STATES ON $BE$-ALGEBRAS

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Abstract: In this paper, we study the Bosbach state on $BE$-algebra and show that there is a Bosbach state via $X/\theta$ while, $\theta$ is congruence relation induced by a filter of $X$. We prove that, a Bosbach state $s$ on $X$ is a nontrivial state morphism if and only if ker($s$) is an obstinate filter of $X$. Moreover, we define a new upper bound for two elements of $BE$-algebra $X$ and investigate the relationship between obstinate filters with this upper bound. We prove that $X$ is an upper semi-lattice relative to this upper bound if and only if $X$ is commutative. We show that the state morphisms are Bosbach states, but the converse is not valid.

1. Introduction and preliminaries

In 1933 Kolmogorov introduced axiomatization of probability and both probability and statistics had developed into major fields. But new areas of science have appeared during the last century, such as quantum mechanics, which do not satisfy the Kolmogorov axioms. These new fields of science require a probability theory based on non-classical logics. We know multiple-valued logics are non-classical logics and became popular in computer science since it was understood that they play a fundamental role in fuzzy logics. In analogous to probability measure, the states on multiple-valued algebras proved to be the most suitable models for averaging the truth-value in their corresponding logics. States or measures give a probabilistic interpretation of randomness of events of given algebraic structures and was first introduced by Kôpka and Chovanec for $MV$-algebras and by Riecan for $BL$-algebras in [18]. Mundici introduced states (an analogue of probability measures) on $MV$-algebras in 1995, as averaging of the truth-value in Lukasiewicz logic [21]. Since middle 1990's, mainly after Mundici's paper [21], on probability theory on $MV$-algebras, there has been an increasing amount of studies on generalizations of probability theoretical concepts, most notable states, on various logic origin algebraic structures. Indeed, first on $BL$-algebras, then on $MTL$-algebras and finally on non-commutative residuated lattices; although

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these generalizations were questioned in [20, 25], where it was demonstrated that states on these generalized structures are determined completely through states on their MV–center. The question thus arises, how meaningful all these generalizations are. In the present paper, states are studied on what are called $BE$–algebras. These structures are even more general than residuated lattices, see [15], and also related to hoops studied e.g. in [4]. Indeed, substituting the operation "$*$" by "$	o$" in Definition 1.1, we immediately observe that residuated lattices, and thus also hoops, are $BE$–algebras. Bosbach state was introduced by B. Bosbach in [6, 7, 8]. The notion of a Bosbach state has been studied for other algebras of fuzzy structures such as pseudo $BL$–algebras, [16], bounded non-commutative $R^\ell$–monoids, [11, 12, 13], residuated lattices, [10], pseudo $BCK$–semi-lattices and pseudo $BCK$–algebras, [19]. In [9], C. Buşneag developed theory State-morphism on Hilbert algebra and get some results relative to the theory of Bosbach states on bounded and unbounded Hilbert algebras. $BE$–algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant $1$ which is considered as the logical value "true".

The notion of $BE$–algebras was introduced by H. S. Kim and Y. H. Kim [17], which was deeply studied by S. S. Ahn and et al., in [1, 2, 3], Walendziak in [26], A. Rezaei and et al., in [22, 23, 24].

In this paper, we continue study on the $BE$–algebras and define Bosbach states and state morphism on $BE$–algebras. Furthermore, we define a new upper bound on $BE$-algebra $X$ and prove that $X$ is an upper semi-lattice relative to $\sqcup$ if and only if $X$ is commutative. Also, we consider a filter $F$ of self distributive $BE$–algebra $X$ and construct a quotient $BE$–algebra $X/\theta$, while $\theta$ is a congruence relation induced by filter $F$. We show that, a Bosbach state $s$ on $X$ is a nontrivial state morphism if and only if ker$(s)$ is an obstructive filter of $X$.

**Definition 1.1** [17]. An algebra $(X; *, 1)$ of type $(2, 0)$ is called a $BE$–algebra if the following axioms hold:

$(BE1)$ \[ x \ast x = 1, \]
$(BE2)$ \[ x \ast 1 = 1, \]
$(BE3)$ \[ 1 \ast x = x, \]
$(BE4)$ \[ x \ast (y \ast z) = y \ast (x \ast z), \text{ for all } x, y, z \in X. \]

