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What is This?
Modeling of a Nonlinear Euler–Bernoulli Flexible Beam Actuated by Two Active Shape Memory Alloy Actuators

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ABSTRACT: There are two different ways of using shape memory alloy (SMA) wire as an actuator for shape control of flexible structures: it can be either embedded within the composite laminate or externally attached to the structure. As the actuator can be placed at different offset distances from the beam, external actuators produce more bending moment and, consequently, considerable shape changes with the same magnitude of actuation force compared with the embedded type. Such a configuration also provides faster heat transfer rate owing to convection, which is very important in shape control applications that require a high-frequency response of SMA actuators.

Although combination and physics-based modeling of externally attached SMA actuator wires and strips have been considered by many researchers, these studies have some drawbacks, which, if neglected, result in a number of errors in the theoretical results compared with experimental results. These shortcomings are considering the linear Euler–Bernoulli beam theory, not deriving the actuation force of the SMA from the constitutive equations, and taking into account the effect of only one SMA wire actuation force on, for example, the structure. These assumptions may lead to erroneous theoretical modeling results compared with experimental results.

In this study, the aforementioned difficulties of attaching SMA actuators to the smart structures have been addressed. In other words, instead of linear beam theory, nonlinear beam theory is used in system modeling and, therefore, the proposed method and results are valid in large deflection and rotation behavior of beam. Also, in comparison to many other analyses that the effect of only one SMA wire is investigated, in the present research the effect of all active and inactive SMA wires is considered. Accordingly, the result of this paper can easily be generalized to the structure with several SMA wire actuations. Moreover, with the purpose of having practical applications in modeling and control, the heat transfer equations of all SMA wires are considered in the analysis and, as a result, the control inputs of the presented model are SMA wire electric currents rather than SMA temperatures.

First, a flexible beam actuated by two active SMA actuators is modeled. Then, the Brinson constitutive equations and thermoelectric equations for SMA materials are coupled with the nonlinear beam behavior, and the coupled system of equations is numerically solved for some particular practical cases. Finally, the numerical results of the model simulation are verified against the experimental results using a test setup to validate the proposed model prediction. The implemented method used in this paper can be easily extended to the more complex smart structure with numerous externally attached SMA wires.

Key Words: Nonlinear modeling, flexible beam, shape memory alloy (SMA) actuators.

INTRODUCTION

SMART materials have been extensively used in recent years for their great potential to revolutionize engineering applications and design, particularly for active and passive control of structures. Among these materials, shape memory alloys (SMAs) have been receiving considerable attention owing to their ability to develop extremely large recoverable strains, as well as great force. SMAs are applied in a wide variety of fields such as aeronautics, medicine, and civil and mechanical engineering (Lagoudas, 2008).

Having an adaptive structure is another characteristic of these SMA materials that has largely been considered in vibration control rather than shape control. One way of using an SMA as an actuator in these flexible structures is embedding or bonding it to the surface of the host material so that the moment of the actuating force deforms the structure. Although increasing the distance between the bonded actuator and the neutral axis of the
structure enhances the moment, as a linear function, the flexural stiffness of the structure increases as a square of this distance (Chaudhry and Rogers, 1991). Another difficulty of using an SMA actuator in embedding form is its heat transfer. This problem constrains the high-frequency response of the SMA actuator, which is very important in shape control applications where fast structure response is needed.

To overcome the limitations of bonded or embedded SMA actuators in structures, they can be externally attached to the structure. This configuration can be advantageous in several aspects. First, the actuators can be placed at different offset distances from the neutral axis of the structure and, consequently, produce more bending moment and shape change with the same magnitude of the actuation force (with respect to the embedded type) (Shu et al., 1997). In addition, the increase in the flexural stiffness of the structure can be conditioned. Such a configuration also eliminates the heat transfer problem, mentioned previously, that occurs when embedding SMA within composite laminate.

In order to predict the smart structure behavior, it is essential to have accurate models that can easily map the input(s) effect to the performance of the structure. Two different methods for modeling of these smart structures have been proposed. The first group of models is based on the phenomenological nature and mathematically describes the observed phenomenon without necessarily providing physical insight into the problems (Hughes and Wen, 1997; Samir and Menq, 2000; Tan and Baras, 2002; Mayergoyz, 2003; Ahn and Nguyen, 2006, 2007; Viswamurthy and Ganguli, 2007). Among these models, the Preisach model has found extensive application for modeling hysteresis in smart structures (Hughes and Wen, 1997; Samir and Menq, 2000; Mayergoyz, 2003). The first drawback of this group of models is that they do not provide physical insight into the problem. Also, they cannot be used for designing the smart structures beforehand because they are based on the experimental data and these data are available only after manufacturing the structure. In addition, to have accurate results in the identification and control process, many experimental data are required, and this directly increases the cost of the training process.

The second group of models is derived from the underlying physics of structure and is combined with empirical factors to describe the observed characteristics (Chaudhry and Rogers, 1991; Boyd and Lagoudas, 1994; Brinson et al., 1997; Shu et al., 1997; Moallem, 2003; Sohn, 2009). The main disadvantage of these models is that considerable effort is required in identifying and tuning the model parameters to accurately describe the nonlinearity behavior existing in the smart structures. In this paper the physics-based modeling of a smart structure consisting of a flexible beam and two active SMA actuators is described. Owing to the fact that beams play an important role in structural mechanics, some researchers have paid particular attention to the combination of flexible beams with SMA wires and strips. Chaudhry and Rogers (1991), for example, considered the bending of a beam under external attachment of SMA wire and demonstrated the possibility of using this configuration for shape control applications. The drawback of this research is that the actuation force of the SMA wire was not derived from the constitutive equations and a fixed value of an attached load was used. In addition, in order to solve the derived equations and get a closed-form solution, they made use of the linear Euler–Bernoulli beam assumption. Therefore, the region in which these results were valid was limited and could not be extended to large deflection cases. They showed in their research that by increasing the number of points on the beam through which the actuator passes, the actuation force for getting a specified tip deflection soars accordingly. However, in these configurations the beam behavior is less nonlinear; consequently, the tip deflection or shape control turns out to be easier. Increasing the number of points on the beam through which one SMA actuator passes, as well as increasing the number of the actuators, therefore, is a great advantage for control applications.

Likewise, Shu et al. (1997) developed a thermomechanical model to predict the structural response of a flexible beam actuated with externally attached SMA wire actuators. They first carried out a geometrically nonlinear static analysis to investigate the deformed shape of a flexible cantilever beam caused by an externally attached SMA wire actuated electrically. In that paper, a one-dimensional simplification of the three-dimensional model developed by Boyd and Lagoudas (1994) was used for predicting the thermomechanical response of the attached SMA actuator. The actuation force applied by the SMA actuator to the beam was evaluated by solving a coupled problem combining a thermodynamic constitutive model of SMAs with the heat transfer equation in the SMA and the structural model of the beam. Despite the fact that two SMA wires – one active and another inactive – were attached to the group’s experimental beam set-up, they did not take into account the coupling effect of the inactive wire with the beam structure. They also used a linear constitutive relation between the curvature of the beam and the moment, which means that their results are valid only for the beam with small rotation (consequently small deflection) behavior. In addition, the prediction of the proposed model was not verified against the experimental results.

Similar work was carried out by Brinson et al. (1997). They studied the case of a cantilever beam with externally attached SMA wire. The work took advantage of Brinson’s constitutive law for thermomechanical behavior of SMAs and then coupled it with linear and nonlinear
behavior of the controlled beam. In spite of the fact that they considered the effects imposed by the nonlinear terms, the connection point of the SMA wire to the beam basement was selected at the root end of the beam, and this imposed some restrictions on the results especially for the practical case where this point can be at any position. In addition, only one SMA wire was chosen for the actuation of the beam and the compound effect of the SMA heat transfer equation with the model was ignored. Therefore, their results are not attractive for practical applications in which the number of active SMA wires is not limited to one and also the electric currents of wires (and not their temperatures) are the system inputs. Also, the model simulations are not verified with respect to the experimental results of a test set-up.

