Fracture analysis of U-notched disc-type graphite specimens under mixed mode loading

A.R. Torabi *, M. Fakoor, E. Pirhadi
Fracture Research Laboratory, Faculty of New Science and Technologies, University of Tehran, P.O. Box 13741-4395, Tehran, Iran

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A B S T R A C T

Fracture phenomenon was investigated both experimentally and theoretically for a type of coarse-grained polycrystalline graphite weakened by a U-shaped notch under mixed mode loading. First, 36 disc-type graphite specimens containing a central U-notch, so-called in literature as the U-notched Brazilian disc (UNBD), were prepared for four different notch tip radii and the fracture tests were performed under mode I and mixed mode I/II loading conditions. Then, the experimentally obtained fracture loads and the fracture initiation angles were predicted by using the U-notched maximum tangential stress (UMTS) and the newly formulated U-notched mean stress (UMS) fracture criteria. Both the criteria were developed in the form of the fracture curves and the curves of fracture initiation angle, i.e., in terms of the notch stress intensity factors (NSIFs). The results showed that while the criteria could predict successfully the experimental notch fracture toughness values, the UMS criterion provides slightly better predictions than the UMTS criterion, particularly for shear-dominant deformations. Also, found in this research was that the curves of fracture initiation angle were almost identical for the two criteria which both could predict well the experimental results.

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1. Introduction

Graphite materials are used in aerospace applications mainly because of their protective roles against thermal damages. Because of very low values of the coefficient of thermal expansion, such materials can withstand well against thermal loads. Conversely, graphite materials are so vulnerable to mechanical loads which are normally present in real engineering applications together with the thermal loads. As an example, one can refer to the protective graphite parts that are traditionally fastened to the main aerospace structural component by using a screw-like mechanism with contained normally U and V-shaped threads. The stress concentration around V and U-shaped notches in the presence of material brittleness may cause sudden fracture in protective graphite parts and hence, melting the main component due to the aerodynamic heating. Therefore, some reliable fracture models are seriously needed to be used in design of notched graphite parts, particularly where the notch is subjected in most real cases to mixed mode mechanical loads.

Despite some papers published in the past dealing with fracture of graphite materials in the presence of sharp cracks (see for example Awaï and Sato, 1978; Yamauchi et al., 2000, 2001; Li et al., 1999; Shi et al., 2008; Etter et al., 2004; Ayatollahi and Aliha, 2008), some recent studies can also be addressed. Short crack growth in a type of poly-granular graphite has been quantitatively assessed by Mostafavi and Marrow (2012) using digital image correlation. More recently, Mostafavi et al. (2013) have investigated the three-dimensional crack propagation in poly-granular graphite.

Dealing with notches, however, a few investigations have been conducted in the past regarding fracture in graphite materials. The stress concentration and the notch sensitivity have been studied by Bazaj and Cox (1969) and Kawakami (1985) on various graphite materials. However, taking into account the notch fracture mechanics (NFM) approach which employs the notch stress intensity factors (NSIFs) as the governing fracture parameters, the first attempt to evaluate the resistance of notched graphite parts to mechanical loads has been made by Ayatollahi and Torabi (2010a) who investigated the fracture phenomenon in different V-notched polycrystalline graphite specimens. A large number of V-notched graphite specimens, namely the rounded V-notched Brazilian disc (RV-BD), the rounded V-notched three-point bend (RV-TPB) and the rounded V-notched semi-circular bend (RV-SCB) specimens have been tested under pure mode I loading conditions and the experimental notch fracture toughness values have been predicted by using the mean stress (MS) criterion (Ayatollahi and Torabi, 2010a). In Ayatollahi and Torabi (2011a), numerous experimental results have been provided on mixed mode fracture of V-notched Brazilian disc (V-BD) specimens made of polycrystalline graphite and the notch fracture toughness and...
the fracture initiation angles have been estimated well by using the V-notched maximum tangential stress (V-MTS) criterion. Moreover, the local strain energy density (SED) criterion has been utilized by Ayatollahi et al. (2011a) for predicting mixed mode I/II fracture test results reported by Ayatollahi and Torabi (2011a) and a very good agreement between the experimental and theoretical results was obtained. Additionally, a valuable out-of-plane fracture study on sharp and blunt V-notched round graphite bars has also been performed by Berto et al. (2012a) under torsion.

The first study on brittle fracture of graphite materials in the presence of a U-shaped notch has been published by Berto et al. (2012b) who conducted many of fracture tests on U-notched rectangular isostatic graphite plates under pure mode I and also mixed mode I/II loading conditions. They also made use of the local SED over a control volume as a governing brittle fracture parameter to predict well the experimental results. Note that the SED model has been frequently used by the investigators in the field of brittle fracture of notched components made of different engineering materials (see for instance Ayatollahi et al., 2011a; Torabi, 2010d). The MS criterion in its mixed mode format has been utilized by Ayatollahi et al. (2011a) for predicting mixed mode I/II fracture models are the maximum tangential stress (MTS) and mean stress (MS) criteria. The MTS criterion, proposed originally by Erdogan and Sih (1963) for predicting mixed mode brittle fracture in sharp cracked bodies, has been extended by other researchers to V and U-notched domains and frequently used for predicting the onset of mixed mode brittle fracture in different notched brittle materials (see for example Ayatollahi and Barati, 2011; Berto et al., 2007, 2012c, 2013; Rajad et al., 2009a,b; Berto and Lazzarin, 2009; Lazzarin et al., 2009; Gomez et al., 2007, 2009; etc.).

