Application of the variational iteration method for nonlinear free vibration of conservative oscillators

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Received 12 September 2011; revised 21 February 2012; accepted 24 April 2012

Abstract

In this paper, an analytical solution for the nonlinear free vibration of a conservative oscillator is presented. The nonlinear governing equation is solved by employing the Variational Iteration Method (VIM). This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a function. Obtained results reveal that the proposed method is very effective, simple and exact. In the present investigation, the results of this method are compared with those of the Homotopy Analysis Method (HAM), as well as those predicted by the Runge–Kutta method. The excellent accuracy of the obtained results is demonstrated by comparing them with available analytical and numerical results available in the literature. Furthermore, the numerical results for different particular cases of the problem are presented. The effects of different parameters on the ratio of nonlinear to linear natural frequency of the system are also studied. Consequently, the proposed analytical solution can be used as an efficient tool to study the effects of the material or geometrical parameters in the modeling of devices consisting of nonlinear conservative oscillators for their design and optimization, which requires a large number of simulations.

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1. Introduction

The linearity or nonlinearity of a conservative system is commonly determined by the algebraic relationship between restoring forces and displacement/deflections [1,2]. There are a number of approaches for solving nonlinear equations, which range from completely analytical to completely numerical. Besides all the advantages of using numerical methods, closed form solutions appear more appealing because they reveal physical insights through the physics of the problem. Also, applying analytical methods, parametric studies become more convenient. Moreover, analytical solutions are generally required for the validation of numerical methods and computer software. Although it is now straightforward to find solutions of some problems using computers, it is still rather difficult to solve nonlinear problems employing either analytical or even numerical methods. In addition, in numerical solutions, numerical difficulties may occur if a nonlinear problem possesses singularities or multiple solutions.

Traditional analytical methods, which have been widely used for nonlinear equations, include perturbation methods, such as Lindstedt–Poincare, multiple time scale methods and the generalized averaging method of Krylov–Bogoliubov–Mitropolski [3–5]. Recently, to solve nonlinear problems, new methods have caught broad attention. For example, one can mention the homotopy perturbation method [6,7], the F-expansion method [8,9], and the Exp-function method [10]. Among several analytical methods, the Variational Iteration Method (VIM) is one of the most accurate and efficient for studying nonlinear systems [11]. The variational iteration method, which is developed by Ji-Huan He [12,13], provides an effective and efficient tool for solving an extensive range of nonlinear equations [14–16]. New interpretations, as well
The governing differential equation of motion is obtained as follows:

\[ (1 + 3\varepsilon z u^2) \frac{d^2 u}{dt^2} + 6\varepsilon z u \left( \frac{du}{dt} \right)^2 + \omega_c^2 u + \varepsilon \omega_c^2 u^3 = 0, \tag{1} \]

\[ u(t) = y_2(t) - y_1(t), \tag{2} \]

where parameters \( \varepsilon, \xi, z \) and \( \omega_c \) are defined in the following form:

\[ \varepsilon = \frac{\beta}{K_2}, \tag{3} \]

\[ \xi = \frac{K_2}{K_1}, \tag{4} \]

\[ z = \frac{\xi}{1 + \xi}, \tag{5} \]

\[ \omega_c = \sqrt{\frac{K_2}{m(1 + \xi)}}, \tag{6} \]

The governing equation (1) can be recast in the following form:

\[ \frac{d^2 u}{dt^2} + \Omega^2 u = F[u], \tag{10} \]

Parameters of \( \beta, K_2, K_1, u, m \) and \( \omega_c \) denote the nonlinear spring constant, the linear portion of the nonlinear spring constant, the linear spring constant, the deflection of nonlinear spring, mass and natural frequency, respectively. Moreover, the initial conditions for solving Eq. (1) are:

\[ u(0) = A, \quad \dot{u}(0) = \frac{du}{dt}(0) = 0. \tag{7} \]

It is noteworthy to mention that the initial amplitude, \( A \), can be arbitrarily large. In contrast to some perturbation methods, such as Lindstedt–Poincare, VIM is not limited to small values of initial amplitudes.\[ \text{(16)} \]

It is noted that the same system has been considered by Ganji et al.\ [29] where the first-order approximation of the Iteration Perturbation Method (IPM) has been employed to solve mechanical systems, with single degrees of freedom, containing flexible components consisting of a combination of one linear and one nonlinear spring in series. The overall trends of the predicted results are in agreement with those predicted by IPM.

