Prime and primary hyperideals in Krasner
\((m, n)\)-hyperrings

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**A B S T R A C T**

We introduce and study the quotient structure and an \(n\)-ary hyperintegral domain of a Krasner \((m, n)\)-hyperring, proving some results regarding them. Also, we introduce some important hyperideals such as Jacobson radical, \(n\)-ary prime and primary hyperideals, nilradical, and \(n\)-ary multiplicative subsets of Krasner \((m, n)\)-hyperrings. Finally we investigate the links between these notions.

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**1. Introduction**

The theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. This theory was first initiated by Marty in 1934 ([12]) when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. Later on, many researchers have observed that the theory of hyperstructures also have many applications in both pure and applied sciences. Semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Some review of the theory of hyperstructures can be found in [1,4] and [15], respectively.

\(n\)-ary semigroups and \(n\)-ary groups are algebras with one \(n\)-ary operation which is associative and invertible in a generalized sense. The idea of investigations of \(n\)-ary algebras seems to go back to Kasner's lecture ([7]) at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. But the first paper concerning the theory of \(n\)-ary groups was written by Dörente in 1928 ([6]). Afterward, the \((m, n)\)-rings and their quotient structure were introduced by Crombez and Timm in [2,3]. The concept of an \(n\)-ary hypergroup is defined by Davvaz and Vougiouklis in [5], which is a generalization of the concept of a hypergroup in the sense of Marty and a generalization of an...
n-ary group, too. The notation of \((m, n)\)-hyperrings was defined by Mirvakili and Davvaz ([13]) and they obtained \((m, n)\)-rings from \((m, n)\)-hyperrings by using fundamental relations. For more study on n-ary structures and n-ary hyperstructures refer to [9,10] and [11].

Now in this paper, first we define the quotient structure of Krasner \((m, n)\)-hyperrings. Afterward, we introduce the n-ary integral domain and the n-ary prime hyperideal, and we derive some results linking them. Also, the concept of Jacobson radical, nilradical and \(n\)-ary multiplicative subset of Krasner \((m, n)\)-hyperrings are introduced. Finally, \(n\)-ary primary hyperideals of Krasner \((m, n)\)-hyperrings are defined and some properties in this context are investigated.

2. Preliminaries

In this section we give some definitions and results of n-ary hyperstructures which we need to develop our paper.

A mapping \(f : \prod H \longrightarrow P^n(H)\) is called an n-ary hyperoperation, where \(P^n(H)\) is the set of all the non-empty subsets of \(H\). An algebraic system \((H, f)\), where \(f\) is an n-ary hyperoperation defined on \(H\), is called an n-ary hyperrigroup.

We shall use the following abbreviated notation.

The sequence \(x_1, x_2, \ldots, x_j\) will be denoted by \(x^j\). For \(j < i\), \(x^j\) is the empty symbol. In this convention

\[ f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n) \]

will be written as \(f(x_1^j, y_{i+1}^n)\). In the case when \(y_i = \cdots = y_j = y\) the last expression will be written in the form \(f(x_1^j, y^n, z_{j+1}^n)\).

If \(f\) is an \(n\)-ary hyperoperation and \(t = (n - 1) + 1\), then \(t\)-ary hyperoperation \(f_{(t)}\) is given by

\[ f_{(t)}(x_1^{(n-1)+1}) = f(f(\ldots, f(x_1^n, x_{n+1}^{(n-1)+1}), \ldots), x_1^{(n-1)+1}), x_1^{(n-1)+1}). \]

For non-empty subsets \(A_1, \ldots, A_n\) of \(H\) we define

\[ f(A_1^n) = f(A_1, \ldots, A_n) = \bigcup \{f(x_i^n) \mid x_i \in A_i, i = 1, \ldots, n\}. \]

An \(n\)-ary hyperoperation \(f\) is called associative if

\[ f(x_1^{i-1}, f(x_i^{j-1}), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{j-1}), x_{n+1}^{2n-1}), \]

hold for every \(1 \leq i < j \leq n\) and all \(x_1, x_2, \ldots, x_{2n-1} \in H\). An \(n\)-ary hypergroupoid with the associative \(n\)-ary hyperoperation is called an \(n\)-ary semihypergroup.

An \(n\)-ary hypergroupoid \((H, f)\) in which the equation \(b \in f(a_1^{(n-1)+1}, x_1, a_{n+1}^{(n-1)+1})\) has a solution \(x_1 \in H\) for every \(a_1^{(n-1)+1}, a_{n+1}^{(n-1)+1}, b \in H\) and \(1 \leq i \leq n\), is called an \(n\)-ary quasihypergroup, when \((H, f)\) is an \(n\)-ary semihypergroup, \((H, f)\) is called an \(n\)-ary hypergroup.

An \(n\)-ary hypergroupoid \((H, f)\) is commutative if for all \(\sigma \in S_n\), the group of all permutations of \([1, 2, 3, \ldots, n]\), and for every \(a_1^{(n-1)+1}\) we have \(f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

If \(a_i^n \in H\) we denote \(a_1^{(n-1)+1}\) as the \((a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

**Definition 2.1 ([14]).** Let \((H, f)\) be an \(n\)-ary hypergroup and \(B\) be a non-empty subset of \(H\). \(B\) is called an \(n\)-ary subhypergroup of \((H, f)\), if \(f(x_1^n) \subseteq B\) for \(x_1^n \in B\), and the equation \(b \in f(b_1^{(n-1)+1}, x_1, b_{n+1}^{(n-1)+1})\) has a solution \(x_1 \in B\) for every \(b_1^{(n-1)+1}, b_{n+1}^{(n-1)+1}, b \in B\) and \(1 \leq i \leq n\).