Despite the SED failure concept, two other well-defined brittle fracture models are the maximum tangential stress (MTS) and the mean stress (MS) criteria. The MTS criterion, proposed originally by Erdogan and Sih (1963) for predicting mixed mode brittle fracture in sharp cracked bodies, has been extended by other researchers to V and U-notched domains and frequently used for predicting the onset of mixed mode brittle fracture in different notched brittle materials (see for example Ayatollahi and Barati, 2011; Berto et al., 2007, 2012c, 2013; Rajad et al., 2009a,b; Berto and Lazzarin, 2009; Lazzarin et al., 2009; Gomez et al., 2007, 2009; etc.). The mean stress (MS) criterion has been already used by the researchers mainly in its pure mode I format. For example, the MS criterion has been utilized to assess mode I failure in V-notched polycrystalline graphite samples (see Ayatollahi and Torabi, 2010a) and V-notched PMMA and ceramic specimens (see Ayatollahi and Torabi, 2010d). The MS criterion in its mixed mode format has been previously formulated by Seweryn and Lukaszewicz (2002) for V-notched components and presented as a closed-form expression.

Recently, Torabi (2013a) has successfully made use of the point stress (PS) and the mean stress (MS) fracture criteria for predicting mode I failure of U-notched graphite plates tested and reported by Berto et al. (2012b). Moreover, he utilized the MTS failure curves to estimate theoretically the fracture loads of the U-notched graphite specimens reported by Berto et al. (2012b) under mixed mode I/II loading conditions and found very good agreement between the theoretical and the experimental results (see Torabi, 2013b). The latest published work on brittle fracture of U-notched graphite components is a paper published more recently by Torabi et al. (2013) in which the load-carrying capacity of U-notched Brazilian disc (UNBD) specimens made of a coarse-grained polycrystalline graphite have been successfully predicted under mode I loading by using the mean stress (MS) and the point stress (PS) criteria.

In the present research, first, several U-notched disc-type specimens, so-called in literature as the U-notched Brazilian disc (UNBD) specimen, made of a type of coarse-grained polycrystalline graphite were fabricated and the mixed mode fracture tests were carried out for different notch tip radii and various mode mixity ratios. Then, the U-notched MTS (UMTS) criterion, presented originally by Ayatollahi and Torabi (2009), was reformulated for the tested polycrystalline graphite in terms of the notch stress intensity factors (NSIFs) and the theoretical fracture curves and the curves of fracture initiation angles were plotted. Similarly, the MS concept was applied to the stress distribution around the U-shaped notch and the failure curves of the U-notched mean stress (UMS) criterion were also plotted for the polycrystalline graphite. Finally, the theoretical results were compared with the experimental results obtained from the fracture tests of UNBD specimens. It was found that while both the criteria could estimate well the onset of brittle fracture in tested specimens, the UMS model provides better predictions than the UMTS model, especially for high mode mixity ratios. Also, it was demonstrated that both the criteria provide almost equal curves for fracture initiation angle which both, however, are well accurate.

2. Experiments

2.1. Material

The material tested was a type of coarse-grained commercial polycrystalline graphite with the properties presented in Table 1.

<table>
<thead>
<tr>
<th>Material property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus, E (GPa)</td>
<td>8.05</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.2</td>
</tr>
<tr>
<td>Ultimate tensile strength (MPa)</td>
<td>27.5</td>
</tr>
<tr>
<td>Plane-strain fracture toughness (MPa m^{0.5})</td>
<td>1.0</td>
</tr>
<tr>
<td>Bulk density (kg/m^3)</td>
<td>1710</td>
</tr>
<tr>
<td>Mean grain size (μm)</td>
<td>320</td>
</tr>
</tbody>
</table>
brittle fracture under mode I, mixed mode I/II and also pure mode II loading conditions (see for example Awaji and Sato, 1978; Yamauchi et al., 2000; Ayatollahi and Aliha, 2008). Fig. 1 represents the UNBD specimen, schematically.

In Fig. 1, $\beta$ is the angle between the loading direction and the notch bisector line. The parameters $D$ and $P$ denote the disc diameter and the applied compressive load, respectively. When the direction of the applied load $P$ is along the notch bisector line (i.e., $\beta = 0$), the central notch is subjected to pure mode I loading conditions. When $\beta$ enhances gradually from zero, the loading condition varies from pure mode I towards pure mode II. For a particular angle, called $\beta_{II}$, pure mode II deformation is obtained. The angle $\beta_{II}$ is always less than 90° and depends upon the notch length and the notch tip radius. This angles can be determined by using the finite element (FE) method as presented in Ayatollahi and Torabi (2010c) and Torabi and Jafarinezhad (2012).

The diameter, the overall notch length (i.e., the tip-to-tip distance) and the thickness of the UNBD specimens were 60 mm, 18 mm and 10 mm, respectively. The notch tip radii ($\rho$) were equal to 0.5, 1, 2 and 4 mm. To fabricate the specimens, first, a graphite block was provided. Then, three slices of 10 mm thick were cut from the block by using a cutter blade. The geometry of each sample was given to a high-precision 2-D CNC water jet cutting machine and finally, the UNBD specimens were fabricated. Before performing the fracture tests, the specimens were polished by using a fine abrasive paper in order to remove possible local stress raisers remained from the manufacturing process. Fig. 2 displays the UNBD graphite specimens fabricated (not polished).

