Nonlinear Structural Dynamics Analysis Using Weighted Residual Integration

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Nonlinear Structural Dynamics Analysis Using Weighted Residual Integration

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A new time integration method is proposed for solving a differential equation of motion of structural dynamics problems with nonlinear stiffness. In this method, it is assumed that order of variation of acceleration is quadratic in each time step and, therefore, variation of displacement in each time step is a fourth-order polynomial that has five coefficients. Other than implementing initial conditions and satisfying equation of motions at both ends, weighted residual integral is also used in order to evaluate the unknown coefficients. By increasing order of acceleration, more terms of Taylor series are used, which were expected to have better responses than the other classical methods. The proposed method is non-dissipative and its numerical dispersion showed to be clearly less than the other methods. By studying stability of the method, it reveals that critical time step duration for the proposed method can be larger than the critical time step duration obtained from other classical methods; also, computational time for analysis with critical time step duration for the proposed method becomes less in comparison with linear acceleration and central difference methods. The order of accuracy of the proposed method is four, which showed to be very accurate in almost all practical problems.

Keywords: nonlinear structural dynamics, weighted residual, numerical stability, numerical accuracy, dissipation, dispersion

Nomenclature

- $C$ = damping matrix
- $K$ = stiffness matrix
- $M$ = mass matrix
- $n$ = time step number
- $P$ = force vector
- $t$ = time
- $T$ = time period
- $u$ = displacement value for SDOF system
- $U$ = displacement vector
- $\alpha$ = order of accuracy
- $\lambda$ = Eigenvalue
- $\Delta t$ = time step duration
- $\Delta t_c$ = critical time step duration
- $\tau$ = variable of time into each time step
- $K_n$ = stiffness matrix at $n$th time step
- $U$ = velocity vector
- $\dot{U}$ = acceleration vector
- $\dot{u}$ = velocity value for SDOF system
- $\ddot{u}$ = acceleration value for SDOF system
- $u_{exact}$ = exact value of displacement
- $\omega$ = natural frequency
- $\bar{\omega}$ = numerical natural frequency
- $\xi$ = damping ratio
- $\xi_n$ = numerical damping
- $\mu_n$ = stiffness ratio at $n$th time step
- $T$ = numerical time period
- $\Omega$ = phase
- $\bar{\Omega}$ = numerical phase
- $A_n$ = magnification matrix for $n$th time step
- $X_{n+1}$ = recursive matrix for the end of $n$th time step
- $\rho(A)$ = spectral radius
- $error^t$ = relative error percent at time $t$

1. Introduction

A procedure for calculating the dynamic responses of structures is the direct time integration method, which can be applied for nonlinear stiffness by using a numerical step by step method. In this method, time is divided into several steps and in a stepwise manner, displacement, velocity, and acceleration responses at the end of each time step are calculated based on previous values [1–5]. In nonlinear analysis with variable stiffness, stiffness is calculated at the beginning of each time step and responses are evaluated at the end of this time step by assuming that calculated stiffness at the beginning is constant throughout each time step. Nonlinearity of stiffness is considered by calculating stiffness value again at the beginning of the next time step. Therefore, it can be said that nonlinear behavior is replaced with several linear segments in each time step [1, 2, 6, 7].

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Time integration algorithms are divided into two categories: explicit and implicit methods. In explicit methods, equation of motion is written at the beginning of each time step and responses are calculated at the end of it. But in implicit methods, equation of motion must be written at the end of each time step for calculating the responses at the same points [4, 6, 8–10]. Among explicit methods, central difference method is the simplest procedure for applying in the nonlinear dynamics problems and among implicit methods, Newmark’s methods are the most common procedure used in structural dynamics [1, 2, 4, 11, 12]. Some other known explicit methods are Runge-Kutta, predictor-corrector, stiffly stable, and Taylor series methods [4].

Relative period error (dispersion), numerical damping (dissipation), and order of accuracy are evaluated for studying the accuracy of numerical methods. Relative period error shows the numerical errors with elongating or shortening the natural period of vibration compared to exact values, and numerical damping is numerical error with decreasing or increasing the amplitude of vibration compared to exact values [13, 14].