An element \(e \in H\) is called a scalar neutral element if \(x = f(e^{(n-1)+1}, x, e^{(n-1)+1})\), for every \(1 \leq i \leq n\) and for every \(x \in H\). An element \(0\) of an \(n\)-ary semihypergroup \((H, g)\) is called a zero element if for every \(x_1^n \in H\) we have

\[ g(0, x_1^n) = g(x_2, 0, x_1^n) = \cdots = g(x_2^n, 0) = 0. \]

If \(0\) and \(0'\) are two zero elements, then \(0 = g(0', 0^{(n-1)+1}) = 0'\) and so the zero element is unique.
Definition 2.2 ([9]). Let \((H, f)\) be a commutative \(n\)-ary hypergroup. \((H, f)\) is called a canonical \(n\)-ary hypergroup if

1. there exists a unique \(e \in H\), such that for every \(x \in H\), \(f(x, e^{(n-1)}) = x\);
2. for all \(x \in H\) there exists a unique \(x^{-1} \in H\), such that \(e \in f(x, x^{-1}, e^{(n-2)})\);
3. if \(x \in f(x^n)\), then for all \(i\), we have \(x_i \in f(x, x^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1})\).

We say that \(e\) is the scalar identity of \((H, f)\) and \(x^{-1}\) is the inverse of \(x\). Notice that the inverse of \(e\) is \(e\).

Mirvakili and Davvaz define \((m, n)\)-hyperrings and obtained several results in this respect [13]. Moreover, they introduce Krasner \((m, n)\)-hyperrings as subclasses of \((m, n)\)-hyperrings and as a generalization of Krasner hyperrings, as follows.

Definition 2.3 ([14]). A Krasner \((m, n)\)-hyperring is an algebraic hyperstructure \((R, f, g)\) which satisfies the following axioms:

1. \((R, f)\) is a canonical \(m\)-ary hypergroup;
2. \((R, g)\) is a \(n\)-ary semigroup;
3. the \(n\)-ary operation \(g\) is distributive with respect to the \(m\)-ary hyperoperation \(f\), i.e., for every \(d_1^{n-1}, a_1^n, x_1^m \in R\), and \(1 \leq i \leq n\),
   \[ g(d_1^{n-1}, f(x_1^m), a_1^n, x_{i+1}) = f(g(d_1^{n-1}, x_1, a_1^n), \ldots, g(d_1^{n-1}, x_m, a_1^n)); \]
4. \(0\) is a zero element (absorbing element) of the \(n\)-ary operation \(g\), i.e., for every \(x_2^n \in R\) we have
   \[ g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0) = 0. \]

It is clear that every Krasner hyperring is a Krasner \((2, 2)\)-hyperring. Also, every Krasner \((m, 0)\)-hyperring is a canonical \(m\)-ary hypergroup and every Krasner \((0, n)\)-hyperring is an \(n\)-ary semigroup.

Example 2.4 ([14]). Let \((R, +, \cdot)\) be a ring and \(G\) be a normal subgroup of \((R, \cdot)\), i.e., for every \(x \in R\), \(xG = Gx\). Set \(R = \{\bar{x} \mid x \in R\}\) where \(\bar{x} = xG\) and define the \(m\)-ary hyperoperation \(f\) and \(n\)-ary multiplication \(g\) as follows:

\[
\begin{align*}
f(\bar{x_1}, \ldots, \bar{x_m}) &= \{\bar{\tilde{z}} \mid \tilde{z} \subseteq \bar{x_1} + \cdots + \bar{x_m}\} \\
g(\bar{x_1}, \ldots, \bar{x_n}) &= \bar{x_1} \cdot \bar{x_2} \cdots \bar{x_n}.
\end{align*}
\]

It can be verified obviously that \((\bar{R}, f, g)\) is a Krasner \((m, n)\)-hyperring.

A non-empty subset \(S\) of \(R\) is called a subhyperring of \(R\) if \((S, f, g)\) is a Krasner \((m, n)\)-hyperring. Let \(I\) be a non-empty subset of \(R\), we say that \(I\) is a hyperideal of \((R, f, g)\) if \((I, f)\) is an \(m\)-ary subhypergroup of \((R, f)\) and \(g(x_1^{i-1}, I, x_i^n, x_{i+1}) \subseteq I\), for every \(x_i^n \in R\) and \(1 \leq i \leq n\).

Lemma 2.5 ([14]). If \(I\) is a hyperideal of Krasner \((m, n)\)-hyperring \((R, f, g)\) and \(a_2^n \in I\), then \(f(1, a_2^n) = I\).

Definition 2.6 ([14]). Let \((R_1, f_1, g_1)\) and \((R_2, f_2, g_2)\) be two Krasner \((m, n)\)-hyperrings. A mapping \(\varphi : R_1 \to R_2\) is called a homomorphism if for all \(x_1^n \in R_1\) and \(y_1^n \in R_1\) we have

\[
\begin{align*}
\varphi(f_1(x_1, \ldots, x_m)) &= f_2(\varphi(x_1), \ldots, \varphi(x_m)) \\
\varphi(g_1(y_1, \ldots, y_n)) &= g_2(\varphi(y_1), \ldots, \varphi(y_n)).
\end{align*}
\]

3. The quotient structure of Krasner \((m, n)\)-hyperrings

In this section, we introduce the concept of the quotient structure of a Krasner \((m, n)\)-hyperring. This definition helps us to investigate the link between these notions and those which will be introduced in the next sections.
Moreover, for all $y_1^{n_1} \in R$

$$
\pi (g(y_1^{n_1})) = f(g(y_1^{n_1}), I, 0^{(m-2)}) = f(g(y_1^{n_1}), I, g(0^{(n)}(m-2))) = g(f(y_1, I, 0^{(m-2)}), \ldots, f(y_n, I, 0^{(m-2)})) = g(\pi(y_1), \ldots, \pi(y_n)).
$$

Therefore the proof is complete. \qed

Now, we construct the quotient structure of Krasner $(m, n)$-hyperring by using the hyperideals. We can obtain isomorphism theorems for Krasner $(m, n)$-hyperrings and some results in this context.

4. $n$-ary prime and primary hyperideals

In this section, we define the $n$-ary hyperintegral domain, $n$-ary prime hyperideal, $n$-ary multiplicative subset, Jacobson radical, nilradical and an $n$-ary primary hyperideal of a Krasner $(m, n)$-hyperring and we investigate some results in relation to them.