From now on, $X$ is a $BE$–algebra, unless otherwise is stated. A subset $F$ of $X$ is called a filter of $X$ if $(F1)$ $1 \in F$ and $(F2)$ $x \in F$ and $x \ast y \in F$ imply $y \in F$. We denote by $F(X)$ the set of all filter of $X$ and $Max(X)$ the set of all maximal filter of $X$. Let $A$ be a non-empty subset of $X$, then the set

\[ \langle A \rangle = \bigcap \{ G \in F(X) \mid A \subseteq G \} \]
is called the filter generated by $A$, written $\langle A \rangle$. If $A = \{a\}$, we will denote $\langle\{a\}\rangle$, briefly by $\langle a \rangle$, and we call it a principal filter of $X$. For $F \in F(X)$ and $a \in X$, we denote by $F_a$ the filter generated by $F \cup \{a\}$. A algebra $X$ is said to be self distributive if $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$, for all $x, y, z \in X$, (Example 8., [17]). In a self distributive $BE$-algebra $X$, $F_a = \{x \in X : a \ast x \in F\}$, ([3]). We say that $X$ is commutative if $(x \ast y) \ast y = (y \ast x) \ast x$, for all $x, y \in X$.

We introduce a relation "$\leq$" on $X$ by $x \leq y$ if and only if $x \ast y = 1$. We note that "$\leq$" is reflexive by $(BE1)$. If $X$ is self distributive, then relation "$\leq$" is a transitive order set on $X$, because if $x \leq y$ and $y \leq z$, then

$$x \ast z = 1 \ast (x \ast z) = (x \ast y) \ast (x \ast z) = x \ast (y \ast z) = x \ast 1 = 1$$

Hence $x \leq z$. If $X$ is commutative, then by Proposition 3.3, [26], relation "$\leq$" is antisymmetric. Hence if $X$ is a commutative self distributive algebra, then "$\leq$" is a partial order set on $X$, (Example 3.4., [3]).

A mapping $f : X \to Y$ of $BE$–algebras is called a $BE$–homomorphism if $f(x \ast y) = f(x) \ast f(y)$, for all $x, y \in X$.

**Proposition 1.2** [17]. The following properties hold:

(i) $x \ast (y \ast x) = 1$,

(ii) $y \ast ((y \ast x) \ast x) = 1$, for all $x, y \in X$.

**Proposition 1.3** [22]. Let $X$ be a self distributive. If $x \leq y$, then

(i) $z \ast x \leq z \ast y$ and $y \ast z \leq x \ast z$,

(ii) $y \ast z \leq (z \ast x) \ast (y \ast x)$, for all $x, y, z \in X$.

**Theorem 1.4** [26, 23]. Let $X$ be commutative. Then it is a semi-lattice with respect to $\vee$.

**Definition 1.5** [24]. $X$ is said to be an implicative if $x = (x \ast y) \ast x$, for all $x, y \in X$.

**Definition 1.6** [5]. A filter $F$ of $X$ is called an obstinate filter if $x, y \notin F$ imply $x \ast y \in F$ and $y \ast x \in F$, for any $x, y \in X$.

2. A new upper bound in $BE$–algebras

For two elements $x, y \in X$, we denote

$$x \uplus y = (x \ast y) \ast ((y \ast x) \ast x).$$

By Proposition 1.2, it is obvious that $x \vee y \leq x \uplus y$. 

Example 2.1. Let $X = \{1, a, b, c, d\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
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<td>1</td>
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<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>a</td>
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<tr>
<td>d</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $(X; *, 1)$ is a BE-algebra, and

\[ a \lor d = (d * a) * a = 1 * a = a \neq 1 = d * a = (a * d) * ((d * a) * a) = a \cup d \]

Proposition 2.2. Let $X$ be a self distributive BE-algebra. Then

(i) $x, y \leq x \cup y$ (i.e. $x \cup y$ is an upper bound for $x$ and $y$),
(ii) $x \cup x = x$, $x \cup 1 = 1$,
(iii) $x \cup (x * y) = 1$,
(iv) $x * (y \cup z) = (x * y) \cup (x * z)$, for all $x, y, z \in X$.

Proof. By Definition 1.1 and Propositions 1.2 and 1.3, the proofs are clear. \qed

Note. In the following example we can see that the condition self distributivity in Proposition 2.2, is necessary.

Example 2.3. Let $X = \{1, a, b, c\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<td>1</td>
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<td>a</td>
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<td>1</td>
<td>1</td>
<td>a</td>
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<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>a</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $(X, *, 1)$ is a BE-algebra but it is not self distributive because $c * (a * b) = c * a = 1 \neq (c * a) * (c * b) = 1 * a = a$. Now,

\[ c \cup (c * b) = (c * (c * b)) * (((c * b) * c) * c) = 1 * (a * c) = 1 * a = a \neq 1. \]

Proposition 2.4. Let $X$ be implicative. Then $\lor = \sqcup$.