To perform fracture tests under mixed mode loading conditions, the angle $\beta_{II}$ should first be determined so that we could select appropriate intermediate angles between $\beta = 0$ (pure mode I) and $\beta = \beta_{II}$ (pure mode II). The values of $\beta_{II}$ for different values of the relative notch length (RNL) and the relative notch tip radius (RNR) have been presented in Torabi and Jafarinezhad (2012) for the UNBD specimen. Considering the RNL (i.e., the ratio of the overall notch length to disc diameter) equal to 0.3 for the specimens, $\beta_{II}$ can be obtained from Torabi and Jafarinezhad (2012) to be the values in the range of 31 to 33 (deg.) for different notch tip radii. Therefore, the angles $\beta = 0$, 10 and 20 (deg.) were selected for pure mode I and mixed mode I/II fracture tests. For each notch tip radius, nine tests were carried out; three for each value of $\beta$, under displacement-control conditions with a loading rate of 0.1 mm/min and the fracture load of the samples were recorded. 36 test results were totally provided under pure mode I and mixed mode I/II loading conditions. Figs. 3 and 4 show the UNBD graphite specimens during mode I and mixed mode I/II tests, respectively.

Two broken specimens are shown in Fig. 5. The experimentally recorded fracture loads for UNBD graphite specimens are presented in Table 2.

The load–displacement graphs recorded from the tests were completely linear up to final fracture and the fracture for each specimen was seen to occur suddenly. Thus, brittle fracture criteria based on the linear elastic fracture mechanics (LEFM) are allowable. Fig. 5 displays two sample load–displacement curves for the graphite UNBD specimens (see Fig. 6).

In the next section, two well-known brittle fracture models are described and formulated with the aim to estimate the experimental results. The first one is the U-notched maximum tangential stress (UMTS) (Ayatollahi and Torabi, 2009) and the second one is the U-notched mean stress (UMS) criteria.

3. Mixed mode fracture criteria

3.1. The U-notched maximum tangential stress (UMTS) criterion

This criterion has been originally suggested by Ayatollahi and Torabi (2009) for predicting mixed mode I/II brittle fracture in
blunt V-notches. The closed-form expressions of the elastic stress distributions derived by Lazzarin and Tovo (1996) have been reformulated by Filippi et al. (2002) with the aim to achieve more accurate expressions. The distribution of the tangential stress around a U-shaped notch can be written as (Filippi et al., 2002)

$$\sigma_{\text{w}} = \frac{1}{2\sqrt{2\pi r}} \left\{ K_I^{\text{U}} \left[ \left( \frac{3}{2} + \rho \right) \cos \frac{\theta}{2} + \frac{1}{2} \cos \frac{3\theta}{2} \right] + K_{II}^{\text{U}} \left[ \left( \frac{3}{2} - \rho \right) \sin \frac{\theta}{2} + \frac{3}{2} \sin \frac{3\theta}{2} \right] \right\}$$

(1)

The parameters $K_I^{\text{U}}$ and $K_{II}^{\text{U}}$ are the mode I and mode II notch stress intensity factors (NSIFs), respectively. \( \rho \) is the notch tip radius and the parameters \( r \) and \( \theta \) denote the local polar coordinate system located at the distance \( r_0 = \rho/2 \) behind the U-notch tip.

According to the UMTS fracture concept, the tangential stress at a critical distance around the notch tip should be a maximum at the onset of fracture. Thus:

$$\frac{\partial \sigma_{\text{w}}}{\partial \theta} = 0 \Rightarrow \theta = \theta_0$$

(2)

The angle \( \theta_0 \) is the fracture initiation angle (sometimes referred to as the notch bifurcation angle) which determines the location of crack initiation on the U-notch border with respect to the local polar coordinate system. Eq. (2) provides a set of positive and negative roots. To reach to a maximum, the second derivative of \( \sigma_{\text{w}} \) must be a negative value. Therefore, only negative \( \theta_0 \) values are valid. According to Fig. 7, the point of fracture initiation locates always on the lower half border of U-notch. It is necessary to note that \( \theta_0 \) does not give the crack path since it is measured from the local coordinate origin and not from the U-notch center. The crack path is always radial (i.e., perpendicular to the U-notch semi-circle) because it is perpendicular to the direction of the maximum tangential stress in accordance with the UMTS model. In order to plot the crack path, one can simply draw a line from the notch center to the crack initiation point obtained from Eq. (2) (see the angle \( \theta_0^* \) in Fig. 8). Details of the above are represented in Fig. 8.

