapes. It is to be noted that this conclusion is not necessarily true for in-plane vibrations.

5.3. The transient dynamics of microplates

Studying transient behavior of micro systems is important in design process. Dynamical behavior of these systems is influenced by many parameters. Pressure, voltage, dimensions and configuration of layers are some of these parameters.

Finite element model of plate dynamics obeys Eq. (15). Newmark time integration scheme is used to discretize this equation. There are some well-known schemes for

| Table 2 | Resonance frequencies of a single layer microplate \((l = 800\mu m, \ h = 20\mu m, \ E_1 = 149GPa)\) |
| --- | --- | --- |
| Thickness \((\mu m)\) | 2.25 | |
| Resonance frequency (kHz) | Experimental results \([11]\) | C–F–F–F plate \((10\ elements)\) |
| First | 4.8 | 4.665 |
| Second | 30.2 | 29.573 |
| Third | 85.0 | 84.438 |

| Table 3 | Vibrational analysis of a C–F–C–F multi layer microplate \((l = 250\mu m, \ h_1 = 3\mu m, \ h_2 = 0.5\mu m, \ E_1 = 250GPa, \ E_2 = 53GPa, \ E' = 8, \ 25 \ elements \ in \ length \ and \ four \ elements \ in \ width)\) |
| --- | --- | --- |
| Mode number | Natural frequencies (kHz) |
| | 50 | 15 | 6 |
| Width to height ratios | \(b > 1/10\) | \(1/10 > b > 1/20\) | \(1/20 > b\) |
| 1st | 342.204 | 338.989 | 338.126 |
| 2nd | 943.034 | 934.383 | 931.990 |
| 3rd | 1145.331 | 1831.827 | 1826.828 |
| 4th | 1850.739 | 3028.458 | 3019.296 |
| 5th | 2359.576 | 3215.302 | 4509.403 |
| 6th | 3063.671 | 4524.871 | 5851.882 |
| 7th | 3703.286 | 6321.560 | 6297.046 |
| 8th | 4582.282 | 6445.002 | 8382.383 |
| 9th | 5224.337 | 8419.200 | 10,765.945 |
| 10th | 6406.176 | 9703.507 | 11,705.112 |
Fig. 6. Mode shapes of multi-layer microplates with different widths (a) $b = 6\, \mu m$, (b) $b = 15\, \mu m$, (c) $b = 50\, \mu m$ ($l = 250\, \mu m$, $h_1 = 3\, \mu m$, $h_2 = 0.5\, \mu m$, $E_1 = 250GPa$, $E_2 = 53GPa$, $\rho_1 = 3100\, kg/m^3$, $\rho_2 = 19300\, kg/m^3$).
Newmark method [8]. In this paper, the constant-average acceleration method which is a stable scheme is used. The pressure is governed by the discrete nonlinear or linearized Reynolds equations (Eqs. (31) and (35)). Finite difference method (FDM) is used to solve these equations step-by-step.

In studying dynamical behavior of multi-physics systems, step-input voltages are imposed. A time step has to be chosen that satisfy finite difference stability relation (Eq. (32)). Using this time step, in each step, values of deflections in the previous step are used to calculate electrostatic force and pressure, and solve plate equations using finite element method. Solution results in determining magnitudes of deflection in the current time step. Using deflections of this step and deflections and pressures of the previous step, pressures at the current step can be calculated using FDM. It is evident that in the first step there is no dynamic pressure. Also, in the first step, electrostatic force does not depend on deflections. More explanation is presented in Table 4.

Using this method, response of multi-layer microplates to step-input voltage is computed. In Fig. 7 deflections of plate center point vs. time for various values of ambient pressure is presented.

In this diagram, it is assumed that the pressure is governed by nonlinear Reynolds equation. As it can be seen, increasing ambient pressure, results in reduction of altitude and period of oscillations. In high-ambient pressure, there is no oscillation and system does not respond as under-damped systems.