Moallem (2003), on the other hand, proposed a nonlinear control scheme for deflection control of a flexible beam system using SMA wires. Taking the equation of linear Euler–Bernoulli beam and thermal characteristics of SMA wire into account, he developed a control scheme in order to regulate the force exerted by an SMA actuator attached to a flexible beam. Although two SMA wires—one active and another inactive—in a diagnostic configuration were used, he neglected the effect of inactive wire in the system of equations.

Elsewhere, Sohn et al. (2009) investigated the control performance of a flexible beam structure by adopting SMA actuator with robust control algorithm. Here, an antagonistic type of actuator using two SMA wires is installed to a flexible beam structure. The governing equation of motion of the proposed flexible structure is obtained via Hamilton’s principle by considering the linear Bernoulli beam and thermal characteristics of the SMA wire actuator are experimentally identified and incorporated with the governing equation. Although the proposed model has the inadequacies of the aforesaid research, it is sufficient and accurate for vibration control where the structure is in small deflection mode.

In the present research, the above-mentioned limitations of attaching SMA actuators to the smart structures have been resolved, and nonlinear modeling of a flexible beam actuated by two active SMA actuators is carried out. In other words, instead of linear beam theory, the nonlinear beam theory is used in modeling and, therefore, the results are valid in large deflection and rotation behavior of beam. Also, unlike many analyses that consider the effect of only one SMA wire (despite the fact that more than one SMA wire is connected to the structure), in this research the effect of all active and inactive SMA wires is considered. Accordingly, the result of this paper can easily be generalized to the structure with several SMA wire actuations. Moreover, with the purpose of having practical application in modeling and control, the heat transfer equations of all SMA wires are considered in the analysis and, as a result, the inputs of the current model are SMA wire electric currents and not their temperatures.

In the first section of this paper, nonlinear formulation for a flexible beam, under two applied forces, is derived. To consider the generality of the modeling, the connection point of the SMA wires to the beam basement is selected at any position in the plane. In the second section, the Brinson thermomechanical constitutive equation of SMA wires (with the corrected evolution kinetics developed by Chung et al. (2007)) is reviewed because of its simplicity and its applicability to the entire range of thermomechanical conditions. Thermoelectric heat transfer equation of SMA actuators and the response of one SMA wire under the step and ramp input electric current in free-stress case are investigated in the third section. Finally, in the last section the thermoelectric heat transfer and thermomechanical equations of SMA wires coupled with non-linear load deflection behavior of a beam are solved numerically and the beam, as well as wire behavior, is shown for some practical cases. The results of the proposed model are also verified with respect to the PC-based experimental set-up values. The experimental results show that the presented model can predict the behavior of the smart structure with externally attached SMA wires with moderate accuracy.

**NON-LINEAR ANALYSIS OF A FLEXIBLE BEAM UNDER TWO AXIAL FORCES**

A real smart structure can be made of numerous actuators and complex structural members. However, because in this paper the tip position of the structure is supposed to be controlled by two SMA actuators, a smart structure composed of an elastic beam and two different SMA wires is considered. The arrangement of the beam and the two SMA wires prior to and following the deformation is schematically shown in Figure 1.

As most of the SMAs undergo a change in behavior under cycling loading (Brinson et al., 1997), it is assumed that the SMA wires have been initially stabilized and then attached to the beam. Incidentally, before attaching the SMA wires to the beam they are subjected to a tensile stress in order to induce some prestains in wires. Next, one side of wire 1 and wire 2 are each....

**Figure 1.** The schematic illustration of the beam structure and SMA wires before and after deformation.
attached to the beam with offset distances $d_1$ and $d_2$, respectively. Wire 2 is attached to the tip of the beam while wire 1 is attached to the middle. As stated before, in order to have generality in modeling, the other sides of the wires are fixed to the base at $(x_{01}, y_{01})$ and $(x_{02}, y_{02})$ positions, respectively. Increasing the temperature of SMA wires cause martensite to austenite transformation in wires, which creates stress and strain in the beam and, as a result, deflects it.

Let $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ be the coordinates of points on the beam in the wires are attached (points A1 and A2 in Figure 1) and $(x_1, y_1)$ and $(x_2, y_2)$ are the coordinates of wire end-points in the fixed $X-Y$ Cartesian coordinate system, respectively, as shown in Figure 2. Also, assume that the pair $(x, w(x))$ defines the coordinate of any points on the beam in the mentioned coordinate system in which the $w(x)$ is the deflection of the point $x$ from the $x$-axis. Moreover, $w'(x)$ is the slope of the any point of the beam with coordinate $x$.

It is clear from this figure that the relations between $x_1, y_1, x_2, y_2$ and $\xi_1, \eta_1, \xi_2, \eta_2$ can be written as:

$$
\begin{align*}
    x_1 &= \xi_1 - d_1 \sin(\tan^{-1}(w'(\xi_1))), \\
    y_1 &= \eta_1 + d_1 \cos(\tan^{-1}(w'(\xi_1))), \\
    x_2 &= \xi_2 - d_2 \sin(\tan^{-1}(w'(\xi_2))), \\
    y_2 &= \eta_2 + d_2 \cos(\tan^{-1}(w'(\xi_2)))
\end{align*}
$$

Moreover, the cosines and sines of the parameters $\theta_1$ and $\theta_2$, denoted respectively by $C_1, S_1, C_2, S_2$, are:

$$
\begin{align*}
    C_1 &= \cos(\theta_1) = \frac{x_1 - x_{01}}{\sqrt{(x_1 - x_{01})^2 + (y_1 - y_{01})}}, \\
    S_1 &= \sin(\theta_1) = \frac{y_1 - y_{01}}{\sqrt{(x_1 - x_{01})^2 + (y_1 - y_{01})}}, \\
    C_2 &= \cos(\theta_2) = \frac{x_2 - x_{02}}{\sqrt{(x_2 - x_{02})^2 + (y_2 - y_{02})}}, \\
    S_2 &= \sin(\theta_2) = \frac{y_2 - y_{02}}{\sqrt{(x_2 - x_{02})^2 + (y_2 - y_{02})}}
\end{align*}
$$

By getting moment about point $(x, w(x))$, the distribution of bending moment, $M$, along the beam span is obtained as:

$$
M = \begin{cases} 
(F_1 C_1 + F_2 C_2) w(x) - (F_1 S_1 + F_2 S_2) x \\
+ F_1 (S_1 x_1 - C_1 y_1) + F_2 (S_2 x_2 - C_2 y_2) & 0 \leq x \leq \xi_1 \\
F_2 C_2 w(x) - F_2 S_2 x + F_2 (S_2 x_2 - C_2 y_2) & \xi_1 < x \leq \xi_2 
\end{cases}
$$

The differential equation governing the bending moment and the beam deflection is:

$$
\frac{-M}{EI} = \kappa = \frac{w''(x)}{(1 + w'(x))^3}^{3/2}
$$

Knowing the two axial forces $F_1$ and $F_2$, one cannot find the deflection $w(x)$ of the beam because there are four unknowns $\xi_1, \eta_1, \xi_2, \eta_2$ whose finding requires four equations. These equations are as follows:

$$
\begin{align*}
    w(\xi_1) &= \eta_1 \\
    w(\xi_2) &= \eta_2 \\
    \int_0^{\xi_1} \sqrt{1 + (w'(x))^2} \, dx &= L_1 \\
    \int_0^{\xi_2} \sqrt{1 + (w'(x))^2} \, dx &= L_2
\end{align*}
$$

Equations (5a and b) indicates that pair points $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ are on the beam and therefore should satisfy the beam bending equation. Equations (5c and d) are the inextensional hypothesis of the beam, indicating that the length of the beam remains unchanged in the beam axial direction after bending.