The numerical stability depends on degree of error propagation from one step to the next time step, in which the solution will diverge from exact responses if a considerable value of error from the previous time step propagates to the next time step [13, 14]. Numerical method is named conditionally stable if instability occurs when the size of time step duration is more than critical time step duration. But if instability never happens by each size of time step duration, it is named as an unconditionally stable method. Central difference and linear acceleration methods (Newmark’s method) are two conditionally stable methods that have the highest range of stability than the other methods [4, 10].

Classical methods, such as Newmark’s methods, assume that variation of acceleration throughout each time step is constant or linear [1, 2, 4, 8, 12, 15]. In this article, a new time integration method is introduced, in which acceleration varies quadratically throughout each time step by means of weighted residual integration. It is obvious that by increasing the degree of variation of acceleration in each time step, more terms are used in Taylor series expansion in comparison with the other classical methods [16, 17]. Therefore, degree of accuracy of the proposed method will be higher than the other methods. Dispersion, dissipation, and order of accuracy are determined in order to evaluate accuracy of the proposed method. Also, numerical stability of a proposed method, which is an indication of the largest time step duration to be taken in the analysis, is being analyzed in the present study.

**2. Proposed Method**

In this study, acceleration in each time step is assumed to vary quadratically. Therefore, function of displacement is a fourth-order complete polynomial in each time step. This function has five unknown constants that need to be calculated in each step. These unknowns are found from (a) two initial conditions from the end of the previous time step, (b) satisfying equation of motion at both ends of the step, and (c) using weighted residual integration to be equal to zero. By evaluating these unknowns, displacement, velocity, and acceleration responses will be calculated at the end of each time step.

The governing equation of motion for a structural dynamics system with nonlinear stiffness is described as:

\[ M\ddot{U} + C\dot{U} + KU = P, \]

where \( M, C, \) and \( K \) are mass, damping, and stiffness matrices; \( P \) is applied force vector; and \( U, \dot{U}, \) and \( \ddot{U} \) are displacement, velocity, and acceleration vectors, respectively. Assume that the initial conditions are as follows:

\[ U(0) = U_0, \dot{U}(0) = \dot{U}_0, \]

where \( U_0 \) and \( \dot{U}_0 \) are initial displacement and velocity vectors, respectively. For the \( n \)th time step, variable of time changes as \( \tau = t - t_n, \) where \( t \in [t_n, t_{n+1}] \). By considering the time step as \( \Delta t, \) it can be expressed that \( \tau \in [0, \Delta t] \) in each time step. As mentioned earlier, function of displacement in \( \tau \) interval for the \( n \)th time step can be expressed as:

\[ U = a_n\tau^4 + b_n\tau^3 + c_n\tau^2 + d_n\tau + e_n, \]

where \( a_n \) to \( e_n \) are five unknown coefficients vectors. Therefore, by differentiation, velocity and acceleration can be expressed as:

\[ \dot{U} = 4a_n\tau^3 + 3b_n\tau^2 + 2c_n\tau + d_n, \]
\[ \ddot{U} = 12a_n\tau^2 + 6b_n\tau + 2c_n. \]

Coefficients vectors \( e_n \) and \( d_n \) are calculated by initial conditions of this time step as:

\[ e_n = U_n, d_n = \dot{U}_n. \]

The coefficient vector \( c_n \) is calculated by satisfying equation of motion at the beginning of this time step by substituting Eq. (6) into Eq. (1) as:

\[ M(2c_n) + C(d_n) + K_n(e_n) = P_n. \]

Thus, \( c_n \) is found to be as follows:

\[ c_n = (2M)^{-1}.(P_n - C(d_n) - K_n(e_n)). \]