Proof. By $(BE4)$ and Definition 1.5, the proof is clear. \qed

Theorem 2.5. Let $X$ be a self distributive BE-algebra. Then the following are equivalent:

(i) $X$ is an upper semi-lattice relative to $\sqcup$,
(ii) $X$ is commutative.
PROOF. (i $\Rightarrow$ ii). Suppose that for every $x, y \in X$, $x \lor y$ exists and $x \lor y = x \uplus y$. Since $x, y \leq (y \ast x) \ast x$ we get that $x \uplus y = x \lor y \leq (y \ast x) \ast x$. Since $(y \ast x) \ast x \leq x \uplus y$ we get that $x \uplus y = x \lor y = (y \ast x) \ast x$. Analogously $x \uplus y = x \lor y = (x \ast y) \ast y$. Hence $X$ is commutative.

(ii $\Rightarrow$ i). Suppose that $X$ is commutative. Let $x, y \in X$.

\[
x \uplus y = (x \ast y) \ast ((y \ast x) \ast x)
= (x \ast y) \ast ((x \ast y) \ast y) \quad \text{(By commutativity)}
= ((x \ast y) \ast (x \ast y)) \ast ((x \ast y) \ast y) \quad \text{(By self distributivity)}
= 1 \ast ((x \ast y) \ast y) \quad \text{(By BE1)}
= (x \ast y) \ast y \quad \text{(By BE3)}
\]

By the similar way we have $x \uplus y = (y \ast x) \ast x$. Now, since $X$ is an upper semi-lattice relative to $\uplus$, then $x \uplus y = y \uplus x$. Hence $x \lor y = (x \ast y) \ast y = x \uplus y = y \uplus x = (y \ast x) \ast x = y \lor x$, and so $X$ is commutative. $\square$

For $a \in X$ we denote by $H(a) = \{x \in X : a \ast x = 1\}$. which is called segment section of an element $a$. Since $1, a \in H(a)$, then $H(a)$ is not empty set. S. S. Ahn et al., proved that in commutative algebra $X$ satisfies: $H(a) \cap H(b) = H(a \lor b)$ for all $a, b \in X$ in [3]. Using this notation, we have a similar result for Theorem 3.10, [3] for the case of operation $\uplus$.

**Theorem 2.6.** A self distributive BE-algebra $X$ is commutative iff $H(a) \cap H(b) = H(a \uplus b)$, for all $a, b \in X$.

**Proof.** ($\Rightarrow$). Suppose $X$ is commutative and $a, b \in X$. Since $a, b \leq a \uplus b$, we have $H(a \uplus b) \subseteq H(a) \cap H(b)$. If $x \in H(a) \cap H(b)$, then $a \leq x$ and $b \leq x$. Then

\[
a \uplus b = (a \ast b) \ast b \leq (x \ast b) \ast b = (b \ast x) \ast x = 1 \ast x = x,
\]

so $x \in H(a \uplus b)$. Hence $H(a) \cap H(b) = H(a \uplus b)$.

($\Leftarrow$). Let $a, b \in X$ and $H(a) \cap H(b) = H(a \uplus b)$. Now, we have $(a \ast b) \ast b \in H(a) \cap H(b) = H(a \uplus b)$ by Proposition 1.2. Hence $a \uplus b = (b \ast a) \ast ((a \ast b) \ast b) \leq (a \ast b) \ast b$, that is $a \uplus b = (a \ast b) \ast b$. Analogously we get that $a \uplus b = (b \ast a) \ast a$, and so $(a \ast b) \ast b = (b \ast a) \ast a$. Hence $X$ is commutative. $\square$

**Theorem 2.7.** Let $X$ be a self distributive BE-algebra and $F \in F(X)$. Then the following are equivalent:

(i) $F$ is an obstinate filter;

(ii) If $x, y \in X$ and $x \uplus y \in F$, then $x \in F$ or $y \in F$. 

PROOF. (i ⇒ ii). Let $F$ be an obstinate filter and there exist $x, y \in X$ such that $x \notin F$ and $y \notin F$ by contrary. Then, we get that $x \ast y \in F$ and $y \ast x \in F$. Since $x \sqcup y = (x \ast y) \ast ((y \ast x) \ast x) \in F$, we get that $(y \ast x) \ast x \in F$ and so $x \in F$, which is a contradiction.

(ii ⇒ i). Suppose that $F$ is not an obstinate filter, that is, there exists $x, y \notin F$ such that $x \ast y \notin F$ and $y \ast x \notin F$. Since $x \sqcup (x \ast y) = 1 \in F$ and $x \notin F$, then $x \ast y \in F$, which is a contradiction. On the other hand, since $y \sqcup (y \ast x) = 1 \in F$ and $y \notin F$, then $y \ast x \in F$, which is a contradiction. \qed

**THEOREM 2.8.** Let $X$ be a self distributive BE–algebra, $F \in F(X)$ and $F \neq X$. Then the following are equivalent:

(i) $F$ is an obstinate filter,

(ii) If $x \notin F$, then $x \ast y \in F$, for all $y \in F$.