Knowing the two axial forces $F_1$ and $F_2$, and by solving Equations (3) to (5) iteratively, utilizing the bisection technique, one is able to determine the beam deflections.

**SIMULATION RESULT OF BEAM EQUATIONS**

Before investigating the response of the beam under the thermal actuation of SMA wire strings, first the load deflection behavior of the beam is simulated. In order to visualize the behavior of the beam under different load conditions, a short series of simulations is carried out with the beam arrangement illustrated in Figure 1. The geometric parameters and material properties of an aluminum (7075-T6) beam are given in Table 1 as an example. As, in the case of a beam actuated by one force, the nonlinear theory tracks the experimental data more accurately and the linear theory is valid only for small deflections (Brinson et al., 1997), these simulations are carried out for nonlinear assumptions (i.e. $w'(x) \gg 0$), which have greater generality than the linear assumptions (i.e., $w'(x) \approx 0$).
Table 1. Geometry parameters and material properties of an aluminum (7075-T6) beam used for numerical simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the beam at connection point with wire.1</td>
<td>400</td>
<td>mm</td>
<td>L₁</td>
</tr>
<tr>
<td>Length of the beam at connection point with wire.2</td>
<td>200</td>
<td>mm</td>
<td>L₂</td>
</tr>
<tr>
<td>Width</td>
<td>25</td>
<td>mm</td>
<td>b</td>
</tr>
<tr>
<td>Thickness</td>
<td>1.27</td>
<td>mm</td>
<td>t</td>
</tr>
<tr>
<td>First force offset distance</td>
<td>5</td>
<td>mm</td>
<td>d₁</td>
</tr>
<tr>
<td>Second force offset distance</td>
<td>10</td>
<td>mm</td>
<td>d₂</td>
</tr>
<tr>
<td>Position of first force support</td>
<td>(0, 5)</td>
<td>mm, mm</td>
<td>(x₀₁, y₀₁)</td>
</tr>
<tr>
<td>Position of second force support</td>
<td>(0, 10)</td>
<td>mm, mm</td>
<td>(x₀₂, y₀₂)</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>70</td>
<td>GPa</td>
<td>E</td>
</tr>
<tr>
<td>Yield stress</td>
<td>410</td>
<td>MPa</td>
<td>σₚ₀₂₀</td>
</tr>
</tbody>
</table>

Figure 3 shows the variation of the end point deflection (point A₂ in Figure 1) with respect to the change in \( F₁ \) and \( F₂ \). From this figure two points can be borne in mind. First, in large \( F₁ \) and \( F₂ \) forces the system is much more nonlinear and sensitive to loading variations; that is, position control at large deflections is very difficult. Second, as it transpires that the change in the end point deflection with respect to \( F₂ \) is more severe than with respect to \( F₁ \), i.e., because of the larger moment of \( F₂ \) about the base of the beam, the specified variation of \( F₂ \), when the same variation in \( F₁ \) is enforced leads to more deflection change in the end point. Thus, it can be inferred that to control the position of the end point it is easier to change \( F₁ \) rather than \( F₂ \).

This explains why the configuration shown in Figure 1 uses two forces for control position rather than one force. Although the end point position can be controlled by either \( F₁ \) or \( F₂ \), this brings about two problems. First, if \( F₂ \) is applied, albeit the end point can reach the large deflection, the position control of this point will be difficult. Applying \( F₁ \), plus \( F₂ \), increases not only the end point workspace but also its controllability. Second, although the controllability is great, using \( F₁ \) cannot lead to the large deflection positions.

Therefore, fair controllability in large deflection positions requires that two forces instead of one are applied at the end point of the wire. The detail of position control of this structure in large deflection mode is beyond the scope of the current study.

For evaluating the current analysis of the cantilever beam under the above-mentioned forces, ABAQUS Finite Element Method (FEM) code is also used. In this software, the beam is modeled with 22 three-node quadratic two-dimensional beam elements. Each node has three degrees of freedoms (two translations along the \( x \)- and \( y \)-axes and one rotation about the \( z \)-axis). Boundary conditions and forces in the FEM model are selected similar to the real problem. Therefore, during the simulation, the magnitude and direction of forces are changed in order to have the comparable result. Considering the elastic behavior of beam, for material behavior a simple elastic property of aluminum is allocated to the model. Because the deflection of the beam is about half of its span, a geometric nonlinear analysis should be carried out. For accurate results, this analysis is performed in 50 steps.

The problem is approached in two steps. In the first step, the amount of \( F₂ \) is increased while \( F₁ \) is zero, while in the second step \( F₁ \) has a constant amount of 154 N. The results of two methods, for the two mentioned steps, are depicted in Figure 4. In this figure, the variation of the beam tip deflection during the increase of \( F₂ \) is illustrated. It is clear that, up to 14 cm of beam tip deflection, the results of both methods are very similar, and, up to 18 cm of tip deflection, the difference between the methods is less than 11%. FEM results show that the procedure used to model the beam behavior under the actuation of SMA wire is satisfactory and accurate.

The deformed shape of the beam under \( F₁ = 15N \) and \( F₂ = 13N \) as well as \( F₁ = 15N \) and \( F₂ = 8N \) for the proposed analysis is, also, compared with the results of

![Figure 3](image_url)  
Figure 3. Tip deflection variation with \( F₁ \) and \( F₂ \) changes.

![Figure 4](image_url)  
Figure 4. Comparing the results of the current study with Finite Element Method (FEM) result; the deflection variation of the end point with \( F₂ \) changes while \( F₁ \) is fixed.
the FEM method in Figure 5. It is clear that the presented method can predict the shape of the deformed beam with great accuracy at small deflections as well as good accuracy at large deflections.

Von Mises stress contours, created from the deformed FEM model (under $F_1 = 15N$ and $F_2 = 13N$), are also shown in Figure 6. It can be seen that the maximum amount of beam stress is about 1544 MPa, which is very different from the yield point (>4004 MPa) of AL 7075-T6 that is supposed to be used in the manufacturing of experimental set-up. Thus, the cantilever beam can deflect even more than 18 cm without any plastic deformation.

THERMO-MECHANICAL CONSTITUTIVE RELATIONS FOR SMA WIRES

The phase transformation in SMA materials occurs as a function of both stress and temperature. At zero stress, phase transformation happens at temperatures denoted by $A_s$, $A_f$, $M_s$, and $M_f$, which represent, respectively, austenite start, austenite finish, martensite start, and martensite finish temperatures. Tanaka’s model is one of the first constitutive formulations for SMAs (Tanaka, 1986). It is assumed that strain, temperature, and martensite volume fraction are the only state variables in this model, and the stress is calculated as a function of these variables. Furthermore, phase transformation kinetics is expressed with exponential form and is a function of stress and temperature. Liang and Rogers (1990) formulated a model based on the rate form of the Tanaka constitutive equation, except a cosine model was replaced by the exponential model for the martensite volume fraction.