By satisfying equation of motion at the end of present step, equation of motion can be expressed as follows:

\[ M\ddot{U}_{(n+1)} + C\dot{U}_{(n+1)} + K_nU_{(n+1)} = P_{(n+1)}, \]

which results in:

\[
M(12a_n\Delta t^2 + 6b_n\Delta t + 2c_n) \\
+ C(4a_n\Delta t^3 + 3b_n\Delta t^2 + 2c_n\Delta t + d_n) \\
+ F_{sn} + K_n\left(\left[a_n\Delta t^4 + b_n\Delta t^3 + c_n\Delta t^2 + d_n\Delta t + e_n\right] - e_n\right) = P_{(n+1)},
\]

where \( \Delta t \) is the size of time step duration to be taken in the analysis.
where $K_n$ is the stiffness matrix at the beginning of the present time step (nth time step), $F_{sn}$ is equivalent spring force vector at the beginning of the nth time step, which it is considered as:

$$F_{sn} = F_{s(n-1)} + K_{n-1}(U_n - U_{n-1}),$$

(11)

in which $F_{s(n-1)}$, $K_{n-1}$, and $U_{n-1}$ are equivalent spring force vector, nonlinear stiffness matrix, and displacement vector at the beginning of the previous time step, respectively.

Equation (10) has two unknown coefficients vectors, $a_n$ and $b_n$. One more equation is required in order to obtain these two unknowns. This equation can be found from the fact that the approximate solution has a zero average of residual on solving equation of motion throughout the time step [18, 19] as follows:

$$\int_0^{\Delta t} \text{(residual)} \, d\tau = 0,$$

(12)

so by considering residual within the time step, Eq. (12) can be written as:

$$\int_0^{\Delta t} (M\ddot{U} + C\dot{U} + KU - P) \, d\tau = 0.$$  

(13)

In Eq. (13), variation of load vector within the time step is assumed to be linear for a complicated situation as follows:

$$P = P_n + \frac{\tau}{\Delta t}(P_{n+1} - P_n).$$

(14)

Now values of vectors $a_n$ and $b_n$ can be calculated by using Eqs. (10) and (13) simultaneously in matrix form as follows:

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

(15)

where

$$X_{11} = 12M\Delta t^2 + 4C\Delta t^3 + K_n\Delta t^4,$$

$$X_{12} = 16M\Delta t + 3C\Delta t^2 + K_n\Delta t^3,$$

$$X_{21} = 4M\Delta t^3 + C\Delta t^4 + \frac{K_n\Delta t^5}{5},$$

$$X_{22} = 3M\Delta t^2 + C\Delta t^3 + \frac{K_n\Delta t^4}{4},$$

$$Y_1 = P_{(n+1)} - F_{sn} - (2M + 2C\Delta t + K_n\Delta t^2)c_n - (C + K_n\Delta t) \, d_n,$$

$$Y_2 = I_n - F_{sn} \Delta t - \left(2M\Delta t + C\Delta t^2 + \frac{K_n\Delta t^3}{3}\right) c_n - \left(C\Delta t + \frac{K_n\Delta t^2}{2}\right) d_n,$$

(16)

in which

$$I_n = \int_0^{\Delta t} P \, d\tau = \frac{(P_n + P_{(n+1)})\Delta t}{2}.$$  

(17)

Consequently, the responses at the end of the nth time step are calculated by using five coefficients vectors evaluated.

3. Numerical Stability

A single degree of freedom (SDOF) system with free vibration is considered to evaluate the stability of method in recursive matrix form written at the end of each time step and eigenvalues of magnification matrix are calculated [16, 20]. Recursive matrix form of responses at the end of the nth time step for the proposed method can be written as:

$$X_{n+1} = \begin{bmatrix} u_{(n+1)} \\ \dot{u}_{(n+1)} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_n \\ \dot{u}_n \end{bmatrix} = A_n X_n,$$

(18)

where $A_n$ is magnification matrix for the nth time step. By considering Eqs. (3) and (4) for the proposed method, the component of this matrix can be written as follows:

$$A_{11} = (3K_n^2\Delta t^4 M + 0.33K_n^2C^2\Delta t^4 C - 12K_nMC\Delta t^3$$

$$-104K_nM^2\Delta t^2 + 240M^2 + 120M^2C\Delta t$$

$$+20MC^2\Delta t^2)(W_1)^{-1},$$

$$A_{22} = (3K_n^2M\Delta t^4 - 0.33K_n^2C^2\Delta t^4 C + 12MC\Delta t^3$$

$$-104K_nM^2\Delta t^2 + 240M^2 - 120M^2C\Delta t$$

$$+20MC^2\Delta t^2)(W_1)^{-1},$$

$$A_{21} = -K_n\Delta t(-24K_nM\Delta t^2 + 0.33K_n^2\Delta t^4 + 240M^2)(W_1)^{-1},$$