PROOF. (i ⇒ ii). Let $F$ be an obstinate filter and $x \notin F$. Since $x \sqcup (x \ast y) = 1 \in F$ for every $y \in X$, then by Theorem 2.7, we get that $x \ast y \in F$.

(ii ⇒ i). Suppose that $x, y \notin F$. Then $x \ast y \in F$ and $y \ast x \in F$ by (ii). Hence $F$ is an obstinate filter of $X$. \qed

### 3. States on BE–algebras

**DEFINITION 3.1.** A Bosbach state on $X$ is a function $s : X \to [0, 1]$ such that the following axioms hold:

(i) $s(1) = 1$,

(ii) $s(x) + s(x \ast y) = s(y) + s(y \ast x)$, for all $x, y \in X$.

For a Bosbach state $s : X \to [0, 1]$, define $\ker(s) = \{x \in X : s(x) = 1\}$. Obviously, a function $s : X \to [0, 1]$ such that $s(x) = 1$, for all $x \in X$, is called the trivial Bosbach state on $X$, and the function

$$s(x) = \begin{cases} 1 & \text{if } x = 1 \\ t & \text{otherwise} \end{cases}$$

is called the constant Bosbach state, for some $t \in [0, 1]$.

**EXAMPLE 3.2.** (a). In Example 2.2, define a function $s : X \to [0, 1]$ by $s(1) = 1$, $s(a) = 0.7$ and $s(b) = s(c) = 0.4$. Then $s$ is a Bosbach state on $X$. Let define $s(1) = 1$, $s(a) = 0.7$ and $s(b) = s(c) = 0.3$. Then $s$ is not a Bosbach state on $X$. Because $1.4 = s(a) + s(a \ast b) \neq s(b) + s(b \ast a) = 1.3$.

(b). If $s : X \to [0, 1]$ is a Bosbach state and $X$ is a self distributive BE–algebra, then for every $x \in X$, $s_a : X \to [0, 1]$, $s_a(x) = s(a \ast x)$ is also a Bosbach state on $A$. Indeed, $s_a(1) = s(a \ast 1) = s(1) = 1$ and for all $x, y \in X$,
$s_a(x) + s_a(x \ast y) = s(a \ast x) + s(a \ast (x \ast y))$

$= s(a \ast x) + s((a \ast x) \ast (a \ast y))$

$= s(a \ast y) + s((a \ast y) \ast ((a \ast x))$

$= s(a \ast y) + s(a \ast (y \ast x))$

$= s_a(y) + s_a(y \ast x).$

**Proposition 3.3.** Let $F \in \text{Max}(X)$. Then there is a Bosbach state $s : X \to [0, 1]$ such that $\ker(s) = F$.

**Proof.** Let $F \in \text{Max}(X)$. Then $s : X \to [0, 1]$, which is defined by

$$s(x) = \begin{cases} 
1 & \text{if } x \in F \\
0 & \text{otherwise}
\end{cases}$$

is a Bosbach state such that $\ker(s) = F$. Since $1 \in F$, then $s(1) = 1$.

Let $x, y \in X$. Consider the following cases:

Case 1). If $x, y \in F$, then since filter $F$ is a subalgebra of $X$, so $x \ast y$ and $y \ast x \in F$. Thus

$$s(x) + s(x \ast y) = 1 + 1 = s(y) + s(y \ast x)$$

Case 2). If $x, y \notin F$, then by Theorem 2.8(ii), $x \ast y$ and $y \ast x \in F$, and so

$$s(x) + s(x \ast y) = 0 + 1 = s(y) + s(y \ast x)$$

Case 3). If $x \notin F$ and $y \in F$, then by Theorem 2.8(ii), we have $x \ast y \in F$. Also, $y \ast x \notin F$, by contrary, if $y \ast x \in F$ since $y \in F$ and $F$ is a filter, then $x \in F$, which is a contradiction. Thus

$$s(x) + s(x \ast y) = 1 + 0 = s(y) + s(y \ast x).$$

Therefore, $s$ is a Bosbach state.

**Lemma 3.4.** Let $s : X \to [0, 1]$ be a Bosbach state on $X$. Then

(i) If $x \leq y$, then $s(x) \leq s(y)$ and $s(y \ast x) = 1 + s(x) - s(y)$.

(ii) $s((x \ast y) \ast y) = s((y \ast x) \ast x)$, for all $x, y \in X$.