A Krasner $(m, n)$-hyperring $(R, f, g)$ is commutative if $(R, g)$ is a commutative $n$-ary semigroup. Also, we say that $(R, f, g)$ is with scalar identity if there exists an element $1_R$ such that $x = g(x, 1_R^{(n-1)})$ for all $x \in R$.

**Definition 4.1.** A Krasner $(m, n)$-hyperring $(R, f, g)$ is called an $n$-ary hyperintegral domain, if $R$ is a commutative Krasner $(m, n)$-hyperring and $g(a_1^n) = 0$ implies that $a_1 = 0$ or $a_2 = 0$ or \ldots or $a_n = 0$, for all $a_i^n \in R$.  

**Theorem 4.2.** The projection map $\pi$ is a homomorphism of Krasner $(m, n)$-hyperrings.

**Proof.** Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring and $I$ be a hyperideal of $R$. For all $x_1^{m_1} \in R$ we have

$$
\pi(f(x_1^{m_1})) = f(f(x_1^{m_1}), I, 0^{(m-2)}) = f(f(x_1^{m_1}), I, f(0^{(m)}(m-2)) = f(f(x_1, I, 0^{(m-2)}), \ldots, f(x_m, I, 0^{(m-2)})) = f(\pi(x_1), \ldots, \pi(x_m)).
$$

Moreover, for all $y_1^{n_1} \in R$

$$
\pi(g(y_1^{n_1})) = f(g(y_1^{n_1}), I, 0^{(m-2)}) = f(g(y_1^{n_1}), I, g(0^{(n)}(m-2))) = g(f(y_1, I, 0^{(m-2)}), \ldots, f(y_n, I, 0^{(m-2)})) = g(\pi(y_1), \ldots, \pi(y_n)).
$$

Therefore the proof is complete. \qed

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**Definition 4.1.** A Krasner $(m, n)$-hyperring $(R, f, g)$ is called an $n$-ary hyperintegral domain, if $R$ is a commutative Krasner $(m, n)$-hyperring and $g(a_1^n) = 0$ implies that $a_1 = 0$ or $a_2 = 0$ or \ldots or $a_n = 0$, for all $a_i^n \in R$.  

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**Proof.** Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring and $I$ be a hyperideal of $R$. For all $x_1^{m_1} \in R$ we have

$$
\pi(f(x_1^{m_1})) = f(f(x_1^{m_1}), I, 0^{(m-2)}) = f(f(x_1^{m_1}), I, f(0^{(m)}(m-2)) = f(f(x_1, I, 0^{(m-2)}), \ldots, f(x_m, I, 0^{(m-2)})) = f(\pi(x_1), \ldots, \pi(x_m)).
$$

Moreover, for all $y_1^{n_1} \in R$

$$
\pi(g(y_1^{n_1})) = f(g(y_1^{n_1}), I, 0^{(m-2)}) = f(g(y_1^{n_1}), I, g(0^{(n)}(m-2))) = g(f(y_1, I, 0^{(m-2)}), \ldots, f(y_n, I, 0^{(m-2)})) = g(\pi(y_1), \ldots, \pi(y_n)).
$$

Therefore the proof is complete. \qed

Now, we construct the quotient structure of Krasner $(m, n)$-hyperring by using the hyperideals. We can obtain isomorphism theorems for Krasner $(m, n)$-hyperrings and some results in this context.
Example 4.2. Let \((R, +, \cdot)\) be a Krasner hyperring in which the operation \(\cdot\) is the ordinary multiplication and let \(R\) be a hyperintegral domain (for more details refer to \cite{16}). Then \(R\) endowed with the following \(m\)-ary hyperoperation \(f\) and \(n\)-ary operation \(g\) is a Krasner \((m, n)\)-hyperring.

\[
f(x_1^m) = \sum_{i=1}^{m} x_i \quad \text{and} \quad g(x_i^n) = x_1 \cdots x_n
\]

It is easy to see that \(R\) is an \(n\)-ary hyperintegral domain.

Example 4.3. Consider a Krasner \((m, n)\)-hyperring \((R, f, g)\) which is an \(n\)-ary hyperintegral domain. Let \(b_2, \ldots, b_n, a, c \in R\) and \(b_2, \ldots, b_n \neq 0\) such that \(g(a, b_2^n) = g(c, b_2^n)\). We can show that \(a = c\), because

\[
g(a, b_2^n) = g(c, b_2^n) = f(g(c, b_2^n), 0^{(m-1)})
\]

\[
\Rightarrow 0 \in f(g(c, b_2^n), -g(a, b_2^n), 0^{(m-2)})
\]

\[
\Rightarrow 0 \in f(g(c, b_2^n), -g(a, b_2^n), g(0, b_2^n), \ldots, g(0, b_2^n)) = f(a, b_2^n)
\]

now, since \(b_2, \ldots, b_n \neq 0\) and \(R\) is an \(n\)-ary hyperintegral domain we have \(0 \in f(c, -a, 0^{(m-2)})\). Since \((R, f)\) is a canonical \(m\)-ary hypergroup, then \(a \in f(c, 0^{(m-1)}) = c\).

Definition 4.4. A hyperideal \(P\) of a Krasner \((m, n)\)-hyperring \((R, f, g)\), such that \(P \neq R\), is called an \(n\)-ary prime hyperideal if for hyperideals \(U_1, \ldots, U_n\) of \(R\), \(g(U_1^n) \subseteq P\) implies that \(U_1 \subseteq P\) or \(U_2 \subseteq P\) or ... or \(U_n \subseteq P\).

Lemma 4.5. Let \(P \neq R\) be a hyperideal of a Krasner \((m, n)\)-hyperring \((R, f, g)\). Then \(P\) is an \(n\)-ary prime hyperideal if for all \(a_1^n \in R\)

\[
g(a_1^n) \in P \iff a_1 \in P \text{ or } \ldots \text{ or } a_n \in P.
\]

Theorem 4.6. If \(P \neq R\) is a hyperideal in a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\), then \(P\) is \(n\)-ary prime if and only if \(R/P\) is an \(n\)-ary hyperintegral domain.