Substituting Eq. (1) into Eq. (2)

$$K_I^{\text{U}} \left[ \left( \frac{3}{4} + \frac{\rho}{2r_{c,\text{U}}} \right) \sin \frac{\theta_0}{2} - \frac{3}{4} \sin \frac{3\theta_0}{2} \right] + K_{II}^{\text{U}} \left[ \left( \frac{3}{4} - \frac{\rho}{2r_{c,\text{U}}} \right) \cos \frac{\theta_0}{2} + \frac{9}{4} \cos \frac{3\theta_0}{2} \right] = 0$$

(3)

Note that the parameter \( r \) in Eq. (1) is substituted with \( r_{c,\text{U}} \) (U-notch critical distance) according to the requirements of the UMTS criterion. In pure mode I loading conditions, crack initiates along the notch bisector line and the fracture initiation angle at the notch border (\( \theta_0 \)) is zero because both the geometry and loading are symmetric. In pure mode II loading conditions (i.e., pure in-plane shear deformation of the notch), $K_{II}^{\text{U}}$ is zero. Thus, Eq. (3) is simplified to

U-notched components. A brief description of the UMTS criterion is presented herein.

A U-notch is schematically depicted in Fig. 7 including its polar coordinate system. The origin of the coordinate is located at the distance \( r_0 = \rho/2 \) behind the notch tip on the notch bisector line. Creager and Paris (1967) derived the elastic stress field equations for blunt cracks. Lazzarin and Tovo (1996) have also evaluated the elastic stress distributions in the vicinity of cracks and

<table>
<thead>
<tr>
<th>( \beta ) (deg.)</th>
<th>( \rho ) (mm)</th>
<th>( P_1 ) (N)</th>
<th>( P_2 ) (N)</th>
<th>( P_3 ) (N)</th>
<th>( P_{av} ) (N)</th>
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<tr>
<td>0</td>
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<tr>
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<tr>
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<tr>
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<td>4</td>
<td>4000</td>
<td>4188</td>
<td>3722</td>
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</tr>
<tr>
<td>10</td>
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<td>4101</td>
<td>3836</td>
<td>4203</td>
<td>4047</td>
</tr>
<tr>
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<td>3398</td>
<td>3000</td>
<td>3359</td>
</tr>
</tbody>
</table>

The experimentally recorded fracture loads for UNBD graphite specimens.
Eq. (4) implies that mode II fracture initiates from a point on the notch border that its angular position from the local polar coordinate system is recognized by the angle $\theta_{II}$ which depends on the critical distance $r_{c,U}$ and the notch tip radius $q$. Another important requirement of the UMTS criterion proposes that brittle fracture occurs when the tangential stress attains necessarily the critical value $(\sigma_{\infty})_{II}$. Therefore, Eq. (1) in critical conditions can be written as

$$
(3 - \frac{\rho}{2r_{c,U}}) \cos \theta_{II} + \frac{9}{4} \cos \frac{3\theta_{II}}{2} = 0 \Rightarrow \theta_{II} = \theta_{II}
$$  \hspace{1cm} (4)

A simple relationship has been obtained by Ayatollahi and Torabi (2009) between $(\sigma_{\infty})_{II}$ and the mode I notch fracture toughness $K_{lc}^{II}$. It is

$$
(\sigma_{\infty})_{II} = \left(2 + \frac{\rho}{r_{c,U}}\right)\frac{K_{lc}^{II}}{2\sqrt{2\pi r_{c,U}}}
$$  \hspace{1cm} (6)

Eq. (6) has been also written in some papers before Ayatollahi and Torabi (2009). For example, Gomez et al. (2006) have published a paper in which several mode I brittle fracture models have been suggested and utilized to predict numerous experimental results. Eq. (6) with a bit difference in shape has been one of the proposed criteria (Gomez et al., 2006). The idea has been fundamentally taken from Erdogan and Sih (1963). Substituting Eq. (6) into Eq. (5) gives:
To plot the curves of fracture initiation angle, a useful parameter called the mode mixity parameter \( M_U^e \) is defined herein

\[
M_U^e = \frac{2}{\pi} \tan^{-1} \left( \frac{K_{IIU}^e}{K_{I}^e} \right)
\]  

(8)

The value of \( M_U^e \) varies from zero (for pure mode II) to one (for pure mode I). By extracting \( K_{IIU}^e/K_{I}^e \) from Eq. (3) and substituting into Eq. (8), one can obtain:

\[
M_U^e = \frac{2}{\pi} \tan^{-1} \left( \frac{(3 - \rho \cos \theta_0) \cos \frac{\theta_0}{2} + \frac{3}{2} \cos \frac{3\theta_0}{2}}{(3 - \rho \cos \theta_0) \sin \frac{\theta_0}{2} + \frac{3}{2} \sin \frac{3\theta_0}{2}} \right)
\]  

(9)

In order to plot the curves of fracture initiation angle (i.e., a graph consists of a horizontal axis for \( M_U^e \) and a vertical axis for \( \theta_0 \)) for given values of the notch tip radius \( \rho \) and the notch critical distance \( r_{c,U} \), one can follow the steps below:

1. Choose an arbitrary value of \( M_U^e \) between zero and one.
2. Substitute \( M_U^e \) into Eq. (9).
3. Solve Eq. (9) and determine the fracture initiation angle \( \theta_0 \).
4. Repeat steps 1 to 3 for other values of \( M_U^e \).
5. Draw the curve of fracture initiation angle utilizing the points calculated in step 4.