The major drawback of both the Tanaka and the Liang and Rogers models in their original form is that they describe only the phase transformation from martensite to austenite and its reverse transformation. These models cannot be applied to the detwinning of the martensite that is responsible for the SME at lower temperature. This problem was solved by Brinson’s model (Brinson, 1993). In this model the martensite volume fraction ($\xi$) is separated into stress-induced ($\xi_s$) and temperature-induced components ($\xi_T$):

$$\xi = \xi_s + \xi_T$$

The first form of the constitutive equation in this model, relating the state variable stress ($\sigma$), strain ($\epsilon$), and temperature ($T$), was:

$$\sigma - \sigma_0 = E(\dot{\epsilon})\epsilon - E(\dot{\epsilon}_0)\epsilon_0 + \Omega(\dot{\xi}_s - \dot{\xi}_0)\xi_0 + \Theta(T - T_0)$$

where ($\sigma_0$, $\epsilon_0$, $\xi_0$, $T_0$) represent the initial state or original condition of the material and $\Theta$ is the thermal coefficient of expansion. In this equation, $E$ is the module of elasticity and assumed to be a linear function of the martensite volume fraction:

$$E(\dot{\xi}) = E_A + \dot{\xi}(E_M - E_A)$$

and $\Omega$ is called phase transformation coefficient and is defined:

$$\Omega(\dot{\xi}) = -\epsilon_L E(\dot{\xi})$$

where $\epsilon_L$ is the maximum recoverable strain. It was shown by Brinson and Huang (1996) that this constitutive equation could be reduced to the simplified form of:

$$\sigma = E(\dot{\xi}(\epsilon - \epsilon_L \xi_s) + \Theta(T - T_0)$$

The critical stress–temperature profiles used in Brinson’s model are shown in Figure 7. The evolution equations for calculation of the martensite fractions according to temperature and stress can now be represented in conjunction with Figure 7 as below.

Conversion to detwinned martensite

For $T > M_s$ and $\sigma_s^{CR} + C_M(T - M_s) < \sigma < \sigma_f^{CR} + C_M(T - M_s)$

$$\xi_s = \frac{1 - \xi_{00}}{2} \cos \left( \frac{\pi}{\sigma_s^{CR} - \sigma_f^{CR}} \left( \sigma - \sigma_f^{CR} - C_M(T - M_s) \right) \right) + \frac{1 + \xi_{00}}{2}$$

$$\xi_T = \xi_{f0} - \frac{\xi_{f0}}{1 - \xi_{00}}(\xi_s - \xi_{00})$$

For $T < M_s$ $\sigma_s^{CR} < \sigma < \sigma_f^{CR}$

Figure 5. Shape of the deformed beam for FEM method and the proposed analysis.
\[ s = \frac{1 - s_{d0}}{2} \cos \left( \frac{\pi}{s_{cr}^f - s_{cr}^f} (\sigma - \sigma_{cr}^f) \right) + \frac{1 + s_{d0}}{2} \]

\[ \xi_T = \frac{\xi_{T0}}{1 - s_{d0}} (\xi_s - \xi_s\_d) + \Delta_T \]

where, if \( M_f < T < M_s \) and \( T < T_0 \):

\[ \Delta_T = \frac{1 - \xi_{T0}}{2} \left[ \cos \left( a_M (T - M_f) \right) + 1 \right] \]

else

\[ \Delta_T = 0 \]

Conversion to austenite

For \( T > A_s \) and \( C_A(T - A_s) < C_A(T - A_f) \)

\[ \xi_s = \frac{\xi_{T0}}{1 - s_{d0}} (\xi_s - \xi_s\_d), \xi_T = \frac{\xi_{T0}}{\xi_{T0} - \xi_{T0}} (\xi_s - \xi_s) \]

In these equations the constants \( a_A \) and \( a_M \) are two material constants in terms of transition temperatures \( A_s, A_f, M_s, \) and \( M_f \) as:

\[ a_A = \frac{\pi}{(A_f - A_s)}, a_M = \frac{\pi}{(M_s - M_f)} \]

The original Brinson model (Brinson, 1993), in the case where \( M_f < T < M_s \) and \( \sigma_{cr}^f < \sigma < \sigma_{cr}^f \), can be used only with specific initial conditions, otherwise it gives rise to a physically inadmissible volume fraction (\( \xi > 1 \)). Although after Brinson’s original model, Bekker and Brinson developed a kinetics that was robust and did not permit volume fractions to exceed unity (Brinson and Huang, 1996), Brinson’s original model is still widely used in formulae describing the behavior of SMAs. To meet the above conditions, a modification of Brinson’s martensite kinetics by Chung et al. (2007) is developed. In this formulation, Equation (12) is revised as:

For \( T < M_s \) and \( \sigma_{cr}^f < \sigma < \sigma_{cr}^f \):

\[ \xi_s = \frac{1 - s_{d0}}{2} \cos \left[ \frac{\pi}{s_{cr}^f - s_{cr}^f} (\sigma - \sigma_{cr}^f) \right] + \frac{1 + s_{d0}}{2} \]

\[ \xi_T = \Delta_T - \frac{\Delta_T}{1 - s_{d0}} (\xi_s - \xi_s\_d) \]

where, if \( M_f < T < M_s \) and \( T < T_0 \):
wire actuator with resistive heating is investigated in the current study.

Shu et al. (1997) showed that for long and fine wires, used in many control applications, the temperature distribution is uniform along the length of the wires (except at the two ends) and in the cross-section. Thus, the effect of thermal conductivity of wires can be ignored. Recently, Mirzaeifar, DesRoches and Yavari (2011) showed that temperature distribution in a cross-section of SMA wires is affected by loading rate, ambient condition, and size of the specimen. Because in this study the SMA wire diameter in use is so small, and assuming the uniform temperature in the wire cross-section does not lead to great differences between the simulation and experimental results, these effects are ignored. In such cases, the equation governing the aforementioned problem can be reduced into the following simplified equation:

\[
C_v(T, \sigma) \frac{\partial T(t)}{\partial t} = -\frac{A}{D} h(T(t) - T_{amb}) + \rho_e J^2
\]  

where \( T(t) \) is the temperature of the wire at time \( t \), \( D \) is the diameter of the wire, \( \rho_e \) is the electrical resistivity of the wire, \( J \) is the magnitude of the current density (i.e. \( \frac{I}{A} \), where \( A \) is the wire cross-section area), \( T_{amb} \) is the ambient temperature, \( C_v(T, \sigma) \) is the heat capacity, and \( h(T, D) \) is the convective heat coefficient.

Bhattacharyya et al. (1995) proposed an empirical relation describing the dependence of \( C_v \) on \( T \) and \( \sigma \) as follows:

\[
C_v = \begin{cases} 
C_v^0 + q \left[ \frac{\ln(100)}{(M_f - M_s)} \right] 
& \text{if } M_f + \frac{\sigma}{\sigma^e} \leq T \leq M_s + \frac{\sigma}{\sigma^e} \\
\text{else} & \Rightarrow C_v^0
\end{cases}
\]

\[
C_v = \begin{cases} 
C_v^0 + q \left[ \frac{\ln(100)}{(A_f - A_s)} \right] 
& \text{if } A_s + \frac{\sigma}{\sigma^e} \leq T \leq A_f + \frac{\sigma}{\sigma^e} \\
\text{else} & \Rightarrow C_v^0
\end{cases}
\]

for the forward transformation, and

for the reverse transformation. It should be mentioned here that these equations are somewhat different from the equations used in their paper because, as previously discussed, stress has a profound effect on the transformation temperature and consequently on heat capacity. However, in their study, Bhattacharyya et al. (1995) overlooked this effect and in their model stress has no influence on heat capacity.

In this paper, for analysis of the Shape Memory Effect (SME) and superelastic behavior of SMA components, the Brinson model with the corrected evolution kinetics developed by Chung et al. (2007) is applied. In order to obtain the required thermomechanical properties of a nickel–titanium (Ni–Ti) alloy, experimental measurements were carried out on a Flexinol™ actuator wire, manufactured by Dynalloy Inc (CA, USA). For the experiment, one-way shape memory, 0.01-inch-diameter (0.254 mm), low-temperature (70°C), Ni–Ti SMA actuator wire has been selected. The details of these tests are reported in a separate paper (Zakerzadeh and Salehi, 2009) and the experimentally derived parameters related to the Brinson model can be seen in Table 2.