Proof. Let \(P\) be an \(n\)-ary prime hyperideal and \(f(a_1^{1(i−1)}, P, a_1^m_{1(i+1)}), \ldots, f(a_n^{1(i−1)}, P, a_n^m_{n(i+1)}) \in R/P\) for all \(a_1^m, \ldots, a_n^m \in R\) such that

\[
g\left(f(a_1^{1(i−1)}, P, a_1^{m_{1(i+1)}}, \ldots, f(a_n^{1(i−1)}, P, a_n^{m_{n(i+1)}})\right) = P = 0_{R/P},
\]

then by the definition of the quotient Krasner \((m, n)\)-hyperring we have

\[
f(g(a_1^n), \ldots, g(a_1^{n(i−1)}), P, g(a_1^{m(i+1)}), \ldots, g(a_1^{m}), g(a_1^{n}), \ldots, g(a_1^{m}) = P
\]

\[
\Rightarrow f(g(a_1^n), \ldots, g(a_1^{n(i−1)}), P, 0, g(a_1^{m(i+1)}), \ldots, g(a_1^{m}), \ldots, g(a_1^{m}) \subseteq P
\]

\[
\Rightarrow g(f(a_1^{1(i−1)}, P, 0, a_1^{m(i+1)}), \ldots, f(a_1^{1(i−1)}, P, 0, a_1^{m(i+1)}) \subseteq P
\]

\[
\Rightarrow f(a_1^{1(i−1)}, P, a_1^{m(i+1)}) \subseteq P \quad \text{or} \quad f(a_1^{1(i−1)}, P, a_1^{m(i+1)}) \subseteq P
\]

\[
\Rightarrow f(a_1^{1(i−1)}, P, a_1^{m(i+1)}) = P \quad \text{or} \quad f(a_1^{1(i−1)}, P, a_1^{m(i+1)}) = P.
\]

Therefore \(R/P\) is an \(n\)-ary hyperintegral domain.
Conversely, suppose that $R/P$ is an $n$-ary hyperintegral domain. Let $g(a^n_i) \in P$ for all $a_1, \ldots, a_n \in R$. Then by Lemma 2.5, we have $f(g(a^n_i), P, 0^{(m-2)}) = P$. Hence

\[ \Rightarrow f(g(a^n_i), g(0^{(n)}), \ldots, g(0^{(n)}), P, g(0^{(n)}), \ldots, g(0^{(n)})) = P \]

\[ \Rightarrow g(f(a_1, P, 0^{(m-2)}), \ldots, f(a_n, P, 0^{(m-2)})) = P = 0_{R/P} \]

Since $R/P$ is an $n$-ary hyperintegral domain we have

\[ \Rightarrow f(a_1, P, 0^{(m-2)}) = P \text{ or } \ldots \text{ or } f(a_n, P, 0^{(m-2)}) = P \]

\[ \Rightarrow f(a_1, 0^{(m-1)}) \subseteq P \text{ or } \ldots \text{ or } f(a_n, 0^{(m-1)}) \subseteq P \]

\[ \Rightarrow a_1 \in P \text{ or } \ldots \text{ or } a_n \in P. \]

Consequently, $P$ is an $n$-ary prime hyperideal.

**Example 4.7.** Consider the Krasner $(m, n)$-hyperring $(\tilde{R}, f, g)$ constructed in Example 2.4. The hyperideal of $\tilde{R}$ is of the form $I$ such that $G \subset I < \tilde{R}$. Let $P$ be a prime ideal of $R$ such that $G \subset P$, then the $n$-ary prime hyperideals of $R$ are of the form $\bar{P}$, because for $\bar{x}_1^n \in \tilde{R}$, $g(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{P}$ implies that $x_1 \cdots x_n \in P$. Since $P$ is prime, we have $x_1 \in P$ or $\ldots$ or $x_n \in P$, and so

\[ \bar{x}_1 \in \bar{P} \text{ or } \ldots \text{ or } \bar{x}_n \in \bar{P}. \]

Hence $\bar{P}$ is an $n$-ary prime hyperideal of $(\tilde{R}, f, g)$.

**Example 4.8.** Suppose that $R = \{0, 1, 2, 3\}$ and define a 2-ary hyperoperation “+” on $R$ as follows:

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It follows that $(R, +)$ is a canonical 2-ary hypergroup. Let $g$ be a 4-ary operation on $R$ such that

\[ g(x_1, x_2, x_3, x_4) = \begin{cases} 2, & \text{if } x_4 \in \{2, 3\} \\ 0, & \text{otherwise.} \end{cases} \]

Then, $(R, +, g)$ is a Krasner $(2, 4)$-hyperring. Consider $I = \{0\}$. It is clear that $I$ is a hyperideal of the above Krasner $(2, 4)$-hyperring. By the definition of 4-ary operation $g$, it is seen that $g(1, 1, 2, 3) = 0 \in I$, but none of the elements 1, 2 and 3 are not in $I$. Therefore, $I = \{0\}$ is not a 4-ary prime hyperideal of $R$.

**Remark 4.9.** Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring. By Definition 2.3, we have

\[ g(0, x^n_1) = g(x_2, 0, x^n_3) = \cdots = g(x^n_2, 0) = 0 \]

for $x^n_1 \in R$. Therefore, the hyperideal $\{0\}$ is an $n$-ary prime hyperideal in a Krasner $(m, n)$-hyperring, if for all nonzero elements $x^n_1 \in R$, we have $g(x^n_1) \neq 0$, i.e., $R$ is an $n$-ary hyperintegral domain.

In the following example, we will investigate the above remark.