The notch critical distance \( r_{c,U} \) which is measured from the origin of the local polar coordinate system (not from the notch tip) can be calculated by solving the equation below (Ayatollahi and Torabi, 2009)

\[
(4r_{c,U}^2 + \rho^2 + 4\rho r_{c,U}) \left( \frac{K_{IIU}^e}{K_{I}^e} \right)^2 - 8\pi r_{c,U}^3 = 0
\]  

(10)

The mean values of \( r_{c,U} \) and \( K_{IIU}^e \) for tested U-notched graphite specimens are presented in Table 3. The critical distance has been employed in the past together with T-stress for estimating fracture in various brittle materials like rocks containing a sharp crack under mixed mode I/II loading. In this area, one can refer to Aliha et al. (2010) in which the generalized maximum tangential stress (GMTS) model has been used to estimate the fracture trajectory in a limestone rock under mixed mode loading.

### 3.2. The U-notched mean stress (UMS) criterion

Similar to the UMTS criterion, the U-notched MS (UMS) criterion can also be formulated based on the tangential stress distribution around a U-notch (see Eq. (1)). Brittle fracture takes place in accordance with the mean stress (MS) failure concept when the mean value of the tangential stress over a specified critical distance attains a critical value. To formulate the criterion mathematically, the mean value of the tangential stress over a specified critical distance should first be determined. The critical distances for UMS criterion are denoted by \( d'_c \) and \( d_c \) in Fig. 9 which are measured from the coordinate origin and from the notch tip, respectively (note that a U-notch is geometrically a blunt V-notch with zero notch angle). Fig. 9 shows that \( d'_c = d_c + r_0 \) where \( r_0 \) is equal to \( \rho/2 \) for U-notches (see Ayatollahi and Torabi, 2009, 2010c; Torabi and Jafarinezhad, 2012).

### Table 3

<table>
<thead>
<tr>
<th>( \rho ) (mm)</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{IIU}^e ) (MPa√m)</td>
<td>1.333</td>
<td>1.644</td>
<td>1.825</td>
<td>2.107</td>
</tr>
<tr>
<td>( r_{c,U} ) (mm)</td>
<td>0.692</td>
<td>1.163</td>
<td>1.740</td>
<td>2.770</td>
</tr>
</tbody>
</table>
The mean stress over the critical distance can be easily written as

$$\sigma_m = \frac{1}{d_c} \int_{r=r_0}^{r=d_c} \sigma_m dr$$

Eq. (11) has been proposed for the first time by Seweryn (1994) who referred to a research by Novozhilov (1969). Substituting Eq. (1) into Eq. (11) and integrating gives

$$\sigma_m = \frac{K_{I}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \cos \theta + \cos \frac{3\theta}{2} \right) \left( \sqrt{d_c^2 - r_0^2} - 2\rho \cos \theta \left( \frac{1}{\sqrt{d_c^2 - r_0^2}} - 1 \right) \right) \right\}$$

$$+ \frac{K_{II}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \sin \theta + 2 \sin \frac{3\theta}{2} \right) \left( \sqrt{d_c^2 - r_0^2} + 2\rho \sin \theta \left( \frac{1}{\sqrt{d_c^2 - r_0^2}} + 1 \right) \right) \right\}$$

To present Eq. (12) more conveniently, the following parameters are defined:

$$X = \sqrt{d_c^2 - r_0^2}, \quad Y = \frac{1}{\sqrt{d_c^2 - r_0^2}}$$

Thus, Eq. (12) can be rewritten as

$$\sigma_m = \frac{K_{I}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \cos \theta + \cos \frac{3\theta}{2} \right) X - 2\rho \cos \theta Y \right\}$$

$$+ \frac{K_{II}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \sin \theta + 2 \sin \frac{3\theta}{2} \right) X + 2\rho \sin \theta Y \right\}$$

The procedure to formulate the UMS criterion is completely the same with that for the UMTS criterion except that \(\sigma_m\) is replaced by \(\sigma_m\). Therefore

$$\frac{\partial \sigma_m}{\partial \theta} = 0 \rightarrow \theta = \theta_0$$

The angle \(\theta_0\) is the fracture initiation angle for the UMS criterion. Substituting Eq. (14) into Eq. (15) gives

$$\frac{K_{I}^{\rho}}{K_{II}^{\rho}} \left( \left( \frac{3}{2} \sin \theta_0 + \frac{3}{2} \sin \frac{3\theta_0}{2} \right) X + \rho Y \sin \theta_0 \right)$$

$$+ \frac{K_{II}^{\rho}}{K_{II}^{\rho}} \left( \left( \frac{3}{2} \cos \theta_0 + \frac{3}{2} \cos \frac{3\theta_0}{2} \right) X + \rho Y \cos \theta_0 \right) = 0$$

Under mode I loading conditions, the fracture initiation angle is trivially equal to zero because of the symmetry in geometry and loading. Under pure mode II, \(K_{II}^{\rho}\) becomes zero and hence, Eq. (16) is simplified to

$$\frac{3}{2} \cos \theta_0 + \frac{3}{2} \cos \frac{3\theta_0}{2} X + \rho Y \cos \theta_0 = 0$$

The root of Eq. (17) is, in fact, the fracture initiation angle predicted by the UMS criterion under pure mode II loading conditions (herein after referred to as \(\theta_{II_0}\)).