**Proof.** (i). Let $s$ be a Bosbach state on $X$ and $x \leq y$. Then $x \ast y = 1$. Since $s(x) + s(x \ast y) = s(y) + s(y \ast x)$, then we have $s(y \ast x) = 1 + s(x) - s(y)$ and $s(y) - s(x) = 1 - s(y \ast x) \geq 0$, and so $s(y) \geq s(x)$.

(ii). Since by Proposition 1.2, $y \leq x \ast y$ and $x \leq y \ast x$, then by (i) we get $s((x \ast y) \ast y) = 1 + s(y) - s(x \ast y)$ and $s((y \ast x) \ast x) = 1 + s(x) - s(y \ast x)$. 
Hence
\[ s((x \ast y) \ast y) - s((y \ast x) \ast x) = 1 + s(y) - s(x \ast y) - 1 - s(x) + s(y \ast x) = 0. \]
Therefore, \( s((x \ast y) \ast y) = s((y \ast x) \ast x). \)

PROPOSITION 3.5. Let \( s : X \rightarrow [0, 1] \) be a function on \( X \) such that \( s(1) = 1 \) and \( s((x \ast y) \ast y) = s((y \ast x) \ast x) \), for all \( x, y \in X \). Then the following are equivalent:

(i) \( s \) is a Bosbach state on \( X \).
(ii) \( x \leq y \) implies \( s(y \ast x) = 1 + s(x) - s(y) \).
(iii) \( s(y \ast x) = 1 - s((y \ast x) \ast x) + s(x), \) for all \( x, y \in X \).

PROOF. (i \( \Rightarrow \) ii). By Lemma 3.4, the proof is clear.
(ii \( \Rightarrow \) iii). Since \( x \leq y \ast x \), then \( s((y \ast x) \ast x) = 1 + s(x) - s(y \ast x) \).
Hence \( s(y \ast x) = 1 + s(x) - s((y \ast x) \ast x) \).
(iii \( \Rightarrow \) i). Let \( x, y \in X \). Then
\[
\begin{align*}
  s(x) + s(x \ast y) &= s(x) + 1 - s((x \ast y) \ast y) + s(y) \quad \text{(By (iii))} \\
  &= 1 - s((y \ast x) \ast x) + s(x) + s(y) \quad \text{(By assumption)} \\
  &= s(y \ast x) + s(y).
\end{align*}
\]
Hence \( s \) is a Bosbach state.

PROPOSITION 3.6. If \( s : X \rightarrow [0, 1] \) is a Bosbach state on \( X \), then \( \ker(s) \) is a filter of \( X \).

PROOF. Since \( s(1) = 1 \), then \( 1 \in \ker(s) \). Let \( x \ast y \in \ker(s) \) and \( x \in \ker(s) \).
So \( s(x \ast y) = s(x) = 1 \). Since by Proposition 1.2, \( x \leq y \ast x \) then by Lemma 3.4, we have \( s(x) \leq s(y \ast x) \) and \( 1 = s(x) \leq s(y \ast x) \leq 1 \). Hence \( s(y \ast x) = 1 \).
Also, since \( s(x) + s(x \ast y) = s(y) + s(y \ast x) \), then \( 1 + 1 = s(y) + 1 \) and so \( s(y) = 1 \). Hence \( y \in \ker(s) \).

THEOREM 3.7. Let \( f : X \rightarrow Y \) be a BE-homomorphism and \( s : Y \rightarrow [0, 1] \) be a Bosbach state on \( Y \). Then there exists an unique Bosbach state \( t : X \rightarrow [0, 1] \), such that, the following diagram is commutative (i.e. \( t = s \circ f \)).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\nearrow t & & \searrow s \\
& & [0, 1]
\end{array}
\]

PROOF. Let \( t := s \circ f \). It is clear that \( t \) is well defined. Now, we show that \( t \) is a Bosbach state. First, we can see that \( t(1) = (s \circ f)(1) = s(f(1)) = s(1) = 1 \). Now, let \( x, y \in X \). Then
\[ t(x) + t(x \ast y) = (s \circ f)(x) + (s \circ f)(x \ast y) \]
\[ = s(f(x)) + s(f(x \ast y)) \]
\[ = s(f(x)) + s(f(x) \ast f(y)) \]
\[ = s(f(y)) + s(f(y) \ast f(x)) \]
\[ = s(f(y)) + s(f(y \ast x)) \]
\[ = (s \circ f)(y) + (s \circ f)(y \ast x) \]
\[ = t(y) + t(x \ast y). \]