**Example 4.10.** Consider the set $S = \{0, 1, 2\}$ with 3-ary hyperoperation $f$ and 3-ary operation $g$ as follows:

\[ f(0, 0, 0) = 0, \quad f(1, 1, 1) = 1, \quad f(0, 0, 2) = 2, \quad f(1, 2, 2) = S \]

\[ f(0, 0, 1) = 1, \quad f(1, 1, 2) = S, \quad f(0, 2, 2) = 2, \quad f(2, 2, 2) = 2 \]

\[ f(0, 1, 1) = 1, \quad f(0, 1, 2) = S \]
and 
\[
g(0, x_1, x_2) = 0, \quad \text{for } x_1, x_2 \in S \\
g(1, 1, 1) = 1, \\
g(1, 1, 2) = g(1, 2, 2) = g(2, 2, 2) = 2
\]
such that \(f\) and \(g\) are commutative. Then \((S, f, g)\) is a Krasner \((3, 3)\)-hyperring, and \(I = \{0\}\) and \(K = \{0, 2\}\) are two hyperideals of \(S\).

(1) \(I\) is a 3-ary prime hyperideal, since \(g(0, x_1, x_2) = 0 \in I\) for \(x_i^2 \in S, 0 \in I\) and \(g(x_i^2) \neq 0\) for \(x_i \neq 0\) and \(1 \leq i \leq 3\).

(2) \(K\) is a 3-ary prime hyperideal, since for nonzero elements \(x_1\) and \(x_2\) of \(S\), we have \(g(0, x_1, x_2) = 0 \in K\) and \(g(x_1, x_2, 2) = 2 \in K\).

**Definition 4.11.** Let \((R, f, g)\) be a Krasner \((m, n)\)-hyperring. Then a hyperideal \(M\) of \(R\) is said to be maximal if for every hyperideal \(N\) of \(R, M \subseteq N \subseteq R\) implies that \(N = M\) or \(N = R\).

**Definition 4.12.** The Jacobson radical of a Krasner \((m, n)\)-hyperring \((R, f, g)\) is the intersection of all maximal hyperideals of \(R\) and it is denoted by \(J_{(m,n)}(R)\). If \(R\) does not have any maximal hyperideal, we let \(J_{(m,n)}(R) = R\).

We say that an element \(a \in R\) is invertible if there exists \(b \in R\) such that \(1_R = g(a, b, 1_R^{(n-2)})\). Also, the subset \(U\) of \(R\) is invertible if and only if every element of \(U\) is invertible.

**Definition 4.13.** Let \((R, f, g)\) be a commutative Krasner \((m, n)\)-hyperring with the scalar identity \(1_R\). For every element \(a \in R\), the hyperideal generated by "\(a\)" is denoted by \(\langle a \rangle\) and defined as follows:

\[
\langle a \rangle = g(R, a, 1_R^{(n-2)}) = \{g(r, a, 1_R^{(n-2)}) \mid r \in R\}.
\]

**Theorem 4.14.** Let \(I\) be a hyperideal of a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) with the scalar identity \(1_R\). Then \(I \subseteq J_{(m,n)}(R)\) if and only if every element of \(f(1_R, I, 0^{(m-2)})\) is invertible.

**Proof.** Let \(I \subseteq J_{(m,n)}(R)\) and suppose that there exists \(t \in f(1_R, I, 0^{(m-2)})\) such that \(t\) is not invertible. Clearly \(t \in f(1_R, a, 0^{(m-2)})\) for some \(a \in I\). Since \(t\) is not invertible, there is a maximal hyperideal \(M\) of \(R\) such that \(t \in M\). But \(t \in f(1_R, a, 0^{(m-2)})\) implies that \(1_R \in f(t, -a, 0^{(m-2)}) \subseteq M\). Hence \(1_R \in M\), which is a contradiction with maximality of \(M\). Therefore, every element of \(f(1_R, I, 0^{(m-2)})\) is invertible.

Conversely, let every element of \(f(1_R, I, 0^{(m-2)})\) be invertible, but \(I \not\subseteq J_{(m,n)}(R)\). Thus \(I \not\subseteq M\) for some maximal hyperideal of \(R\), then there exists \(a \in I \setminus M\). Therefore \(f(M, (a), 0^{(m-2)}) = R\). So \(1_R \in f(m, g(r, a, 1_R^{(n-2)}), 0^{(m-2)})\) for some \(r \in R\) and \(m \in M\), hence, since \((R, f)\) is a canonical \(m\)-ary hypergroup, we have

\[
m \in f(1_R, -g(r, a, 1_R^{(n-2)}), 0^{(m-2)}) \subseteq f(1_R, I, 0^{(m-2)})
\]

Thus \(m\) is invertible, which is a contradiction. \(\square\)

**Corollary 4.15.** Let \((R, f, g)\) be a commutative Krasner \((m, n)\)-hyperring with the scalar identity \(1_R\). Then

\[
a \in J_{(m,n)}(R) \iff f(1_R, -g(r, a, 1_R^{(n-2)}), 0^{(m-2)}) \text{ is invertible for all } a \in R.
\]

**Corollary 4.16.** An element \(a\) in commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) with scalar identity, is invertible if and only if \(f(a, J_{(m,n)}(R), 0^{(m-2)})\) is invertible in \(R/J_{(m,n)}(R)\).

**Definition 4.17.** A non-empty subset \(S\) of a Krasner \((m, n)\)-hyperring \((R, f, g)\) is called \(n\)-ary multiplicative, if \(g(s_i^2) \in S\) for \(s_1, \ldots, s_n \in S\).
Example 4.18. (i) Consider Krasner \((2, 4)\)-hyperring \((R, +, g)\) in Example 4.8. The set \(S = \{1, 2, 3\}\) is not a 4-ary multiplicative subset of \(R\), since \(g(1, 2, 2, 3) = 0 \not\in S\). But \(S' = \{2, 3\}\) is a 4-ary multiplicative subset of \(R\).

(ii) It is easy to see that \(L = \{1, 2\}\) is a 3-ary multiplicative subset of Krasner \((3, 3)\)-hyperring \((S, f, g)\) in Example 4.10.