The UMS criterion suggests that \(\sigma_m\) should attain its critical value (normally the ultimate tensile strength of material) at brittle fracture instance. Therefore

$$\left( \frac{\sigma_m}{\sigma_c} \right)_c = \frac{K_{I}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \cos \theta_0 + \cos \frac{3\theta_0}{2} \right) X - 2\rho Y \cos \theta_0 \right\}$$

$$+ \frac{K_{II}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \sin \theta_0 + 2 \sin \frac{3\theta_0}{2} \right) X + 2\rho \sin \theta_0 Y \right\}$$

Similar to Eq. (6), a closed-form expression is needed for \(\left( \sigma_m \right)_c\) in terms of the critical distance and the mode I notch fracture toughness of material (i.e., \(K_{I}^{\rho}\)), in order to present the equations on the basis of the dimensionless NSIFs (i.e., \(K_{I}^{\rho}/K_{II}^{\rho}\) and \(K_{II}^{\rho}/K_{II}^{\rho}\)). For this purpose, one can apply the requirements of mode I fracture to Eq. (18) as follows:

$$\theta_0 = 0$$

$$K_{I}^{\rho} - K_{II}^{\rho} \Rightarrow \left( \frac{\sigma_m}{\sigma_c} \right)_c = \frac{K_{I}^{\rho}}{2d_c \sqrt{2\pi}} \left\{ \left( 3 \cos \theta_0 + \cos \frac{3\theta_0}{2} \right) X - 2\rho Y \cos \theta_0 \right\}$$

$$K_{II}^{\rho} = 0$$

By substituting Eq. (19) into Eq. (18), it is obtained

$$\frac{K_{I}^{\rho}}{K_{II}^{\rho}} \frac{\sqrt{d_c}}{4d_c} \left\{ \left( 3 \cos \theta_0 + \cos \frac{3\theta_0}{2} \right) X - 2\rho Y \cos \theta_0 \right\}$$

$$+ \frac{K_{II}^{\rho}}{K_{II}^{\rho}} \frac{\sqrt{d_c}}{4d_c} \left\{ \left( 3 \sin \theta_0 + 2 \sin \frac{3\theta_0}{2} \right) X + 2\rho \sin \theta_0 Y \right\} = 1$$

By dividing both sides of Eq. (16) by \(K_{II}^{\rho}\), one can obtain

$$\frac{K_{I}^{\rho}}{K_{II}^{\rho}} \left( \left( \frac{3}{2} \sin \theta_0 + \frac{3}{2} \sin \frac{3\theta_0}{2} \right) X + \rho Y \sin \theta_0 \right)$$

$$+ \frac{K_{II}^{\rho}}{K_{II}^{\rho}} \left( \left( \frac{3}{2} \cos \theta_0 + \frac{3}{2} \cos \frac{3\theta_0}{2} \right) X + \rho Y \cos \theta_0 \right) = 0$$

Eqs. (20) and (21) form a simple linear system of equations in which \(K_{I}^{\rho}/K_{II}^{\rho}\) and \(K_{II}^{\rho}/K_{II}^{\rho}\) are unknown. Once the critical distance and mode I notch fracture toughness are known (Note: \(K_{II}^{\rho}\) is obtained from mode I fracture tests (see Table 3) and \(d_c\) is computed using Eq. (19)), the equations can be solved simultaneously for different values of \(\theta_0\) between zero and \(\theta_{II_0}\), and finally, the fracture curves of the UMS model are achieved in terms of \(K_{I}^{\rho}/K_{II}^{\rho}\) and \(K_{II}^{\rho}/K_{II}^{\rho}\).

Analogous to UMTS criterion, the mode mixity parameter \(\theta_{II_0}\) can be defined for the UMS model by using Eq. (16) as follows
The procedure of plotting the curves of fracture initiation angle is completely the same with that for the UMTS criterion elaborated after Eq. (9).

The notch critical distances of tested graphite material associated with the UMTS criterion (d, and d,) are presented in Table 4 for different notch tip radii.

As shown above, both the UMTS and the UMS criteria utilize critical distances to predict the onset of brittle fracture in U-notched graphite components. Some other contributions have also been suggested in the past for engineering failure analysis regarding critical distances (see for instance Tanaka (1983) and Taylor (1999)). The application of critical distances to the static fatigue has also been proposed previously by Susmel and Taylor (2008).

In the next section, the experimental results of the mixed mode notch fracture toughness and the fracture initiation angles for tested U-notched graphite specimens are theoretically predicted by using the fracture curves and the curves of fracture initiation angles of the UMTS and UMS criteria.

4. Results and discussion

In order to make a comparison between the theoretical predictions with the experimental results, the recorded fracture loads of the U-notched graphite samples should be converted to the corresponding values of the critical NSIFs, i.e., the mixed mode I/II notch fracture toughness values. For this purpose, one can use the following equations:

\[ K_{Ic}^{U} = \frac{\sqrt{\pi d}}{2} \sigma_{f} \left( \frac{d}{L} \right) \]  

\[ K_{IIc}^{U} = \text{Lim}_{d \to 0} \frac{\sigma_{f} \left( \alpha_{o} \right)}{2 \left( \frac{d}{L} \right)} \]  

Eq. (23) has been first provided by Glinka (see Glinka, 1985; Glinka and Newport, 1987). Eq. (24) is a reduced form of an expression provided originally by Lazzarin and Filippi (2006) for blunt V-notches. Eqs. (23) and (24) have been frequently utilized by the researchers to calculate the in-plane NSIFs (see for instance Torabi and Jafarinezhad, 2012). In Eqs. (23) and (24), \( \sigma_{f}(\alpha_{o}, 0) \) is the tangential stress at the U-notch tip and the parameter \( \sigma_{f}(\alpha_{o})_{\alpha=0} \) is the in-plane shear stress along the notch bisector line. To compute the critical values of the NSIFs, first, a finite element (FE) model should be created for each notched graphite specimen and the fracture load recorded from mode I and mixed mode fracture tests (see for instance Tanaka (1983) and Taylor (1999)). Then, the critical distances for the U-notched graphite specimens should be theoretically predicted by using Eqs. (23) and (24).