**Table 2. Experimentally derived SMA parameters and properties.**

<table>
<thead>
<tr>
<th>Material parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_f )</td>
<td>43.9</td>
<td>°C</td>
</tr>
<tr>
<td>( M_s )</td>
<td>48.4</td>
<td>°C</td>
</tr>
<tr>
<td>( A_f )</td>
<td>68</td>
<td></td>
</tr>
<tr>
<td>( A_s )</td>
<td>73.75</td>
<td>°C</td>
</tr>
<tr>
<td>( C_A )</td>
<td>6.73</td>
<td>MPa/°C</td>
</tr>
<tr>
<td>( C_M )</td>
<td>6.32</td>
<td>MPa/°C</td>
</tr>
<tr>
<td>( \epsilon_f )</td>
<td>4.1</td>
<td>%</td>
</tr>
<tr>
<td>( E_A )</td>
<td>31.5</td>
<td>GPa</td>
</tr>
<tr>
<td>( E_M )</td>
<td>20</td>
<td>GPa</td>
</tr>
<tr>
<td>( \sigma_f )</td>
<td>25</td>
<td>MPa</td>
</tr>
<tr>
<td>( \sigma_M )</td>
<td>78</td>
<td>MPa</td>
</tr>
</tbody>
</table>
The convection coefficient of an isothermal horizontal wire, as a function of its temperature and its diameter, is as follows (Pathak et al., 2010):

\[ h(T, D) = \frac{k}{D} \frac{Nu}{Nu} \]  

(18)

where \( k \) is the thermal conductivity of the surrounding air and \( Nu \) is the average Nusselt number for free convection of the SMA wire. In this empirical model, the dimensionless Nusselt number can be related to the Rayleigh number \( (Ra = Gr Pr) \) as follows:

\[ Nu = C Ra^m \]  

(19)

where the dimensionless Grashof number \( Gr \) and the Prandtl number \( Pr \) are given as (Pathak et al., 2010):

\[ Gr = \frac{2(T - T_\infty)gD^3}{(T + T_\infty)\nu^2}, \quad Pr = \frac{\mu C_p}{k} \]  

(20)

In Equation (20), \( g \) is the gravitational acceleration \( (g = 9.81 \text{ m/s}^2) \), and \( \nu, \mu, C_p, \) and \( k \) are the kinematic viscosity, viscosity, heat capacity at constant pressure, and thermal conductivity of the surrounding air, respectively.

To obtain \( C \) and \( m \) in Equation (19), the value of \( h \) should be correlated with a set of experimental data. Using a generalized reduced gradient algorithm (GRG) to minimize the average fit error with respect to the collected data, Pathak et al. (2010) find the following value for these values for the two Rayleigh regimes:

\[ C = 0.875, \quad m = 0.038 \]  

\[ C = 1.477, \quad m = 0.142 \]  

(21)

As it is found that the \( Ra \) number for the wire in hand is in the second Rayleigh regime, the corresponding \( C \) and \( m \) values are chosen.

It should be mentioned that the physical parameters of Equations (18) and (20) should be evaluated at the average temperature \( (T_{ave} = \frac{T_a + T_{amb}}{2}) \). The properties of air at atmospheric pressure are obtained from Table A-5 of Holman (2001), reproduced here as Table 3.

As the working temperature of the SMA wire in hand is between \( 40^\circ \text{C} \) (below \( M_f = 44^\circ \text{C} \)) and \( 140^\circ \text{C} \) (the austenite finish temperature at \( \sigma = 450 \text{MPa} \)), and assuming the temperature of the surrounding air is \( T_{amb} = 20^\circ \text{C} \), the average temperature is \( 30^\circ \text{C} \leq T_{ave} \leq 80^\circ \text{C} \) and the data of this table are sufficient. By assuming that the value of these parameters is changed linearly between \( 300^\circ \text{C} \) and \( 400^\circ \text{C} \), the value of Grashof number \( Gr \) and the Prandtl number \( Pr \) in any temperature can be easily computed using Equation (20). Then, by using Equations (18), (19), (20), and (21), the convection coefficient can be obtained in each temperature in the mentioned working temperature range.

Figure 8 shows the change of convective coefficient with the temperature of SMA wire. It can be seen that the convective heat transfer increases as the wire temperature is increased. It is due to the fact that by rising the wire temperature (assuming the fixed ambient temperature) the rate of heat transfer climbs as a result of increase in the temperature difference between the actuator and the ambient. It is found that the change of \( h \) with temperature can be fitted by a four-degree polynomial and the maximum error of this fitting is below 0.15°C. This fitted equation is:

\[ h(T) = -4.034 \times 10^{-7}T^4 + 0.0001654T^3 - 0.02586T^2 
+ 2.071T + 41.96 \quad T(\circ \text{C}) \text{ and } h(\text{W.m}^{-2} \cdot \circ \text{C}) \]  

(22)

In order to solve Equation (15), three other parameters \( (C_{en}, q, \text{ and } \rho_e) \) should be obtained. By performing the differential scanning calorimetric (DSC) test, the first and second parameters are obtained as:

\[ C_{en} = 2.046 \times 10^6(\text{J.m}^{-3} \cdot \circ \text{C}^{-1}), \quad q = 0.0656 \times 10^9 \text{J.m}^{-3} \]

Table 3. Properties of air at atmospheric pressure at 300, 350 and 400 kelvin (Holman, 2001).

<table>
<thead>
<tr>
<th>( T(\circ \text{C}) )</th>
<th>( C_p(\text{J} \cdot \text{K}^{-1} \cdot \text{kg}^{-1}) )</th>
<th>( \mu(10^{-5} \text{Pa.s}) )</th>
<th>( \nu(10^{-6} \text{m}^2/s) )</th>
<th>( k(\text{W} \cdot \text{m}^{-1} \cdot \circ \text{C}^{-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>1.0057</td>
<td>1.8462</td>
<td>15.69</td>
<td>0.02624</td>
</tr>
<tr>
<td>350</td>
<td>1.0090</td>
<td>2.075</td>
<td>20.76</td>
<td>0.03003</td>
</tr>
<tr>
<td>400</td>
<td>1.0140</td>
<td>2.286</td>
<td>25.90</td>
<td>0.03365</td>
</tr>
</tbody>
</table>
Although the electrical resistivity is also a function of the wire temperature, its change with temperature is not considerable (below 8%; Dynalloy data sheet, 2009) and lack of reliable data also forces us to assume it is a linear function of temperature difference between the actuator and the ambient. The resistivity of austenite and martensite phases are selected from our data sheet (Dynalloy data sheet, 2009) as $\rho_A = 82 \times 10^{-8} \, \Omega \, m$ and $\rho_M = 76 \times 10^{-8} \, \Omega \, m$.

The related heat transfer parameters are summarized in Table 4.

<table>
<thead>
<tr>
<th>Material parameters</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{v0}$</td>
<td>$2.046 \times 10^6$</td>
<td>$J \cdot m^{-3} \cdot C^{-1}$</td>
</tr>
<tr>
<td>$q$</td>
<td>$0.0656 \times 10^7$</td>
<td>$J \cdot m^{-3}$</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>$82 \times 10^{-8}$</td>
<td>$\Omega , m$</td>
</tr>
<tr>
<td>$\rho_M$</td>
<td>$76 \times 10^{-8}$</td>
<td>$\Omega , m$</td>
</tr>
<tr>
<td>$T_{amb}$</td>
<td>$20$</td>
<td>$^\circ C$</td>
</tr>
<tr>
<td>$h(T)$</td>
<td>$-4.034 \times 10^{-7} \times 10^3 + 0.0001654 \times 10^3$</td>
<td>$W \cdot m^{-3} \cdot C^{-1}$</td>
</tr>
</tbody>
</table>


These equations are helpful only in obtaining the approximate input electric current required to reach the maximum desired temperature and the approximate time for which the current should persist (for step current input) before running the simulation. For example, in special cases, by assuming $C_v = C_{v0} = 2.046 \times 10^6 (J \cdot m^{-3} \cdot C^{-1})$, $h = h(T = 55^\circ C) = 101.4656 (W \cdot m^{-3} \cdot C^{-1})$, $D = 2.54 \times 10^{-4} m$, and by supposing that the maximum desired temperature is $T_{ss} = 140^\circ C$, the following values are obtained for the time constant, settling time and the step input current:

$$\tau \approx 1.29 (sec)$$

$$t_{setting \, time} \approx 5.13 (sec)$$

$$I_{step} \approx 0.85 (A)$$

In stress-free cases (i.e., $\sigma = 0 \, MPa$), and under the step and ramp input electric current, the corresponding temperature profile is obtained (by numerical solving of Equation (15)) and is shown in Figure 9. It should be mentioned that regions A and B, indicated in the temperature profiles, are, respectively, regions where inverse and forward transformations happen.