(19)

in which

$$W_1 = M(240M^2 + 120MC\Delta t + 16MK_n\Delta t^2 + 20C^2\Delta t^2$$

$$+8CK_n\Delta t^3 + K_n^2\Delta t^4).$$

(20)

The characteristic equation of magnification matrix can be derived using the following equation:

$$|A_n - \lambda I| = 0,$$

(21)

where $\lambda$ is eigenvalues of magnification matrix. Equation (21) can be expressed for the proposed method as:

$$\lambda^2 - A_1\lambda + A_2 = 0,$$

(22)

which by considering $\omega_0 = \sqrt{K_0/M}$, $C = 2M\omega_0\zeta$, $K_n = K_n\mu_n$, and $\omega_n\Delta t = \Omega_0$, $A_1$ and $A_2$ can be found as follows:

$$A_1 = \frac{(160\xi_2\Omega_0^2 + 6\mu_n^2\Omega_0^2 + 20\mu_n\Omega_0^2 + 480)(W_2)^{-1}},$$

$$A_2 = \frac{(-240\xi_2\Omega_0 + 16\mu_n\Omega_0^2 + 80\xi_2\Omega_0^2$$

$$-16\mu_n\Omega_0^2 + \mu_n^2\Omega_0^2 + 240)(W_2)^{-1}},$$

(23)

in which

$$W_2 = 240 + 240\xi_2\Omega_0 + 16\mu_n\Omega_0^2 + 80\xi_2\Omega_0^2$$

$$+16\mu_n\Omega_0^2 + \mu_n^2\Omega_0^2.$$

(24)
where $\xi$, $\omega_0$, and $\mu_n$ represent damping ratio, initial natural frequency, and stiffness ratio at the beginning of the $n$th time step over initial stiffness, respectively.

A method is stable if absolute values of eigenvalues are not greater than unit. In other words, Eq. (22) has two eigenvalues as $\lambda_1$ and $\lambda_2$ in which its spectral radius $\rho(A)$ should be less than unit for numerical stability [6, 16, 20] as follows:

$$\rho(A) = \max(|\lambda_1|, |\lambda_2|) \leq 1. \quad (25)$$

As it can be seen in Eq. (23), spectral radius is a function of damping ratio, length of time step duration, and stiffness ratio. Therefore, to compare the stability of the proposed method with other conditionally stable methods, such as central difference and Newmark’s linear acceleration methods, spectral radius curves with various stiffness ratios are plotted against $\Delta t/T_0$. These plots are presented in Figures 1–5 for undamped condition ($\xi = 0$). As it can be seen in Figures 2–5, a small local instability is started in the proposed method and then stability is returned, which can be removed by increasing damping ratios as shown in Figure 6. From Figures 1–6, the following results are obtained.