Theorem 4.19. A hyperideal \(P \neq R\) in a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) is \(n\)-ary prime if and only if \(R/P\) is an \(n\)-ary multiplicative subset of \(R\).

Proof. Let \(P\) be an \(n\)-ary prime, then by the definition of the quotient structure in Section 3, it is easy to see that \(R/P\) is \(n\)-ary multiplicative.

Conversely, suppose that \(g(a_{i}^{1}) \in P\) and \(a_{2}, \ldots, a_{n} \not\in P\). We must show that \(a_{1} \in P\). If \(a_{1} \not\in P\), then by Lemma 2.5, \(P \neq f(a_{1}, P, 0^{(m-2)}), \ldots, P \neq f(a_{n}, P, 0^{(m-2)})\). Since \(f(a_{1}, P, 0^{(m-2)}), \ldots, f(a_{n}, P, 0^{(m-2)}) \in R/P\) and \(R/P\) is \(n\)-ary multiplicative, we have

\[
P \neq f\left( f(a_{1}, P, 0^{(m-2)}), \ldots, f(a_{n}, P, 0^{(m-2)}) \right) \in R/P
\]

\[
\Rightarrow P \neq f\left( g(a_{1}^{1}), P, 0^{(m-2)} \right) \in R/P
\]

Thus \(g(a_{1}^{1}) \not\in P\), which is a contradiction. Then \(a_{1} \in P\) and so \(P\) is \(n\)-ary prime. \(\square\)

Example 4.20. Consider a 3-ary prime hyperideal \(I = \{0\}\) in Example 4.10. Then \(R/I \cong R\) is a 3-ary multiplicative, because of \(g(x_{1}, x_{2}, x_{3}) \in R\) for all \(x_{1}^{3} \in R\).

Theorem 4.21. Let \(S\) be an \(n\)-ary multiplicative subset of a Krasner \((m, n)\)-hyperring \((R, f, g)\) which is disjoint from a hyperideal \(I\). Then there exists a hyperideal \(P\) which is maximal in the set of all hyperideals of \(R\) disjoint from \(S\) containing \(I\). Furthermore any such hyperideal is \(n\)-ary prime.

Proof. The proof is similar to ordinary algebra. \(\square\)

Definition 4.22. Let \(I\) be a hyperideal in a commutative Krasner \((m, n)\)-hyperring \((R,f,g)\) with scalar identity. The radical (or nilradical) of \(I\), denoted by \(\sqrt{I}^{(m,n)}\) is the hyperideal \( \bigcap P \), where the intersection is taken over all \(n\)-ary prime hyperideals \(P\) which contain \(I\). If the set of all \(n\)-ary hyperideals containing \(I\) is empty, then \(\sqrt{I}^{(m,n)}\) is defined to be \(R\).

Theorem 4.23. If \(I\) is a hyperideal in a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) with the scalar identity \(1_{R}\), then

\[
\sqrt{I}^{(m,n)} = \left\{ r \in R \mid \begin{cases} g(r^{(i)}), 1_{R}^{(n-1)} \in I, & t \leq n \\
g_{i_{0}}(r^{(i)}) \in I, & t > n, t = (n-1) + 1 \end{cases} \right\}
\]

for some \(t \in \mathbb{N}\).

Proof. Let \(P\) be an arbitrary \(n\)-ary prime hyperideal such that \(I \subseteq P\) and let \(x \in \sqrt{I}^{(m,n)}\). Then there exists \(t \in \mathbb{N}\) such that \(g(x^{(i)}, 1_{R}^{(n-1)}) \in I\) for \(t \leq n\), or \(g_{i_{0}}(x^{(i)}) \in I\) for \(t = (n-1) + 1\). If \(g(x^{(i)}), 1_{R}^{(n-1)} \in I \subseteq P\), then

\[
g\left( g(x^{(i)}, 1_{R}^{(n-1)}), 1_{R}^{(n-1)} \right) \in P
\]

\[
\Rightarrow g\left( x, g(x^{(t-1)}, 1_{R}^{(n-(t+1))}, 1_{R}^{(n-2)}) \right) \in P \quad \text{(associativity)}
\]

\[
\Rightarrow x \in P \quad \text{or} \quad g\left( x^{(t-1)}, 1_{R}^{(n-(t+1))} \right) \in P \quad \text{(\(P\) \(n\)-ary prime and \(1_{R} \not\in P\))}
\]

\[
\Rightarrow x \in P \quad \text{or} \quad g\left( x^{(t-1)}, g(1_{R}^{(n)}), 1_{R}^{(n-t)} \right) \in P
\]
\[ \Rightarrow x \in P \quad \text{or} \quad g\left(x, g(x^{(t-2)}, 1_\mathbb{R}^{(n-t+2)}), 1_\mathbb{R}^{(n-2)}\right) \in P \]
\[ \Rightarrow x \in P \quad \text{or} \quad x \in P \quad \text{or} \quad g(x^{(t-2)}, 1_\mathbb{R}^{(n-t+2)}) \in P \]
\[ \vdots \]
\[ \Rightarrow x \in P \quad \text{or} \quad \ldots \quad \text{or} \quad x \in P. \]

Therefore \( x \in P \) for all \( I \subseteq P \). Then \( x \in \bigcap_{I \subseteq P} P \) and so \( \sqrt{I}_{(m,n)} \subseteq \bigcap_{I \subseteq P} P \). Also, if \( g(t_0) (x^{(t)}) \in I \subseteq P \), then\[ g(g(\cdots g(g(x^{(n)}), x^{(n-1)}), \ldots), x^{(n-1)}) \in P. \]

Since \( P \) is \( n \)-ary prime, it is easy to see that \( x \in P \) for all \( I \subseteq P \). Similarly, we have \( \sqrt{I}_{(m,n)} \subseteq \bigcap_{I \subseteq P} P \).