It can be seen from Figure 9 that the total cooling time is more than the corresponding heating time (for step input: $t_{heating} \approx 5 \, sec$, $t_{cooling} \approx 10 \, sec$ and for ramp input $t_{heating} \approx 9 \, sec$, $t_{cooling} \approx 14 \, sec$). As a result, the heating is mostly controlled by the magnitude of the input electric current term $\rho_c I^2$, whereas the cooling rate depends only on the heat convection between the SMA wire and surrounding air, and it is a linear function of the temperature difference between the actuator and the ambient temperature (the $\left(-\frac{1}{h} \frac{d}{dT} h(T)(T - T_{amb})\right)$ term).

It should be mentioned here that for heat transfer problems, all the results of the next section are obtained by considering Equation (15), which assumes non-constant heat capacity and non-constant convection coefficient.

**SIMULATION OF STRUCTURAL RESPONSE AND RESULTS**

Computer simulations of the SMA actuated beam model were performed in MATLAB. A block diagram of the model is shown in Figure 10. It can be seen from this figure that the inputs of the SMAs actuated beam model are the wires’ currents while its output is the strain of the corresponding wires. The geometry of the aluminum beam to be investigated in this study was given in Table 1. In addition, the material parameters of the SMA actuator wires were given in Table 2.

As different initial conditions of each SMA wire (different detwinned martensitic volume fraction and pre-stress) as well as different algorithms of heating and cooling of each SMA, affect the behavior of
stress–strain in each wire and the beam behavior, several simulations are studied. In the initial state of each parameter, the superscript ‘1’ refers to wire 1 and the superscript ‘2’ refers to wire 2.

In order to investigate the effect of including or excluding the SMA wire 1 on the variation of parameters, two graphs are illustrated in some figures: one by considering the effect of SMA wire 1 (entitled Case A) and another by neglecting the effect of SMA wire 1 (entitled Case B), i.e., assuming that only wire 2 is connected to the beam. As the results show, in order to better predict the behavior of a smart structure with numerous SMA wires, the effect of all (active and inactive) ones should be considered in the analysis.

In addition, in all simulations the ON/OFF electric current is applied to the wires. The duration of the ON section of the electric current at each simulation is selected in such a way that the temperatures of both wires saturate to their steady states. Therefore, it is not the same for all simulations and there is a slight difference between these duration times. Also, the OFF section duration of the electric current is selected in such a way that the temperatures of both wires reach the ambient temperature.

**First Simulation**

In order to investigate the sole effect of thermal actuation of wire 2 on the beam behavior, in this simulation the temperature of wire 2 is changed by applying the input step electric current 0.85 A while no electric current is applied to wire 1. As a result, the temperature of wire 1 is fixed at 20°C (with 100 MPa initial prestress) and the temperature of wire 2 is increased from 20°C to 136°C (to allow full transformation to take place). The conditions and processes of this simulation are summarized below:

\[
\begin{align*}
\text{Initial Condition :} & \quad \{ \xi_1^{1} = 1, \xi_2^{1} = 0, \sigma_1^{0} = 100 \text{Mpa}, \\
& \quad \{ \xi_1^{2} = 0.5, \xi_2^{2} = 0.5, \sigma_2^{0} = 0 \text{Mpa} \}, \\
& \quad T_1^{1} = 20^\circ \text{C (fixed)}, \\
& \quad T_2^{0} = 20^\circ \text{C}, I_1 = 0.85 \text{A} \\
\text{Heating Algorithm :} & \quad \{ (\text{ON – OFF current}) \}, \\
& \quad \{ t_{\text{heating}} = 13.53 \text{ sec}, \\
& \quad t_{\text{cooling}} = 17.64 \text{ sec} \}
\end{align*}
\]

In Figure 11, the stress of wire 2 is plotted as a function of its temperature changes for Case A and Case B.
The light dashed lines on the plot indicate the transformation strips displayed earlier in Figure 7. It can be seen from this figure that even in this simulation wire 1 is inactive, but ignoring it can contribute to erroneous prediction of stress in wire 2. In other words, in the cases in which the effect of wire 1 is neglected, the model overpredicts the stress of wire 2. As will be shown later, this effect is greater in the cases in which SMA wire 1 has thermal actuation beside SMA wire 2. Similarly, in Figure 12 the stress of wire 1 is shown as a function of SMA wire 2 temperature for the first simulation.

![Figure 11](image1.png)  
**Figure 11.** Stress in SMA wire 2 as a function of its temperature for Case A and Case B in the first simulation (the dashed lines are the SMA transformation strips).

![Figure 12](image2.png)  
**Figure 12.** Stress in SMA wire 1 as a function of SMA wire 2 temperature for the first simulation.

Case B for this special simulation. Likewise, in Figure 15 the temperature change of wire 2 as a function of time is shown for Case A and Case B.

By comparing the results of Case A and Case B in this simulation, it is concluded that for applications where precise modeling is required the effect of all SMA wires should be considered, even if some of them are inactive and have no thermal actuation. Since Case A considers the effect of inactive SMA wire 1 and has more precise results, the subject of the following paragraphs is Case A.

As can be seen from these figures, the initial increase in the temperature of wire 2 causes a slight decrease in wire 1 stress and a slight increase in wire 1 stress (due to escalation of the wire 1 connection point). If unconstrained, wire 2 recovers its strain; however, since it is attached to the beam and the beam provides a resistance to that deformation, the stress in this wire continues to increase as its temperature increases. On the other hand, because the temperature of wire 1 is unchanged, its initial stress continues to decrease because of the rise of wire 1 connection point. At this stage, the end point...
deflection mounts a little and no change in the stress-induced martensite fraction of any wire takes place. It is worth mentioning that, with respect to Equation (13), since there is no temperature change in wire 1, the stress-induced martensite fraction of this wire does not change during the entire heating process. The initial tip deflection of the beam in Case A is also due to the initial prestress of wire 1 before attachment.

When the critical phase transformation temperature of wire 2 is reached, because of the inverse phase transformation to the austenite phase, continual increase in the temperature of this wire causes the stress-induced \( (\xi_s) \) and temperature-induced \( (\xi_T) \) martensite fraction of this wire to plummet (the change of \( \xi_T \) is not shown in these figures). Thus, the increase in stress of wire 2 and the decrease in stress of wire 1 become significant.

In addition, as can be seen from Figure 13, the tip of the beam (point \( A_2 \) in Figure 1) is notably deflected as wire 2 recovers its initial strain. In the middle of this stage, the stress in wire 1 reaches zero and, because the wire components cannot withstand the pressure, it remains zero after this time and then loosens.

This process continues until the mentioned phase transformation completes and all of the stress-induced \( (\xi_s) \) martensite fraction of wire 2 becomes zero. At this point, the sharp increase in the stress of wire 2 and tip deflection ends and the increase in the temperature of wire 2 does not have any significant effect in the rest of the heating process.