For $\mu_n = 0$, which is an indication of ideally plastic material, there is no instability in these three methods and spectral radius is unit for all of them. For $\mu_n = 0.25$, the central difference method with $\Delta t > 0.62 T_0$, linear acceleration method with $\Delta t > 1.11 T_0$, and proposed method with $\Delta t > 2.46 T_0$ become unstable. Local instability for the proposed method near $\Delta t = 1.02 T_0$ to $1.1 T_0$ is removed by applying $\xi \geq 2.4\%$ damping. For $\mu_n = 0.5$, the central difference method with $\Delta t > 0.44 T_0$, linear acceleration method with $\Delta t > 0.76 T_0$, and proposed method with $\Delta t > 1.74 T_0$ become unstable. Local instability for the proposed method near $\Delta t = 0.72 T_0$ to $0.76 T_0$ is removed by applying $\xi \geq 3.4\%$ damping. For $\mu_n = 0.75$ central difference method with $\Delta t > 0.36 T_0$, linear acceleration method with $\Delta t > 0.62 T_0$, and proposed method with $\Delta t > 1.42 T_0$ become unstable. Local instability for the proposed method near $\Delta t = 0.60 T_0$ to $0.62 T_0$ is removed by applying $\xi \geq 4.6\%$ damping. Also, from these figures it is recognized that by increasing the stiffness ratio or reducing the yielding of material, range of stability of the methods presented become shorter and local instability for proposed methods, if it takes place, usually occurs for the lesser ratios of $\Delta t/T_0$. From the results obtained, it can also conclude that for various stiffness ratios, the proposed method has a higher interval of stability than the other conditionally stable methods presented. In the proposed method, for various stiffness ratios of $\mu_n = 0.25, 0.5, 0.75$, and 1, critical time steps duration obtained are $\Delta t_{cr} = 2.46 T_0$, $1.74 T_0$, $1.42 T_0$, and $1.22 T_0$, respectively. Also, in a general case, local instability appearing in this method can be removed by applying $\xi \geq 5\%$ damping.

![Fig. 1. Comparison of the spectral radius for $\xi = 0$ and $\mu_n = 0$.](image-url)
Fig. 2. Comparison of the spectral radius for $\xi = 0$ and $\mu_n = 0.25$.

Fig. 3. Comparison of the spectral radius for $\xi = 0$ and $\mu_n = 0.5$. 
Fig. 4. Comparison of the spectral radius for $\xi = 0$ and $\mu_n = 0.75$.

Fig. 5. Comparison of the spectral radius for $\xi = 0$ and $\mu_n = 1$. 
4. Numerical Accuracy

Numerical accuracy of every time integration method is evaluated by considering numerical dissipation, numerical dispersion, and order of accuracy of the method. If eigenvalues of a characteristic equation be complex conjugate, these values can be written as follows [16, 20]:

$$\lambda_{1,2} = e^{-\xi_n \bar{\Omega}_n} \cos \Omega_n = \frac{A_1 \pm i \sqrt{4 A_2 - A_1^2}}{2},$$

(26)

where $\bar{\Omega}_n = \bar{\omega}_n \Delta t$ is the numerical phase, $\bar{\xi}_n$ is numerical damping, and $i = \sqrt{-1}$. Numerical phase $\bar{\Omega}_n$ and numerical damping (dissipation) $\bar{\xi}_n$, for an undamped system are found to be:

$$\bar{\Omega}_n = \tan^{-1} \left( \frac{\sqrt{4 A_2 - A_1^2}}{A_1} \right),$$

(27)

$$\bar{\xi}_n = -\ln \frac{\|\lambda_{1,2}\|}{\Omega_n},$$

(28)

where $\|\lambda_{1,2}\|$ represents the norm of eigenvalues.

Relative period error (dispersion) is calculated from the following equation:

$$PE = \frac{T_n - T_0}{T_0} = \frac{\Omega_n}{\bar{\Omega}_n} - 1,$$

(29)

where $T_n = \frac{2\pi}{\omega_n}$ and $T_0 = \frac{2\pi}{\omega_0}$ are numerical period and exact period of solution, respectively.

Numerical dampings of several stiffness ratios are plotted in Figures 7–10. From Eq. (28), it can be said that the numerical damping is positive for $\|\lambda_{1,2}\| < 1$ and negative for $\|\lambda_{1,2}\| > 1$ (unstable condition). There is no numerical damping for $\|\lambda_{1,2}\| = 1$. According to these figures, the proposed method for $\mu_n = 1$ has local numerical damping near $\Delta t = 0.51 T_0$ to $0.55 T_0$, which, in a general case, can be said that the proposed method is non-dissipative.