3. Variational iteration method (VIM)

In this section, we briefly review the VIM to solve a general nonlinear initial value problem. In this method, the problem is initially approximated with possible unknowns. Then, a corrected functional is constructed through a general Lagrange multiplier, which can be identified optimally via the variational theory.\[ \text{(15)} \]To illustrate the basic idea of the method, consider the following general nonlinear system:

\[ L[u(t)] + N[u(t)] = g(t), \tag{8} \]

where \( L \) is a linear differential operator, \( N \) a nonlinear analytic operator, and \( g(t) \) an homogeneous term.

The basic character of the method is to construct a correction functional for the system as follows:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) \left( L[u_n(\tau)] + N[\dot{u}_n(\tau)] - g(\tau) \right) d\tau, \tag{9} \]

where \( \lambda \) is a general Lagrange multiplier, which can be obtained optimally via the variational method. Also, \( u_n \) is the nth approximate solution and \( \dot{u}_n \) represents a restricted variation, i.e. \( \delta u_n = 0 \).

4. Implementation of the VIM

The governing equation (1) can be recast in the following form:

\[ \frac{d^2 y}{dt^2} + \Omega^2 y = F[u]. \tag{10} \]
The correctional function may be constructed in the following form:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) \left[ \frac{d^2 u_n(\tau)}{d\tau^2} + \Omega^2 u_n(\tau) - \hat{F}[u_n(\tau)] \right] d\tau. \]  

(12)

\( \hat{F}[u_n] \) is considered as a restricted variation, i.e. \( \delta \hat{F}[u_n] = 0 \). Calculating the variation of Eq. (12) and considering that \( \delta \hat{F}[u_n] = 0 \), the following stationary conditions are calculated:

\[
\begin{cases}
\frac{d^2}{d\tau^2} \lambda(\tau) + \Omega^2 \lambda(\tau) = 0 \\
\lambda(0) = 0 \\
1 - \frac{d\lambda}{d\tau}(\tau) = 0
\end{cases}
\]  

(13)

Solving Eq. (13), the Lagrange multiplier can be identified as:

\[
\lambda(\tau) = \frac{1}{\Omega} \sin \Omega(\tau - t).
\]  

(14)

On the other hand, by taking into consideration relation:

\[
\int_0^t \sin \Omega(\tau - t) \left[ \frac{d^2 u_n(\tau)}{d\tau^2} + \Omega^2 u_n(\tau) \right] d\tau = -\Omega u_n(t) + \Omega u_0(0) \cos \Omega t + \frac{du_0}{dt}(0) \sin \Omega t.
\]  

(15)

Eq. (12) can be rewritten as:

\[
\begin{align*}
\frac{d}{dt} \left( u_n(t) \cos \Omega t \right) + \frac{du}{dt}(t) \sin \Omega t & = -\frac{1}{\Omega} \int_0^t \sin \Omega(\tau - t) \left[ F[u_n(\tau)] \right] d\tau.
\end{align*}
\]  

(16)

Considering the initial conditions, \( u(0) = A \) and \( \dot{u}(0) = 0 \), the correctional function may be further simplified as follows:

\[
\begin{align*}
u_{n+1}(t) = A \cos \Omega t - \frac{1}{\Omega} \int_0^t \sin \Omega(\tau - t) \left[ F[u_n(\tau)] \right] d\tau.
\end{align*}
\]  

(17)

As an initial guess, \( u(0) = A \) can be considered as follows:

\[
u_0(t) = A \cos \Omega t.
\]  

(18)

Expanding \( F[u_0(t)] \), we arrive at:

\[
F[u_0(t)] = \left( A^2 \Omega^2 + \frac{3}{4} \varepsilon \Omega A^2 \Omega^2 - A \omega_n^2 - \frac{3}{4} \varepsilon A^2 \omega_n^2 \right) \cos \Omega t
\]  

+ \left( -\frac{1}{4} \varepsilon A^2 \omega_n^2 + \frac{9}{4} \varepsilon A \Omega^2 \right) \cos 3\Omega t.
\]  

(19)