Hence \( t \) is a Bosbach state. Finally, we prove that \( t \) is unique. Suppose that there exists another Bosbach state \( r : X \to [0, 1] \), such that, \( r = s \circ f \). Then \( r(x) = (s \circ f)(x) = t(x) \), for all \( x \in X \). Hence \( r = t \). \( \square \)

**Theorem 3.8.** Let \( f : X \to Y \) be a bijection BE-homomorphism and \( s : X \to [0, 1] \) be a Bosbach state on \( X \). Then there exists an unique Bosbach state \( t : Y \to [0, 1] \), such that, \( s = t \circ f \).

\[
\begin{array}{c}
X \\
\downarrow s
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
Y \\
\downarrow t
\end{array} \quad [0, 1]
\]

**Proof.** Let \( y \in Y \). Since \( f \) is onto, then there exists \( x \in X \) such that, \( f(x) = y \). Set \( t(y) := s(x) \). Hence \( t(y) = t(f(x)) = (t \circ f)(x) = s(x) \) and so, \( t \circ f = s \). Since \( f \) is one to one, then \( \ker f = \{1\} \), and so \( t(1) = t(f(1)) = (t \circ f)(1) = s(1) = 1 \). Let \( y_1, y_2 \in Y \). Then there exists \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Then

\[
\begin{align*}
t(y_1) + t(y_1 \ast y_2) &= t(f(x_1)) + t(f(x_1) \ast f(x_2)) \\
&= t(f(x_1)) + t(f(x_1) \ast x_2) \\
&= (t \circ f)(x_1) + (t \circ f)(x_1 \ast x_2) \\
&= s(x_1) + s(x_1 \ast x_2) \\
&= s(x_2) + s(x_2 \ast x_1) \\
&= (t \circ f)(x_2) + (t \circ f)(x_2 \ast x_1) \\
&= t(f(x_2)) + t(f(x_2) \ast x_1) \\
&= t(f(x_2)) + t(f(x_2) \ast f(x_1)) \\
&= t(y_2) + t(y_2 \ast y_1).
\end{align*}
\]

Hence, \( t \) is unique.
Hence $t$ is a Bosbach state. Suppose that there exists another Bosbach state $r : X \to [0, 1]$, such that, $s = r \circ f$. Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Now, $r(y) = r(f(x)) = (r \circ f)(x) = s(x)$, but by definition of $t$ we have $r(y) = s(x) = t(y)$. Hence $r = t$.  

Let $F$ be a filter of self distributive $BE$-algebra $X$. Consider relation $\theta$ on $X$ by

$$(x, y) \in \theta \iff x \ast y \in F \text{ and } y \ast x \in F,$$

then $\theta$ is a congruence relation on $X$, [27]. For $x \in X$, we denote by $[x]$ the congruence class of $x$ and we let $X/\theta = ([x] : x \in X)$. Then $X/\theta$ is a $BE$-algebra, where, $[x] \ast [y] = [x \ast y]$ for all $x, y \in X$. Also, we denote $p_\theta : X \to X/\theta$ the canonical surjective morphism of $BE$-algebras, by $p_\theta(x) = [x]$. If $x \in F$, then $[x] = [1] = 1$.

We introduce a relation “$\leq$” on $X/\theta$ by $[x] \leq [y]$ if and only if $[x] \ast [y] = [1] = 1$.

**Proposition 3.9.** Let $s : X \to [0, 1]$ be a Bosbach state on $X$. Then

(i) $[x] \leq [y] \iff s(x \ast y) = 1 \iff s((x \ast y) \ast y) = s(y)$.

(ii) $[x] = [y] \iff s(x \ast y) = s(y \ast x) = 1 \iff s(x) = s(y) = s((x \ast y) \ast y)$.

**Proof.** It follows from Proposition 3.5.

**Theorem 3.10.** Let $F$ be a filter of self distributive $X$ and $s : X \to [0, 1]$ be a Bosbach state on $X$. Then there exists a unique Bosbach state $t : X/\theta \to [0, 1]$, such that, the following diagram is commutative (i.e. $s = t \circ p_\theta$), in fact, $\theta$ is congruence relation induced by filter $F$.

\[
\begin{array}{ccc}
X & \xrightarrow{p_\theta} & X/\theta \\
\downarrow s & & \downarrow \exists! t \\
[0, 1] & & \\
\end{array}
\]

**Proof.** By Theorem 3.8, the proof is clear.

**Corollary 3.11.** Let $X$ be self distributive and $s : X \to [0, 1]$ be a Bosbach state on $X$. Then there exists an unique Bosbach state $t : X/\theta \to [0, 1]$, such that, the following diagram is commutative (i.e., $s = t \circ p_\theta$), in fact, $\theta$ is congruence relation induced by filter $\ker(s)$.

\[
\begin{array}{ccc}
X & \xrightarrow{p_\theta} & X/\theta \\
\downarrow s & & \downarrow \exists! t \\
[0, 1] & & \\
\end{array}
\]
Proof. By Proposition 3.6 and Theorem 3.10, the proof is clear. □

Consider the real interval \([0,1]\) of reals with the operation "\(*\)" defined by
\[
x * y = \min\{1 - x + y, 1\}, \quad \text{for all } x, y \in X.
\]
Then \([0,1]; *, 1\) is a \(BE\)-algebra. We denote by \([0,1]\) the standard \(BE\)-algebra of real unit interval \([0,1]\).