Now, let \( x \in \bigcap_{I \subseteq P} P \) and let \( x \not\in \sqrt{I}_{(m,n)} \). Then for every \( t \in \mathbb{N} \) we have \( g(x^{(t)}, 1_\mathbb{R}^{(n-t)}) \notin I \). Consider the set
\[ S = \{ 1_\mathbb{R}, x, g(x^{(2)}, 1_\mathbb{R}^{(n-2)}), g(x^{(3)}, 1_\mathbb{R}^{(n-3)}), \ldots \} \]

It is clear that \( S \cap I = \emptyset \) and \( S \) is an \( n \)-ary multiplicative, because
\[ g\left(x, g(x^{(2)}, 1_\mathbb{R}^{(n-2)}), g(x^{(3)}, 1_\mathbb{R}^{(n-3)}), 1_\mathbb{R}^{(n-3)}\right) = g\left(g(x^{(6)}, 1_\mathbb{R}^{(n-6)}), 1_\mathbb{R}^{(n-1)}\right) = g(x^{(6)}, 1_\mathbb{R}^{(n-6)}) \in S. \]

By Theorem 4.21, there exists an \( n \)-ary prime hyperideal \( P \) such that \( I \subseteq P \) and \( S \cap P = \emptyset \). Since \( x \in S \), then \( x \not\in P \), which is a contradiction with \( x \in \bigcap_{I \subseteq P} P \). Hence \( x \in \sqrt{I}_{(m,n)} \). Therefore \( \sqrt{I}_{(m,n)} = \bigcap_{I \subseteq P} P. \)

**Definition 4.24.** A hyperideal \( Q \neq R \) in a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) with the scalar identity \( 1_\mathbb{R} \) is said to be \( n \)-ary primary if \( g(a_i^n) \in Q \) and \( a_i \notin Q \) implies that \( g(a_i^{n-1}, 1_\mathbb{R}, a_{i+1}^n) \in \sqrt{Q}_{(m,n)} \). Or, for all \( 1 \leq i \leq n \),
\[ g(a_i^n) \in Q \implies a_i \in Q \quad \text{or} \quad g(a_i^{n-1}, 1_\mathbb{R}, a_{i+1}^n) \in \sqrt{Q}_{(m,n)}. \]

**Remark 4.25.** Every \( n \)-ary prime hyperideal is \( n \)-ary primary.

Indeed, let \( P \) be an \( n \)-ary prime hyperideal and \( g(a_i^n) \in P \). Since \( P \) is \( n \)-ary prime we have \( a_1 \in P \) or \( \ldots \) or \( a_i \in P \) or \( \ldots \) or \( a_n \in P \). Then \( a_i \in P \) or \( g(a_i^{n-1}, 1_\mathbb{R}, a_{i+1}^n) \in P \) (since \( P \) is a hyperideal). Hence there exists \( 1 \in \mathbb{N} \) such that
\[ a_i \in P \quad \text{or} \quad g\left(g(a_i^{n-1}, 1_\mathbb{R}, a_{i+1}^n), 1_\mathbb{R}^{(n-1)}\right) \in P \]
\[ \Rightarrow a_i \in P \quad \text{or} \quad g(a_i^{n-1}, 1_\mathbb{R}, a_{i+1}^n) \in \sqrt{P}_{(m,n)} \]
\[ \Rightarrow P \text{ is } n \text{-ary primary.} \]

**Lemma 4.26.** Let \((R, f, g)\) be a commutative Krasner \((m, n)\)-hyperring with the scalar identity \( 1_\mathbb{R} \). If \( t = l(n-1) + 1 \), then for \( a_1, \ldots, a_n \in R \) we have
\[ g(t_0) (g(a_i^n (t))) = g\left(g(t_0) (g(a_1^n, 1_\mathbb{R}, a_{i+1}^n) (t)), g(t_0) (a_i^n), 1_\mathbb{R}^{(n-2)}\right). \]

By the following example, we investigate Lemma 4.26, for two small numbers \( n \) and \( t \) such that \( t = l(n-1) + 1 \).
Example 4.27. Consider a commutative Krasner \((m, 3)\)-hyperring \((R, f, g)\) with the scalar identity \(1_R\). Let \(t = 5\), then by \(t = l(n - 1) + 1\) we have \(l = 2\). Hence by the definition of \(g_{(i)}\) and associativity of \(g\) for \(a_1, a_2, a_3 \in R\), we have

\[
g_{(2)}(g(a_1^{(5)})) = g\left(g\left(g\left(g(a_1^{(3)}), g(a_1^{(2)})\right), g(a_1^{(2)})\right), 1_R\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(3)}), g(a_1^{(2)}), 1_R\right), 1_R\right), 1_R\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(3)}), g(a_1^{(2)}), 1_R, a_2^{(2)}\right), a_2^{(2)}\right), a_2^{(2)}\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(3)}), g(a_1^{(2)}), a_2^{(2)}\right), 1_R, a_2^{(2)}\right), a_2^{(2)}\right)
\]

On the other hand

\[
g\left(g(a_1^{(3)})\right) = g\left(g\left(g\left(g(a_1^{(3)}, 1_R\right), g(a_1^{(2)}), 1_R\right), 1_R\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(2)}, g(a_1^{(2)}), 1_R\right), 1_R\right), 1_R\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(2)}, g(a_1^{(2)}), a_2^{(2)}\right), a_2^{(2)}\right), a_2^{(2)}\right)
\]

\[
= g\left(g\left(g\left(g(a_1^{(2)}, a_2^{(2)}), 1_R, a_2^{(2)}\right), a_2^{(2)}\right), a_2^{(2)}\right)
\]

Hence

\[
g_{(2)}(g(a_1^{(5)})) = g\left(g\left(g\left(g(a_1^{(3)}, 1_R, a_3^{(3)}), g(a_2^{(3)}), 1_R\right), g(a_1, 1_R, a_3^{(2)}), a_2^{(2)}\right), a_2^{(2)}\right)
\]

\[
= g\left(g\left(g\left(g(a_1, 1_R, a_3^{(3)}), g(a_1, 1_R, a_3^{(2)}), g\left(g(a_2^{(3)}, a_2^{(2)}), 1_R\right)\right), 1_R\right)
\]

\[
= g\left(g\left(g\left(g(a_1, 1_R, a_3^{(3)}), g(a_1, 1_R, a_3^{(2)}), 1_R\right), 1_R\right), 1_R\right)
\]

Theorem 4.28. If \(Q\) is an \(n\)-ary primary hyperideal in a commutative Krasner \((m, n)\)-hyperring \((R, f, g)\) with the scalar identity \(1_R\), then \(\sqrt{Q}^{(m,n)}\) is \(n\)-ary prime.