Relative period errors are plotted versus $\Delta t/T_0$ in Figures 11–14 for various stiffness ratios, which show that the proposed method has much less dispersion than the other methods. The period of linear acceleration and proposed methods will be elongated and the central difference method will be shortened. It can be concluded from Figures 7–14 that the dissipation and dispersion of methods increase with increasing stiffness ratios. Order of accuracy can be evaluated by using a local truncation error [16, 21] as follows:

$$|u - u_{\text{exact}}| = \beta \left( \frac{\Delta t}{T_0} \right)^{\alpha},$$

(30)

where $u$ and $u_{\text{exact}}$ are numerical, and exact values of displacement at time $t$, $\alpha$, and $\beta$ are constants that can be calculated from regression analysis. In Eq. (30), $\alpha$ is an order of accuracy of the numerical method. Order of accuracy of the proposed method obtained from regression analysis varies between 3.5 to 4.5 in comparison with linear acceleration and central
Fig. 7. Comparison of the numerical damping (dissipation) for $\mu_n = 0.25$.  
Fig. 8. Comparison of the numerical damping (dissipation) for $\mu_n = 0.5$.  

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**Fig. 7.** Comparison of the numerical damping (dissipation) for $\mu_n = 0.25$.  

**Fig. 8.** Comparison of the numerical damping (dissipation) for $\mu_n = 0.5$.  

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Fig. 9. Comparison of the numerical damping (dissipation) for $\mu_n = 0.75$.

Fig. 10. Comparison of the numerical damping (dissipation) for $\mu_n = 1$. 
Fig. 11. Comparison of the relative period error (dispersion) for $\mu_n = 0.25$.

Fig. 12. Comparison of the relative period error (dispersion) for $\mu_n = 0.5$. 
Fig. 13. Comparison of the relative period error (dispersion) for $\mu_n = 0.75$.

Fig. 14. Comparison of the relative period error (dispersion) for $\mu_n = 1$. 
difference methods, which are about two. It can be said that the proposed method is more accurate (dissipative, dispersion, and order of accuracy) than the other methods being normally used in the nonlinear dynamic analysis.

5. Examples

In order to see overall behavior as well as to confirm numerical stability and accuracy of the proposed method, three classical examples are being considered.

**Example 1. Single Degree of Freedom Problem [22]**

A second-order nonlinear differential equation of single degree of freedom system is considered as follows:

\[ \ddot{u} + 0.1\dot{u} + u^3 = 2\sin\left(\frac{4\pi t}{15}\right), \]

(31)

with initial conditions of \( u(0) = 0 \) and \( \dot{u}(0) = 0 \).

Results of displacement responses obtained by central difference, linear acceleration, and proposed methods are shown in Figure 15 for \( \Delta t = 0.1 \) sec.

Let’s define relative error percent at time \( t \) (error') as follows:

\[ \text{error'} = \left| \frac{u' - u'_{\text{exact}}}{u_{\text{exact}}} \right| \times 100, \]

(32)

where \( u' \) and \( u'_{\text{exact}} \) are numerical and exact solution (displacement responses) at time \( t \), respectively. The relative error percent versus time is shown in Figure 16. It can be seen from this figure that the proposed method has better response and less error in comparison with the other methods for the same time step increment.

**Example 2. Two-Story Shear Building [20]**

A two-story shear building is considered in this example with initial conditions \( U_0 = [0, 0]^T \) and \( \dot{U}_0 = [0, 0]^T \). This building has flexurally rigid floor beams and slabs. Nonlinear story stiffness for each story is defined as:

\[ k = k_0 \left[ 1 + \eta (\Delta U)^2 \right], \]

(33)

where \( \Delta U \) and \( k_0 \) are story drift and initial stiffness, respectively. The bottom story has \( k_0 = 10^8 \text{N/m} \) and \( \eta = -100 \), and the top story has \( k_0 = 10^5 \text{N/m} \) and \( \eta = -0.1 \). Negative values for coefficient \( \eta \) reveal that this building may show some softening behavior in its stiffness. The system is excited by a ground acceleration of \( 100 \sin(\frac{4\pi t}{15}) \) at the base of the building and lumped masses are considered to be \( 10^4 \text{kg} \) and \( 10^3 \text{kg} \) for bottom and top stories, respectively. Natural frequencies of the system are found to be 9.995 and 100.05 rad/sec, respectively. Displacement responses obtained from linear acceleration method (Newmark’s method) by a time step increment of 0.001 sec resulted in exact solutions. Figures 17–20 show the comparison of displacement responses with \( \Delta t = 0.02 \) sec.