At the end of the heating process (i.e., \( T_2 = 136^\circ C \)), the tip deflection value reaches 68.9 mm and the stress of wires 1 and 2 reaches 0 and 2264 MPa, respectively. This maximum stress in wire 2 is on the order of the selected Ni-Ti wire in reversible phase transformation regions, and it is significantly below the plastic deformation stress. These results verify the good selection of the beam geometry in the design stage.

The beam response in the cooling process of wire 2 is similar to the heating process. That is, at the beginning of this process the stress-induced \( (\xi_s) \) martensite fraction of both wires does not change. Furthermore, the stress in wire 2 and the end point deflection decrease slightly. However, the stress in wire 1 remains unchanged because at the end of heating process it had loosened so its corresponding connection point first should come to the position that this wire begins to extend.

This trend continues until the forward phase transformation to the martensite phase for wire 2 begins. Here, the stress-induced martensite fraction \( (\xi_s) \) of wire 2 starts to climb, leading to dramatic decline in the wire 2 stress as well as the end point deflection. Nonetheless, as wire 1 is still loose, its stress remains unaffected until the middle of this state when its beam connection point comes to a position where further cooling of wire 2 enhances wire 1 stress.

The increase in the \( \xi_s \) of wire 2 continues until its temperature diminishes to the martensite start temperature \( M_s \) where cooling of wire 2 does not change its stress-induced \( (\xi_s) \) martensite fraction, considering Equation (14) and its stress levels until the end of the process. Needless to say, as expected and discussed before, the duration of the heating process, that is 13.53 s, is less than the duration of the cooling process (17.64 s). In addition, inverse and forward transformation regions in temperature profiles (A and B regions in Figure 9) last longer in this simulation with respect to the free-stress simulation shown in Figure 9. It is because of this that the forward and inverse transformations in the free-stress case occur between \( M_s \rightarrow M_f \) and \( A_s \rightarrow A_f \) while in this simulation these transformations, respectively, occur between 87°C → Mf and 71.6°C → 113.5°C. This is also seen in other simulations.

It can be clearly seen from these figures that after this first thermal cycle, the magnitude of the stress and stress-induced martensite fraction \( (\xi_s) \) in wires, as well as the end point deflection, do not reach their original initial value. That is, the cold states of the wire after the first cycle do not yield to the same beam shape as their initial states; consequently, the beam does not recover its original shape. This occurs only in the first cycle of heating and cooling of wires (provided the SMA material has stable cyclic behavior) and, as will be shown, in the subsequent cycles the initial and final shapes of the beam are the same. This result is also obtained in by Brinson et al. (1997). This behavior is because when the SMA wires are attached to the beam they have special initial conditions, such as martensite fraction and prestress. Following the first cycle of heating and cooling of SMA wires, their corresponding beam connecting points do not have enough stiffness and flexural rigidity to bring them back to their initial conditions. Therefore, their final conditions are different from their initial conditions and this yields to a different beam shape before and after the first cycle.

This is simulated to show the sole effect of thermal actuation of one wire on the beam behavior.
To investigate the behavior of the beam under both wire thermal actuations, there are infinite patterns. The same thermal actuation for both wires turns out to be the simplest way in the first glance, but it has ample practical applications, especially in the control processes. Further explanation of this is beyond the scope of this paper.

In addition, there are infinite selections for initial stress-induced martensite fraction ($\xi_s$) of wires. However, as the aim of this paper is to reach the larger deflection of beam, we focus on the fully detwinned ($\xi_s = 1$) martensite for the initial state of both SMA wires. When the simulation results for this case (initial condition: $\xi_{s0}^1, \xi_{s0}^2 = 1, \sigma_{0}^1, \sigma_{0}^2 = 0$ and heating algorithm: $I_1 = I_2 = 0.85\text{A} (\text{ON} - \text{OFF current})$) are investigated, it is seen (Figure 16) that the full phase transformation in the heating process occurs only for the second wire, and the first wire does not exit from the phase transformation state (between martensite and austenite phases) even with increasing the wire’s temperature up to 150°C. This is because the stress–temperature profile for this wire is nearly parallel to phase transformation strips, and increasing the wire temperature only increases the stress in wire 1 without departing it from this phase transformation state. Therefore, it is concluded that this simulation is not an appropriate case in this study.

To solve this problem, after several simulations it is concluded that for the same thermal actuation of both wires, two solutions can be proposed. First, decreasing the initial stress-induced martensite fraction of wire 1 and, second, enlarging the length of wire 2 from the back of the base in such a way that the connection point of wire 2 with the base (point $Q_2$ in Figure 1) does not change and the comparison with the previously obtained data remains valid.

Second Simulation

As discussed previously, the initial stress-induced martensite fraction of wire 1 is decreased to 0.6 in this case in order to have full phase transformation for both wires in the heating and cooling process. The conditions and processes of this simulation are summarized below:

- Initial Condition: $\left\{ \begin{array}{l} \xi_{s0}^1 = 0.6, \xi_{s0}^1 = 0.4, \sigma_{0}^1 = 0 \\ \xi_{s0}^2 = 1, \xi_{s0}^2 = 0, \sigma_{0}^2 = 0 \end{array} \right\}$
- Heating Algorithm: $I_1 = I_2 = 0.85\text{A} (\text{ON} - \text{OFF current}) (t_{\text{heating}} = 21.18\text{ sec} \quad t_{\text{cooling}} = 38.82\text{ sec})$

Figure 17 shows the stress of SMA wire 1 as a function of its temperature when the effect of this wire is not neglected (Case A). Also, in Figure 18 the stress of SMA wire 2 is plotted as a function of its temperature for Case
A and Case B. Similarly, in Figure 19 the variation of the wire’s temperature is plotted as a function of time in Case A and Case B. Correspondingly, Figure 20 shows the deflection of the end point as a function of time in the entire heating and cooling process for both aforementioned cases.

As both of the SMA wires are active and both have thermal actuation, the difference between the results of Case A and Case B is significant. In other words, when the effect of SMA wire 1 is neglected in the simulation (Case B) the model overpredicts the stress of SMA wire 2 while underpredicting the deflection of the end point. It is comprehensible because when the sole actuation of one wire is considered in analysis, the whole of the actuation task should be done by one wire and its predicted stresses raised with respect to a case where two wires have thermal actuation. In addition, it cannot yield to a larger deflection of beam, and the deflection of the beam is underpredicted. It clearly shows the advantage of using two active SMA wires for deflecting a beam: reaching a larger deflection with less stress in each SMA wire.

As can be seen from Figures 17 and 18 (Case A), both wires have full phase transformation in the heating process, but since wire 2 experiences the end of this inverse full phase transformation (i.e. $\xi_s = 0$) sooner, its stress decreases at the end of the heating process. In addition, since this SMA wire enters the forward phase transformation region in the cooling process later, this decrease wipes out at the middle of the cooling process. It is worth mentioning that, similar to the first simulation, none of the variables, such as stress in the wire as well as the end point deflection, goes back to its initial value at the end of the cooling process in the first cycle of heating and cooling of wires.

It is also important to point out that despite the stress-induced ($\xi_s$) martensite fraction of wire 1 being only 0.6
and not 1, the maximum end point deflection for this simulation (Case A) is 127.4 mm, about 18 mm more than the first simulation where only one wire actuation existed. It is also seen that, as previously observed in the first simulation, the duration of the cooling process is longer than the duration of the heating process.

Third Simulation and Validation with Experimental Data

The previous simulations were presented only to show the power of the current analysis in the prediction of beam behavior as a result of SMA wire actuations. As was shown in the previous section, if the properties of the SMA wires as well as the beam are experimentally obtained, then the stress, strain, and temperature of SMA wires can easily be computed as a result of known SMA electric currents.