Figs. 10 and 11 represent respectively the fracture curves and the curves of fracture initiation angle of UMTS and UMS failure criteria together with the experimental results of the UNBD graphite specimens for four various notch tip radii.

Fig. 10 represents that for a constant notch tip radius, as the contribution of mode II deformation increases, the deviation between the results of the UMTS and UMS criteria enhances (the maximum deviation is achieved under pure mode II). For mode I dominant loading conditions, both the criteria provide almost identical predictions. Also, Fig. 10 implies that the deviation of the two curves decreases in the entire domain from mode I to mode II as the notch tip radius increases. For \( \rho = 4 \text{ mm} \), almost no difference can be seen between the UMTS and UMS fracture curves. It is seen in Fig. 10 that for pure mode II, the ratio on y-axis tends always to a value greater than 1.0. This is because \( K_{IIc}^{U} \) is always greater than \( K_{IIc} \) from pointed notches also for small values of the notch tip radius resulting from the particular applications of the Creager and Paris (1967) equation for U-notches. This point should be underlined. It can be seen in Fig. 11 that for all of the notch tip radii, the curves of fracture initiation angle provided by the two criteria are almost identical which both predict the experimental results accurately.

In order to compare the experimental and theoretical results quantitatively under mixed mode loading conditions, a dimensionless parameter, called the equivalent relative notch fracture toughness (ERNFT) \( K_{IIc}^{U} / K_{IIc} \) is defined herein as:

\[ K_{IIc}^{U} = \sqrt{\frac{\left( K_{IIc}^{U} / K_{IIc} \right)^{2} + \left( K_{IIc}^{U} / K_{IIc} \right)^{2}}{2}} \]  

Note that the value of \( K_{IIc}^{U} \) is, in fact, the magnitude of the cord drawn from the coordinate origin in Fig. 10 to the intercept point on the UMTS and the UMS fracture curves (theoretical values) and to that point on each individual experimental point. The theoretical values of ERNFT together with the mean values of the experimental ERNFTs are presented in Table 5 including the mean discrepancies. Additionally, a quantitative comparison of the theoretical and experimental fracture initiation angles can be found in Table 6.

As can be seen in Tables 5 and 6, both the UMTS and the UMS fracture criteria provide very good predictions with very close total accuracies.

In the engineering design, the engineers' main goal is to design the notched graphite member in order to withstand against sudden fracture. The UMTS and the UMS convenient and straightforward procedures for determining the load-bearing capacity of a notched graphite member under tensile/shear loading conditions can be simply explained as follows:

1. Apply a unit load to the FE model of the U-notched member and compute the ratio \( (K_{IIc}^{U} / K_{IIc}^{U}) \) for it by using Eqs. (23) and (24). Note that such ratio is independent of the magnitude of the load.

2. Draw a line with the slope of \( (K_{IIc}^{U} / K_{IIc}^{U}) \) on the plane of fracture curve. The first point of the line is the origin of the coordinate system and the second one is the intercept point between the curve and the line.

3. Read the horizontal and the vertical components of the intercept point (i.e., \( (K_{IIc}^{U} / K_{IIc}^{U}) \) and \( (K_{IIc}^{U} / K_{IIc}^{U}) \)).

4. Multiply both the obtained components by \( K_{IIc}^{U} \) (see Table 3) to achieve critical \( K_{IIc}^{U} \) and \( K_{IIc}^{U} \).

5. Increase the initially applied unit load to the greater values till NSIFs reach to the values computed in the step 4. The load associated with the critical NSIFs obtained in this step is, in fact, the load-bearing capacity of the U-notched component.

As previously mentioned, the critical distances for both the UMTS and UMS criteria were directly determined by using the results of mode I fracture tests (see Tables 3 and 4). The UMTS critical distances, lie on the material, were 0.42, 0.66, 0.74, 0.77 mm for the notch tip radii 0.5, 1, 2, 4 mm, respectively. Those critical distances of the UMS criterion (i.e., \( d_{c} \) values) were equal to 1.24, 1.77, 1.80, 1.74 mm. Comparing the first and the second group values of critical distances with the critical distances of the MTS and MS criteria
for a sharp crack $r_c = 1/2\pi (K_c/\sigma)^2 = 0.21$ mm and $d_c = 2/\pi (K_c/\sigma)^2 = 0.84$ mm demonstrates that assuming the notch critical distance equal to that for sharp crack may result in inaccurate theoretical predictions. This forced the authors to utilize initially the actual notch critical distances in theoretical computations.