Proof. Let \(g(a_1^n) \in \sqrt{Q}^{(m,n)}\) and \(a_1^{i-1}, a_1^n \notin \sqrt{Q}^{(m,n)}\) for \(a_1 \in R\). We show that \(a_1 \in \sqrt{Q}^{(m,n)}\). Since \(g(a_1^n) \in \sqrt{Q}^{(m,n)}\), there exists \(t \in \mathbb{N}\) such that if \(t \leq n\), then \(g(g(a_1^n), 1_R^{(n-t)}) \in Q\). Hence by associativity we have

\[
g\left(a_1^{(t)}, g(a_1^{(n-t)}, 1_R^{(n-2t)})\right) \in Q
\]

\[
\Rightarrow g\left(a_1^{(t)}, g(a_1^{(n-t)}, 1_R^{(n-2t-1)})\right) \in Q
\]

Since \(Q\) is \(n\)-ary primary, then

\[
g(a_1^{(t)}, 1_R^{(n-t)}) \in \sqrt{Q}^{(m,n)}\quad \text{or} \quad g(g(a_1^{(n-t)}, 1_R^{(n-t)}), 1_R^{(n-t)}) \in Q
Let \( g(a_1^{i-1}, a_{i+1}^n)Q(t), 1_R^{(n-1)} + 1R, an_{i+1}(t), 1_R^{(n-2)}) \in Q \), then we have
\[
g\left(a_1^{(t)}, g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-2i)}\right) \right) \in Q
\]
\[
\Rightarrow g\left(g\left(a_1^{(t)}, 1_R^{(n-1)}, g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \right) \in Q
\]
\[
\Rightarrow g\left(a_1^{(t)}, 1_R^{(n-1)} \right) \in Q \text{ or } g\left(g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in \sqrt{Q}
\]
\[
\Rightarrow a_1 \in \sqrt{Q} \text{ or } g\left(g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in \sqrt{Q}.
\]

Since \( a_1 \in \sqrt{Q} \) is a contradiction with the assumption, then
\[
g\left(g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in \sqrt{Q}
\]
\[
\Rightarrow \exists s \in \mathbb{N} : g\left(g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in Q
\]
\[
\Rightarrow g\left(g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in Q
\]
\[
\Rightarrow g\left(g\left(a_3^{(t+1)}, 1_R^{(n-1)}, g\left(a_2^{i-1}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \right) \in Q
\]
\[
\Rightarrow a_2 \in \sqrt{Q} \text{ or } g\left(g\left(a_3^{(t+1)}, 1_R^{(2)}, a_{i+1}^n(t), 1_R^{(n-1)}\right) \right) \in \sqrt{Q}
\]
\[
\Rightarrow \cdots \text{ or } a_n \in \sqrt{Q}
\]

which is a contradiction with \( a_1^{i-1}, a_{i+1}^n \notin \sqrt{Q} \). Therefore, \( g(a_1^{(t)}, 1_R^{(n-1)}) \in \sqrt{Q} \), and so there exists \( c \in \mathbb{N} \) such that
\[
g\left(g\left(a_1^{(t)}, 1_R^{(n-c)} \right) \right) \in Q
\]
\[
\Rightarrow g\left(a_1^{(t+1)}, 1_R^{(n-c)} \right) \in Q \Rightarrow a_i \in \sqrt{Q}.
\]

Now, if \( t = \frac{l(n - 1) + 1}{l(n - 1) + 1} = l(n - 1) + 1 \), then \( g(a_1^{(t)} \in \sqrt{Q} \) implies that \( g_{00} g(a_1^{(t)}) \in Q \). Hence by Lemma 4.26, we have
\[
g\left(g_{n0} g(a_1^{(i-1)}, 1_R, a_{i+1}^n(t), g_{n0}(a_1^{(t)}), 1_R^{(n-2)}) \right) \in Q
\]
since \( Q \) is \( n \)-ary primary we have
\[
g\left(g_{n0} g(a_1^{(t)}) \right) \in \sqrt{Q} \text{ or } g_{n0} g(a_1^{(i-1)}, 1_R, a_{i+1}^n(t)) \in Q.
\]

If \( g_{n0} g(a_1^{(i-1)}, 1_R, a_{i+1}^n(t)) \in Q \), then similar to the before part we arrive at a contradiction. Hence, \( g_{n0} g(a_1^{(t)}) \in \sqrt{Q} \) and so there exists \( b \in \mathbb{N} \) such that \( b = l(n - 1) + 1 \) and
\[
g_{n0} g\left(g_{n0} g(a_1^{(t)})^{(b)} \right) \in Q \Rightarrow a_i \in \sqrt{Q}.
\]

Therefore \( \sqrt{Q} \) is an \( n \)-ary prime hyperideal of Krasner \( (m, n) \)-hyperrrings \( R \). □

5. Conclusion

The study of hyperrings was initiated by Krasner [8] and continued by others, especially in connection with an important hyperstructure, called Krasner \( (m, n) \)-hyperrrings as a subclass of \( (m, n) \)-hyperrrings [14]. In this paper, we consider and analyze some subhyperstructures of this notion. We state the concepts of prime and primary for the hyperideals of Krasner \( (m, n) \)-hyperrrings and
investigate the connections between them, and also their connection with other concepts such as the quotient structure, Jacobson radical and nil radical. The study can be continued for other classes of \((m, n)\)-hyperstructures. As future work, we will study \((m, n)\)-hypermodules.

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