Figures 21 and 22 present a PC-based experimental test set-up and its associated instruments to investigate the capability of the current study in predicting a flexible beam behavior under two SMA wire actuations. The main properties of the SMA wires, the cantilever aluminum beam and the heat transfer parameters of air, respectively, are presented in Tables 1, 2, and 4. The SMA wires are placed horizontally (parallel to the beam’s neutral axis) with one end fixed to the beam (wire 2 at the end and wire 1 at the middle) and the other end to the base of the beam. As the available SMA actuator for this set-up has a moderate maximum recoverable strain (about 4%) and the purpose of this study is achieving a large deflection of the beam, the length of SMA wire 2 is enlarged at the back of the beam base (the added length is 55 cm) in such a way that the connection point of wire 2 with the base (point Q2 in Figure 1) does not change during the cooling and heating processes.

To measure the temperature of each SMA wire, a precision fine wire J-thermocouple (IRC003-BW, Omega, Inc.) with a 0.075-mm bead diameter is bonded to each SMA wire by a two-part epoxy paste (OMEGABOND-200, Omega, Inc.) with very high thermal conductivity and electrical insulation properties. The stress in each wire is also measured individually by means of two S-type load cells (UU-K20, DACELL Co., Ltd). As the tip of the beam does not move on a straight line after the SMA wire actuations, it is connected to a precise

Figure 21. Schematic of the cantilever beam set-up actuated by two active SMA wires.

Figure 22. Experimental test set-up used for verification of the current analysis results.
frictionless rectilinear displacement transducer (PZ12-A-125, GEFRAN, Inc.) while the other side of the transducer is joined to a high-resolution rotary encoder (E50S series, Autonics Corporation). By measuring the length of the transducer and its angle, with respect to their initial quantities, the tip deflection of the beam can easily be computed. In addition, the output voltage of these sensors are fed to a computer-based data acquisition (not shown in Figure 22) using a AD/DA PCI multifunction card (PCI 1711, Advantech Inc.,) and Matlab Data Acquisition Toolbox (Matlab R2008a, Mathworks Ltd.). The activation electric current through the SMA wires are set by the computer-generated voltage controlling a current amplifier that is capable of delivering up to 3 A current. The output electric current of this power supply is proportional to the input voltage.

As connecting the SMA wire with prestress or initial stress-induced martensite fraction lower than unity is experimentally difficult, the conditions and processes of this simulation are selected as following:

Initial Condition: \( \xi_{i0} = 1, \xi_{T0} = 0, \sigma_{0i} = 0; \ i = 1, 2. \)

Heating Algorithm: \( I_1^1 = I_2^2 = 0.85 A \) (ON – OFF current) \( (t_{heating} = 20 \text{ sec} \quad t_{cooling} = 40 \text{ sec}). \)

Figure 23 shows the top view of the deformed structure after the heating process. As it was shown in the previous simulations that ignoring the effect of SMA wire 1 (Case B) has different predictions with the model in which both wires are considered, in this section the data of Case A are compared only with experimental data. The stress profile of SMA wire 1 and SMA wire 2 predicted by the model and that obtained from the experimental actuation of the beam structural system under the on–off input electric currents are shown, respectively, in Figures 24 and 25. The temperature profile of each SMA wire is also compared with experimental data in Figure 26. Moreover, the comparison of the tip deflection profile with data obtained from the experimental set-up is shown in Figure 27.

It can be seen from Figures 24–27 that the results of the proposed model have moderate accuracy with respect to experimental data and that the differences that occur in some regions, as will be explained, are reasonable. In other words, we could rely on the results of the proposed model in the behavior prediction stage of the smart structures that are externally actuated by one or more SMA actuators.

It is clear from Figures 24 and 25 that the experimental stresses of SMA wires are more than the predicted values based on the model at the end of the heating process. Figure 27 also shows that, at the end of the heating process, the experimental value of the beam tip deflection is less than the figure expected from model simulations. These differences are due to two main reasons. First, about 30 mm of each SMA wire is allocated to a thermocouple and, as explained previously, these thermocouples are bonded to the wires through a highly conductive paste. As these thermocouple pastes are rigid after curing, the SMA wires do not have any shape memory features in these regions and it means that in experimental situations we have smaller SMA
wires with respect to the model used in simulations. Therefore, the experimental tip deflection (164.5 mm) is about 10% smaller than the model prediction (181 mm). Second, as is clear from Figures 22 and 23, for connecting the SMA wires as well as linear transducer to the beam, the stiffness of two segments of beam (with 20 mm length) is inevitably increased and, as a result, the overall beam stiffness is higher than the corresponding value of the simulated model. Therefore, at the end of heating process the experimental values of SMA wire stresses are higher than the model prediction while the experimental value of the tip deflection is lower. Also, as a result of this effect, at the end of the cooling process, the experimental values of wire stresses are lower than the corresponding values of proposed model simulation while the tip deflection is lower.

Also, from Figure 27 it is perceived that the experimental values of wire temperatures have no great differences with respect to the corresponding values of the simulated model. It is also important to express the fact that as the temperature of the surrounding air is not controlled, there are some fluctuations in the experimental profiles of wire temperature (especially in SMA wire 2, which is longer) and as a result these fluctuations are also seen in the values of wire stresses and the tip deflection of beam.

In addition, similar to first and second simulations, at the end of the cooling process, none of the variables such as stresses in the wires as well as the end point deflection goes back to their initial values in the first cycle of heating and cooling of the wires.

It should be noted here also that all of the figures shown so far were only the first cycle response of the beam under the wire’s electric actuations. In Figure 28, the experimental tip deflection of the beam is plotted as a function of time by the subsequent second cycle actuation of SMA wires. As can be seen from this figure, at the end of any repeated loading cycle, the beam tip deflection value comes back to its corresponding value at the beginning of that cycle, except in the first cycle.
starting from initial conditions, in which there are some small residual deflections in the beam tip end. This residual behavior and cyclic effects can be seen in the other related parameters and variables of the model too.

CONCLUSION

In this paper, some drawbacks of the research conducted on modeling of smart structures actuated by externally attached SMA actuators have been resolved and the nonlinear modeling of a flexible beam actuated by two active SMA actuators was carried out. Nonlinear formulation of a flexible beam under two applied forces was first derived. Although, in the control prospective, the tip deflection control of a beam can be achieved by only one wire, the sensitivity of the system to the actuation force of that SMA is severe, and this makes it difficult to control the system. Therefore, increasing the number of SMA actuators plays an important role in improving the degree of controllability of a beam. Using more than one actuator also has the advantage of shape control instead of solely position control of one point.

Next, the Brinson thermomechanical constitutive equation of SMA wires was reviewed owing to its simplicity and its applicability to the entire range of thermomechanical conditions. In addition, the thermo-electric heat transfer equation of SMA actuators was presented. Finally, the Brinson constitutive model (with the corrected evolution kinetics developed by Chung et al. (2007)) and heat transfer equations of SMA materials were coupled with the nonlinear beam behavior, and the coupled system of equations was numerically solved for some particular practical cases.

It was seen that in the cases in which there are two active SMA wires, and both have thermal actuation, the results are significantly different from when the main SMA wire is solely considered and the effect of the other is ignored. These different results were observed even in the cases where one of the wires is active and the other is inactive. It means that in order to have precise results the effect of all active and inactive SMA wires should be considered (such as diagnostic configurations).

The results of the proposed model were also verified with respect to the PC-based experimental set-up values. The experimental results showed that the presented model could predict the behavior of the smart structure with externally attached SMA wires with moderate accuracy.

The methodology employed in this paper can be easily extended to the complicated smart structure with externally attached SMA wires. In addition, the appropriate parameters of these smart structures, such as the geometry of structure, the arrangement, and the material properties of SMA actuator wires, can be selected by this method. Furthermore, the behavior of this smart structure can be easily simulated before the manufacture process and the optimal value of those mentioned parameters could be selected easily.

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