Definition 3.12. A state morphism on \(X\) is a function \(f : X \to [0,1]\) such that \(f(x * y) = \min\{1 - f(x) + f(y), 1\}\), for all \(x, y \in X\).

Note. \(f(1) = f(1 * 1) = \min\{1 - f(1) + f(1), 1\} = \min\{1,1\} = 1\).

Example 3.13. Let \(X = \{1, a, b\}\) be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>a</td>
<td>1</td>
</tr>
</tbody>
</table>

Then \((X; *, 1)\) is a \(BE\)-algebra. Define \(f : X \to [0,1]\) by \(f(1) = f(a) = 1\) and \(f(b) = 0.2\). Then \(f\) is a state morphism.

Proposition 3.14. Let \(f : X \to [0,1]\) be a state morphism on \(X\). Then \(f\) is a Bosbach state.

Proof. Let \(x, y \in X\).
\[
f(x) + f(x * y) = f(x) + \min\{1 - f(x) + f(y), 1\}
\]
\[
= \min\{1 + f(y), 1 + f(x)\}
\]
\[
= f(y) + \min\{1 - f(y) + f(x), 1\}
\]
\[
= f(y) + f(y * x),
\]
that is, \(f\) is a Bosbach state. □

The following example shows that the converse of above proposition is not valid.

Example 3.15. In Example 3.2(a), function \(s : X \to [0,1]\) defined by \(s(1) = 1, s(a) = 0.7\) and \(s(b) = s(c) = 0.4\) is a Bosbach state but it is not an state morphism on \(X\). Because \(s(c * b) = s(a) = 0.7 \neq s(c) * s(b) = 0.4 * 0.4 = 1\).
Theorem 3.16. A Bosbach state $f$ on $X$ is an state morphism if and only if $f((y * x) * x) = \max\{f(x), f(y)\}$, for all $x, y \in X$.

Proof. If $f$ is a state morphism on $X$, then by Proposition 3.14, $f$ is a Bosbach state. According to Proposition 3.5 (iii), we have

$$f((y * x) * x) = 1 + f(x) - f(y * x)$$

$$= 1 + f(x) - \min\{1 - f(y) + f(x), 1\}$$

$$= 1 + f(x) + \max\{-1 + f(y) - f(x), -1\}$$

$$= \max\{1 + f(x) - 1 + f(y) - f(x), 1 + f(x) - 1\}$$

$$= \max\{f(y), f(x)\}.$$

Conversely, assume that $f$ is a Bosbach state on $X$ such that

$$f((y * x) * x) = \max\{f(x), f(y)\}, \text{ for all } x, y \in X.$$

By Proposition 3.5 (iii), we have

$$f(y * x) = 1 - f((y * x) * x) + f(x)$$

$$= 1 + f(x) - \max\{f(x), f(y)\}$$

$$= 1 + f(x) + \max\{-f(x), -f(y)\}$$

$$= \min\{1 + f(x) - f(y), 1 + f(x) - f(y)\}$$

$$= \min\{1, 1 - f(y) + f(x)\}.$$

Hence $f$ is a state morphism on $X$. \qed

Proposition 3.17. Let $s : X \rightarrow [0, 1]$ be a Bosbach state on self distributive BE–algebra $X$. Then the following are equivalent:

(i) $s$ is a nontrivial state morphism,

(ii) ker $s$ is an obstinate filter of $X$.