By taking into account the relation:

\[
\begin{align*}
\int_0^t \sin \Omega(\tau - t) \left[ \cos \Omega n t \right] d\tau & = \begin{cases}
\cos n \Omega t - \cos \Omega t \\
\frac{n}{2} \sin \Omega t - n \Omega t, \quad n \neq 1.
\end{cases}
\end{align*}
\]  

(20)

Avoiding secular terms in the next iterations, the coefficient of the \( \cos \Omega t \) in \( F[u_0(t)] \) should be vanished. Thus, the first approximation of the frequency (i.e., \( \Omega_1 \)) is obtained as follows:

\[
\Omega_1^2 = \frac{\omega_n^2 - \frac{3}{4} \varepsilon A^2 \omega_n^2}{1 + \frac{1}{4} \varepsilon A^2}.
\]  

(21)

From Eqs. (17) and (20), for \( n = 1 \), the first-order approximate solution is obtained as:

\[
u_1(t) = C_1 \cos \Omega_1 t + C_3 \cos 3\Omega_1 t
\]  

where:

\[
C_1 = \frac{A + \frac{9}{32} \varepsilon A^3 - \frac{1}{32} \varepsilon A^3 \omega_n^2}{2 \Omega_1^2},
\]  

(23)

\[
C_2 = \frac{3}{32} \varepsilon A \omega_n^2 \Omega_1^2 - \frac{9}{32} \varepsilon A^3,
\]  

(24)

where \( \Omega_1 \) frequency is given by Eq. (21). Substituting Eq. (22) into Eq. (11), we may obtain:

\[
F[u_1(t)] = C_1 \cos \Omega_1 t + C_4 \cos 3\Omega_1 t + C_5 \cos 5\Omega_1 t + C_6 \cos 7\Omega_1 t + C_7 \cos 9\Omega_1 t
\]  

(25)

where \( C_1, C_4, C_5, C_6, C_7 \) are listed as below (we remark that we used the symbolic mathematic software, Maple 12, to obtain these coefficients):

\[
\begin{align*}
C_1 &= C_1 \Omega_1^2 + \frac{3}{4} \varepsilon \Omega \Omega_1 C_1^2 \Omega_1^2 - \omega_n^2 C_1 + \frac{3}{4} \varepsilon \Omega^2 \omega_n^2 C_1^3 \negthinspace - \frac{3}{4} \varepsilon \omega_n^2 C_1^2 \Omega_1^2 - \frac{3}{2} \varepsilon \omega_n^2 C_1 \Omega_1^2 - \frac{3}{4} \varepsilon \omega_n^2 C_1^3
\end{align*}
\]  

(26)

\[
\begin{align*}
C_4 &= C_4 \Omega_1^2 - \frac{1}{4} \varepsilon \omega_n^2 C_1^3 - \omega_n^2 C_2 + \frac{3}{4} \varepsilon \omega_n^2 C_3 \Omega_1^2 + \frac{9}{4} \varepsilon \Omega \Omega_1 \omega_n^2 C_1^3
\end{align*}
\]  

(27)

\[
\begin{align*}
C_5 &= C_5 \Omega_1^2 - \frac{3}{4} \varepsilon \omega_n^2 C_1 \Omega_1^2 - \frac{3}{2} \varepsilon \omega_n^2 C_1 \Omega_1^2
\end{align*}
\]  

(28)

\[
\begin{align*}
C_6 &= -\frac{3}{2} \varepsilon \omega_n^2 C_1 \Omega_1^2 - \frac{9}{4} \varepsilon \Omega \Omega_1 \omega_n^2 C_1^3
\end{align*}
\]  

(29)

\[
\begin{align*}
C_7 &= -\frac{1}{4} \varepsilon \omega_n^2 C_2^3 + \frac{81}{4} \varepsilon \Omega^2 \omega_n^2 C_2^3
\end{align*}
\]  

(30)

Avoiding the secular term in the next iteration requires:

\[
\begin{align*}
C_1 \Omega_1^2 + \frac{3}{4} \varepsilon \Omega \Omega_1 C_1^2 \Omega_1^2 - \omega_n^2 C_1 + \frac{3}{4} \varepsilon \Omega^2 \omega_n^2 C_1^3 - \frac{3}{4} \varepsilon \omega_n^2 C_1 \Omega_1^2 - \frac{3}{2} \varepsilon \omega_n^2 C_1 \Omega_1^2 - \frac{3}{4} \varepsilon \omega_n^2 C_1^3
\end{align*}
\]  