In general, the mode I notch fracture toughness $K_{Ic}$ and the critical distances can be related to each other by simple expressions like those presented in Eqs. (10) and (19) for UMTS and UMS criteria, respectively. Under mode I, if one wants to predict theoretically $K_{Ic}$ by the criteria like point stress and the mean stress, he/she needs the values of the critical distances which are usually assumed to be equal to that of sharp crack (see for example Ayatollahi and Torabi (2010a,d)). Conversely, if the experimental value of $K_{Ic}$ is available from mode I fracture tests on notched specimens, the actual values of the critical distances can be directly computed (see Eqs. (10) and (19)) other than those for sharp cracks. The actual values of the critical distances (presented in Tables 3 and 4) seem to be dependent on the overall geometry of the test sample which may cause using the criteria questionable. It has been shown by Ayatollahi and Torabi (2010a) that for three V-notched graphite specimens having completely different overall geometries, the mode I notch fracture toughness tests provided very close values of $K_{Vc}$ for the specimens with the same notch angle and the same notch tip radius. It implies that the mode I notch fracture toughness value is almost independent of the overall geometry of the laboratory-scaled test sample and depends only on the material properties and the notch geometry. In this situation, the notch critical distances do not also depend on the overall geometry of the notched specimen (see Eqs. (10) and (19)). Based on this finding, it is expected that similar behavior exists also for U-notched graphite specimens. Trivially, the validity of the present claim should be verified by various tests. Note that the statement above may be valid for laboratory-scaled specimens. For large or very small components, the size effect should be considered. From the statements above, it is suggested that one first selects

Fig. 10. The fracture curves of UMTS and UMS failure criteria together with the experimental results of the UNBD graphite specimens for four various notch tip radii.

Tables 3 and 4) seem to be dependent on the overall geometry of the test sample which may cause using the criteria questionable. It has been shown by Ayatollahi and Torabi (2010a) that for three V-notched graphite specimens having completely different overall geometries, the mode I notch fracture toughness tests provided very close values of $K_{Vc}$ for the specimens with the same notch angle and the same notch tip radius. It implies that the mode I notch fracture toughness value is almost independent of the overall geometry of the laboratory-scaled test sample and depends only on the material properties and the notch geometry. In this situation, the notch critical distances do not also depend on the overall geometry of the notched specimen (see Eqs. (10) and (19)). Based on this finding, it is expected that similar behavior exists also for U-notched graphite specimens. Trivially, the validity of the present claim should be verified by various tests. Note that the statement above may be valid for laboratory-scaled specimens. For large or very small components, the size effect should be considered. From the statements above, it is suggested that one first selects
arbitrarily a laboratory-scaled test specimen capable of testing
mode I notch fracture. Then, he/she performs mode I fracture tests
and measures $K_U$; $q_{lc}$ which depends on the notch tip radius (not on
the overall-geometry of specimen). Finally, the critical distances of
the UMTS and the UMS criteria are calculated by means of Eqs.(10)
and (19), respectively and the fracture curves can be plotted.

5. Conclusions

The experimental fracture loads and the fracture initiation
angles obtained from the mixed mode I/II fracture tests of the
U-notched Brazilian disc (UNBD) graphite specimens were theoretically estimated by means of the two well-known failure concepts,
Table 6
The theoretical and experimental values of the fracture initiation angles (in deg.) including the mean discrepancies.

<table>
<thead>
<tr>
<th>β = 10 (deg.)</th>
<th>p = 0.5 (mm)</th>
<th>p = 1</th>
<th>p = 2</th>
<th>p = 4</th>
<th>β = 20 (deg.)</th>
<th>p = 0.5</th>
<th>p = 1</th>
<th>p = 2</th>
<th>p = 4</th>
</tr>
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<tr>
<td>Mean experimental results</td>
<td>43.3</td>
<td>40.3</td>
<td>36.6</td>
<td>37.6</td>
<td>54</td>
<td>54</td>
<td>55</td>
<td>50.6</td>
<td></td>
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<td>UMTS criterion</td>
<td>46</td>
<td>44</td>
<td>40</td>
<td>41</td>
<td>56</td>
<td>56</td>
<td>53</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>UMS criterion</td>
<td>47</td>
<td>44.5</td>
<td>40</td>
<td>41</td>
<td>57</td>
<td>56</td>
<td>53</td>
<td>51</td>
<td></td>
</tr>
<tr>
<td>Mean discrepancy of UMTS model (%)</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>7.9</td>
<td></td>
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<tr>
<td>TOTAL: 6.2%</td>
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<tr>
<td>Mean discrepancy of UMS model (%)</td>
<td>8.5</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>7.9</td>
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<tr>
<td>TOTAL: 6.9%</td>
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</table>

namely the maximum tangential stress (MTS) and the mean stress (MS). The fracture curves and the curves of fracture initiation angle were developed in terms of the notch stress intensity factors (NSIFs) which are capable of predicting the notch fracture toughness and the bifurcation angle in the entire domain from pure mode I and pure mode II loading conditions. With about 8% and 6% mean discrepancies in the prediction of notch fracture toughness and fracture initiation angle, respectively, the UMTS model was found to be an appropriate failure model. Those discrepancies were about 6.4% and 7% for the UMS model which provides very close estimates to the UMTS results. The results showed that the curves of fracture initiation angle provided by UMTS and UMS criteria are almost identical which both predict the experimental results accurately. This means that from the view point of fracture initiation angle, there is no difference between the two criteria in real engineering applications. It was obtained in this research that the critical distances for U-shaped notches in polycrystalline graphite are significantly different from those for sharp crack suggesting that the use of the values for sharp crack in U-notched domains is impermissible.

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References