Proof. (i $\Rightarrow$ ii). Since $s$ is a nontrivial state morphism, then ker$(s) \neq X$. Let $x, y \in X$ such that $x \sqcup y \in \ker(s)$. Then

$$s(x \sqcup y) = 1 \Rightarrow s((x * y) * ((y * x) * x)) = 1$$

$$\Rightarrow \min\{1 - s(x * y) + s((y * x) * x), 1\} = 1$$

$$\Rightarrow 1 - s(x * y) + s((y * x) * x) \geq 1$$

$$\Rightarrow s((y * x) * x) \geq s(x * y) = \min\{1 - s(x) + s(y), 1\}.$$
Now, if \( s(x) \leq s(y) \), then \( 1 - s(x) + s(y) \geq 1 \) and \( 1 + s(x) - s(y) \leq 1 \). So we get that

\[
s((y \ast x) \ast x) \geq 1 \Rightarrow s((y \ast x) \ast x) = 1
\]

\[
\Rightarrow \min\{1 - s(y \ast x) + s(x), 1\} = 1
\]

\[
\Rightarrow 1 - s(y \ast x) + s(x) \geq 1
\]

\[
\Rightarrow s(x) \geq s(y \ast x) = \min\{1 - s(y) + s(x), 1\}
\]

\[
\Rightarrow s(x) \geq 1 + s(x) - s(y)
\]

\[
\Rightarrow s(y) \geq 1
\]

\[
\Rightarrow s(y) = 1
\]

\[
\Rightarrow y \in \ker(s).
\]

If \( s(y) \leq s(x) \), then by the similar way we have \( x \in \ker(s) \). Hence by Theorem 2.7, \( \ker(s) \) is an obstinate filter of \( X \).

\[(ii \Rightarrow i)\] Suppose \( \ker(s) \) is not an obstinate filter and \( x, y \in X \). Since \( x \cup (x \cdot y) = 1 \in \ker(s) \), then by Theorem 2.7, \( x \in \ker(s) \) or \( x \cdot y \in \ker(s) \).

If \( x \in \ker(s) \), then \( s(x) = 1 \) and so

\[
\min\{1 - s(x) + s(y), 1\} = \min\{1 - 1 + s(y), 1\} = \min\{s(y), 1\} = s(y).
\]

Since \( x \in \ker(s) \) and \( x \leq y \ast x \), then \( y \ast x \in \ker(s) \). On the other hand, since \( s \) is a Bosbach state, then \( s(x) + s(x \cdot y) = s(y) + s(y \ast x) \). So we have \( 1 + s(x \cdot y) = s(y) + 1 \). Then \( s(x \cdot y) = s(y) \). Hence \( s(x \cdot y) = \min\{1 - s(x) + s(y), 1\} = s(y) \).

If \( x \cdot y \in \ker(s) \), then \( s(x \cdot y) = 1 \). Since \( s(x) + s(x \cdot y) = s(y) + s(y \ast x) \), then

\[
s(x) - s(y) = s(y \ast x) - 1 \leq 0 \Rightarrow s(x) \leq s(y)
\]

\[
\Rightarrow \min\{1 - s(x) + s(y), 1\} = 1 = s(x \cdot y).
\]

Hence \( s \) is a state morphism. \( \square \)

**Corollary 3.18.** Let \( s : X \to [0, 1] \) be a Bosbach state on self distributive \( BE \)-algebra \( X \). Then \( \ker(s) \in \text{Max}(X) \).

**Proof.** By Proposition 3.17 and Theorem 3.14 ([5]) the proof is clear. \( \square \)

**Proposition 3.19.** Let \( s : X \to [0, 1] \) be a state morphism and \( f : Y \to X \) be a \( BE \)-homomorphism. Then \( s \circ f \) is a state morphism.
PROOF. Let $s$ be a state morphism and $f$ be a homomorphism on $X$. First, we can see that $(s \circ f)(1) = s(f(1)) = s(1) = 1$. Now, let $y_1, y_2 \in Y$. Then

$$(s \circ f)(y_1 \ast y_2) = s(f(y_1 \ast y_2))$$

$$= s(f(y_1) \ast f(y_2))$$

$$= \min\{1 - s(f(y_1)) + s(f(y_2)), 1\}$$

$$= \min\{1 - (s \circ f)(y_1) + (s \circ f)(y_2), 1\}.$$ 

Hence $s \circ f$ is a state morphism. $\square$

4. Conclusion and future research

In this paper, we introduced the new upper bound for every pair elements of $BE$-algebra $X$. Bosbach state on $BE$-algebras were studied. We recall some results relative to obstructive filters in $BE$-algebras and discuss relations with state morphism. We show that if $f$ is a state morphism on $X$, then $f$ is a Bosbach state, but the converse is not valid.

In our next research, by taking the ideas of state on $BL$-algebra, $BCK$-algebra, Hilbert algebra, we investigate new state on a $BE$-algebra to be local. Also, we will introduce the notion of pseudo-valuation and develop a theory of Bosbach state on non-trivial $BE$-algebras. Furthermore, we can use the notion of state-operator to generalize this work. Beside some classes of state morphism such as simple, semi-simple, prefect, radical, extremal states and local state morphism will be studied. We try to show that the relations between states on $BE$-algebras with other algebraic structures.

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