In Fig. 8a deflections of plate center point vs. time for various values of applied voltage is presented. Nonlinear Reynolds equation is used to consider squeeze film effect. By decreasing voltage, the fundamental frequency of the system is increased. For smaller voltages, steady-state response of the system is achieved sooner in time. By increasing voltage, the dynamical system becomes unstable and no steady-state response exists. This phenomenon is called dynamic pull in and the voltage causes this phenomenon, is called dynamic pull-in voltage \( V_{pid} \) [12]. Dynamic pull-in voltage usually computed ignoring damping effect to achieve minimum value of \( V_{pid} \) but it also can be calculated considering damping as it has been studied in [13]. Results indicate that for the values of voltage beyond \( V_{pid} \) system deflects without any limit. Pressure variations of these systems are presented in Fig. 8b. As it can be seen in this figure, there exists a steady-state response of pressure for low values of applied voltage, whereas at high magnitudes of voltage, after some periods, pressure oscillations increase and similar to the deflection variations, no steady-state response can be observed. In Fig. 8c, deflections of plate center point vs. time for the undamped system is presented. In this case, deflections for voltages lower than \( V_{pid} \) are periodic.

In Fig. 9, there is a comparison between nonlinear Reynolds modeling and linearized Reynolds modeling. In this figure, deflections of plate center point vs. time for different values of voltages are presented. Curves with thick style represent nonlinear damping and curves with normal style represent linear damping. As it can be seen in this figure, for small deflections both linear and nonlinear Reynolds equation can be used. But by increasing voltages, the difference between nonlinear and linearized damping

<table>
<thead>
<tr>
<th>Step</th>
<th>Inputs of the finite element problem</th>
<th>Outputs of the finite element problem</th>
<th>Inputs of the finite difference problem</th>
<th>Outputs of the finite difference problem</th>
</tr>
</thead>
<tbody>
<tr>
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<td>( U_1 )</td>
<td>( U_1 )</td>
<td>( P_1 )</td>
</tr>
<tr>
<td>2</td>
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<td>( U_2 )</td>
<td>( P_2 )</td>
</tr>
<tr>
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<td>( U_k )</td>
<td>( U_k )</td>
<td>( U_k )</td>
<td>( P_k )</td>
</tr>
</tbody>
</table>

Fig. 7. Deflection of plate center point vs. time \( l = 250 \mu m, b = 50 \mu m, h_1 = 1 \mu m, h_2 = 0.5 \mu m, d_{gap} = 1 \mu m, E_1 = 250GPa, E_2 = 53GPa, \sigma_1 = 0, \sigma_2 = 0, \epsilon_1 = 8, \rho_1 = 3100kg/m^3, \rho_2 = 19300kg/m^3, voltage = 4V \).
increases. Consequently there is a relatively large difference between nonlinear and linearized damping for large magnitudes of deflection. It is to be noted that using linearized Reynolds equation results in overestimated values for deflections.

In Fig. 10a, diagram of deflections for a half-period interval at several sequential time steps is presented. Diagram of pressures for this system is also presented in Fig. 10b. Due to the symmetry, only the values of pressure, for half of the plate are shown.

6. Conclusions

Multi-layer microplates subjected to electric field have been modeled. The model is developed to consider in-plane forces, nonlinear electrostatic actuation and nonlinear squeeze film damping using FSDT. Pull-in analysis of single- and multi-layer microplates is performed and results are in good agreement with literature. Also vibrational behavior of single-layer microplates is studied. The model is validated using results of investigation on single-layer microplates reported in the literature. Resonance frequencies and mode shapes of multi-layer microplates are obtained. It is found that as the aspect ratio of the plates (width over length) decreases, twisting frequencies becomes less important and the plate may be modeled by simple beam theory. Using FEM to model the plate and FDM to model squeeze film damping, an algorithm is developed to study dynamical behavior of microplates. Considering nonlinear and linearized Reynolds equation, effect of ambient pressure and different values of voltage on the system is studied. Results indicate as the ambient pressure increases, the oscillatory system is damped drastically and gradually tends toward over damping. This model will be useful in designing coupled-domain multi-layer microplates.
Fig. 10. (a) Diagram of deflections in different time steps (b) diagram of pressures in different time steps ($l = 250\mu m$, $b = 50\mu m$, $h_1 = 1\mu m$, $h_2 = 0.5\mu m$, $\delta_{gap} = 1\mu m$, $E_1 = 250GPa$, $E_2 = 53GPa$, $\sigma_1 = 0$, $\sigma_2 = 0$, $\epsilon_1 = 8$, $P_0 = 0.2$ bar).

References