![Fig. 15. Displacement responses for single degree of freedom system.](image-url)
Fig. 16. Relative error percent for single degree of freedom.

Fig. 17. Displacement responses for bottom story of shear building ($\Delta t = 0.02$ sec).
Fig. 18. Displacement responses for bottom story of shear building ($\Delta t = 0.04$ sec).

Fig. 19. Displacement responses for top story of shear building ($\Delta t = 0.02$ sec).
and $\Delta t = 0.04$ sec to exact solution for bottom and top stories. According to Figure 17, response of the bottom story from central difference method is not converged, and from Figures 17 and 19, the proposed method has more accurate results in comparison with the other methods for the same time step increment. Also, from Figures 18 and 20, it can be said that at first central difference and then linear acceleration methods become unstable for $\Delta t = 0.04$ sec but responses from the proposed method are still stable at the same time step duration.

**Example 3. n-DOF Spring-Mass System [20]**

In this example, consider the general condition for dynamical system of spring-mass, as shown in Figure 21. This system has structural properties, such as $m_i = 100$ kg and $k_i = 10^7[1 - (u_i - u_{i-1})^2]$ N/m in which $i = 1, 2, \ldots, n$. Also, the system is excited by a ground acceleration of $10 \sin(\omega t)$. Three different systems are considered with $n = 50, 100, \text{and } 200$ for comparing results. The lowest natural frequencies for these systems are 9.84, 4.95, and 2.62 rad/sec, respectively, and the highest natural frequency is 632.4 rad/sec for all three different systems. A time step increment of $\Delta t = 0.001$ sec obtained with the linear acceleration method (Newmark’s method) is considered to be an exact solution. Displacement responses versus time are plotted in Figures 22–24 for 50-DOF, 100-DOF, and 200-DOF spring-mass system. Analysis is accomplished by considering critical time step duration for each method in which critical time step duration for central difference, linear acceleration, and proposed methods are 0.003 sec, 0.005 sec, and 0.012 sec, respectively. In order to evaluate and compare computational time efforts required between presented methods, average time calculations of methods have been recorded and tabulated in Table 1 in which analysis is accomplished with Core 2 Due T9300 2.50 GHz CPU. According to presented results in this table, solution time for the proposed method is much less than the other methods. Therefore, it can be concluded that, although in

<table>
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<th>Central difference method</th>
<th>Linear acceleration method</th>
<th>Proposed method</th>
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<td>4.83</td>
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Fig. 22. Displacement responses of 50-DOF system.

Fig. 23. Displacement responses of 100-DOF system.
the proposed method the displacement functions are higher order but due to larger time step increments, computational time to reach the solution becomes less than methods normally used for analysis of nonlinear structural dynamics problems.

Conclusions

A new step by step integration technique was illustrated for structural dynamics problems with nonlinear stiffness. To get better accuracy, acceleration was considered to be quadratic within each time step. Therefore, fourth-order polynomial of displacement with five unknown coefficients is obtained. Those coefficients could be found from initial conditions from satisfying equation of motion at the beginning of the present time step, from satisfying equation of motion at the end of present time step, and equating weighted residual integration to be zero within step. The proposed method has higher stability and accuracy in comparison with other conditionally stable methods. For instance, it was shown that for $\mu_a = 0.5$, critical time step duration for proposed method turns out to be $1.74T_0$, whereas for central difference and linear acceleration methods it was $0.44T_0$ and $0.76T_0$, respectively. The proposed method is non-dissipative and numerical dispersion error is much less than the other methods. Also, order of accuracy of the proposed method turns out to be around four, which is higher than the other methods that are about two. By increasing order of variation of acceleration in each time step as quadratic, responses become more accurate and computational time for analysis with critical time step duration becomes less in comparison with the linear acceleration method.

References