(31)

Substitution of Eqs. (23) and (24) into Eq. (31) results in the second approximate of the frequency (i.e., \( \Omega_2 \)):

\[
\begin{align*}
-\frac{25}{32} \varepsilon A^3 \omega_n^2 + \frac{33}{32} \varepsilon \Omega A^2 \Omega_1^2 + \frac{1}{32} \varepsilon A^3 \Omega_1^2 \Omega_1^2 + \frac{27}{64} \varepsilon^2 \Omega^2 \Omega_1^2 A^5
\end{align*}
\]  

(32)

\[
\begin{align*}
+ \frac{729}{4996} \varepsilon A^3 \Omega_1^2 A^7 \Omega_1^2 + \frac{2178}{65536} \varepsilon^2 A^2 \Omega^2 \Omega_1^2 \Omega_1^2 + \frac{9}{4096} \varepsilon A^3 \Omega_1^2 A^5
\end{align*}
\]  

(33)

\[
\begin{align*}
+ \frac{3}{65536} \varepsilon A^3 \Omega_1^2 A^7 \Omega_1^2 \Omega_1^2 - \frac{3}{64} \varepsilon A^3 \Omega_1^2 \Omega_1^2 \Omega_1^2 - A \Omega_2^2.
\end{align*}
\]  

(34)
5. Results and discussion

To show the accuracy of the proposed method, the approximate frequencies computed by VIM are compared with those obtained by other researchers. The procedures explained in the previous section are applied to obtain some sets of results, which are presented here. There are many parameters, which can be varied in the governing equation. Table 1 gives a comparison of obtained results with those published in the literature for different parameters, $A$, $\varepsilon$, $\xi$, $z$ and $\omega_o$. From Table 1, excellent agreement between the results of the variational iterative method and those reported in the literature were observed.

Figure 2 presents variation of frequency versus amplitude, associated with the influence of $\varepsilon = \left(\frac{\beta}{\xi}\right)$, corresponding to the fourth order variational iteration method for $m = 3$, $K_1 = 10$ and $K_2 = 5$. It can be observed that by increasing the vibration amplitude, $A$, the frequency tends to a constant value, independent of $\varepsilon$.

Figures 3–5 show the deflection of the nonlinear spring, $u(t)$, deflection of the linear spring, $y_1(t)$, and displacement of mass, $y_2(t)$, respectively, for $m = 1$, $A = 2$, $\varepsilon = 0.12$, $K_1 = 5$ and $K_2 = 50$.

In Table 1, VIM-1 and VIM-4 relative errors (defined as $100 \times \frac{\text{error}}{\text{error}_{\text{Ref}}}$, where $\text{error}_{\text{Ref}}$ is the frequency obtained via the Runge–Kutta method) in percent are reported. These comparisons show how the nonlinear frequencies converge to the exact values (obtained by numerical methods). As we expected, the largest deviations occur when the amplitude of vibrations 'A' increases.
As expected, the accuracy of the results predicted by fourth order VIM (VIM-4) is in better agreement compared to that produced by the first order VIM (VIM-1). Moreover, the results have excellent agreement with the Runge–Kutta numerical solution method. This technique can be potentially used for the analysis of strongly nonlinear vibration problems with high accuracy. As a significant conclusion, the obtained results show that the accuracy of the present VIM solution is considered in the presence of vibrations, with even large oscillation amplitudes (larger values of $A'$). Also, according to the results, using the VIM, accuracy and convergence rates of the solution increase.

6. Conclusions

In this study, the variational iteration method (VIM) has been used to obtain an analytical solution to the nonlinear free vibration of a conservative oscillator with inertia and static type cubic nonlinearities. Besides their irreplaceable theoretical value, analytical solutions can also serve as a benchmark to check the results of numerical calculations, and study various computational methods. A comprehensive parametric study of the dominant parameters (coefficient of nonlinear spring force, the linear portion ($K_L$) of the nonlinear spring constant, linear spring constant ($K_s$), deflection of nonlinear spring and mass) was carried out. Moreover, VIM is suitable, not only for weak nonlinear problems, but also, for strongly nonlinear problems. Results reveal that this method can be considered as a viable alternative to conventional methods, to solve highly nonlinear oscillatory systems. Also, it can be used to solve other conservative truly nonlinear oscillators with complex nonlinearities.

References


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